PRIMITIVE HEREDITY IDEALS

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Abstract. Let R be a left Artinian ring. Dlab and Ringel have shown that R is hereditary if and only if every chain of idempotent ideals can be refined to a heredity chain [1]. In particular, if R is a basic hereditary ring, then every primitive ideal is a heredity ideal. The converse to this is clearly false. (See Example 1). We will introduce a class of rings that includes serial rings and monomial algebras, for which the converse does hold.

Throughout this paper, R will be a basic left Artinian ring (with unity) with a basic set of primitive idempotents $\tau = \{e_1, \dots, e_n\}$ and with J the Jacobson radical of R. We will refer to ideals Re_iR as primitive ideals. If M is an R-module, we denote the Loewy length of M by $L(M)$, and the composition length of M by $c(M)$.

Recall that an ideal I of R is *heredity* if $I^2 = I$, ${}_RI$ is projective, and $IJI = 0$. The ring R is *quasi-hereditary* if there exists a chain of ideals,

$$0 = I_0 \subset I_1 \subset \cdots \subset I_l = R,$$

called a *heredity chain*, such that I_k/I_{k-1} is a heredity ideal in R/I_{k-1} for each $k = 1, \dots, l$ [1,2]. If I is an idempotent ideal of R, then $I = ReR$ for some idempotent e in R. Moreover, e may be taken to have the form $e = e_1 + \cdots + e_k$ for some $k \le n$ and a suitable ordering of τ [2]. Furthermore, if $I = R(e_1 + \cdots + e_k)R$ is a heredity ideal, then each ideal Re_iR, $i = 1, \dots, k$, is again a heredity ideal of R [2]. Thus, [2] with a suitable ordering of τ, any heredity chain can be refined to a heredity chain of the form

$$0 \subset Re_1R \subset R(e_1 + e_2)R \subset \cdots \subset R(e_1 + \cdots + e_{n-1})R \subset R.$$

Dlab and Ringel have shown that R is hereditary if and only if every chain of idempotent ideals can be refined to a heredity chain [1]. We first consider the trace of projective left modules in the radical of the ring R. If Re is projective, then $ReJ = \mathrm{Tr}_J(Re)$. Note that ReJ is the right radical of the idempotent ideal ReR.

Lemma 1. Let ReR be a primitive ideal. The following are equivalent.

(a) ReR is a heredity ideal.
(b) ReJ is a projective left R-module.
(c) For each $i = 1, \dots, n$, $\mathrm{Tr}_{Je_i}(Re)$ is either zero or isomorphic to a direct sum of copies of Re.

Proof. We may assume $e = e_1$. For any module ${}_RN$, $Re_1N = \mathrm{Tr}_N(Re_1)$. Thus, the equivalence of (b) and (c) follows immediately from the direct sum decomposition

$$Re_1J = \oplus_{i=1}^n Re_1Je_i = \oplus_{i=1}^n \mathrm{Tr}_{\ Je_i}(Re_1).$$

Since R is basic, $Re_1Re_i = Re_1Je_i$ for $i = 2, \ldots, n$. Hence, we have the decomposition

$$\begin{aligned} Re_1R &= \oplus_{i=1}^n Re_1Re_i \\ &= Re_1 \oplus (\oplus_{i=2}^n Re_1Je_i) \\ &= Re_1 \oplus (\oplus_{i=2}^n \mathrm{Tr}_{Je_i}(Re_1)). \end{aligned}$$

Assume condition (c). From this last decomposition we have that Re_1R is a direct sum of copies of Re_1 and is therefore projective. If $Re_1Je_1 \neq 0$, then $1 \leq c(Re_1Je_1) \leq c(Je_1) < c(Re_1)$, and thus, Re_1Je_1 cannot be isomorphic to a direct sum of copies of Re_1. Hence, we must have $Re_1Je_1 = 0$ and Re_1R is heredity.

To see that (a) implies (b) it suffices to observe that if Re_1R is heredity, then we have that $\mathrm{Tr}_{Je_1}(Re_1) = Re_1Je_1 = 0$.

Dlab and Ringel have shown that the notion of a heredity ideal (and thus the notion of a quasi-hereditary ring) is two-sided. That is, if I is a heredity ideal of R, then I is also projective as a right R-module [1]. Thus, there exists a corresponding version of Lemma 1 for right R-modules. In particular, we have the following corollary.

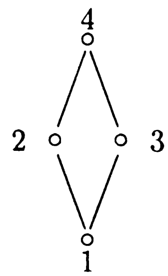
Corollary 1. For a primitive idempotent e in R, ${}_RReJ$ is projective if and only if JeR_R is projective.

Note that a primitive ideal ReR which is projective as a left R-module is not a heredity ideal if and only if $eJe \neq 0$. Moreover, if $eJe \neq 0$, then $1 \leq c(ReJe) \leq c(Je) < c(Re)$ so that $ReJe \neq 0$ and $ReJe$ (and thus, ReJ) is not projective. Thus, we have the following result.

Corollary 2. A primitive ideal ReR is heredity if and only if both ${}_RReJ$ and JeR_R are projective modules.

In particular, if ReR is a primitive heredity ideal, then ${}_RReJ$ (JeR_R) is a direct sum of local left (right) ideals. However, for J to be a direct sum of local left ideals, it does not suffice that each primitive ideal of R be heredity. Consider the following example.

Example 1. Let k be a field and consider the incidence algebra of the poset.



The indecomposable projective left R-modules have diagrams:

1	2	3		4	
	1	1	2		3
				1	

Notice that lgldim $R = 2$ so that R is quasi-hereditary but not left hereditary. It is clear that the primitive ideals Re_iR $(1 \leq i \leq 4)$ are heredity ideals. It is also clear that J is not a direct sum of local left ideals. Observe that $R(e_2+e_3)R/Re_2R$ is not a projective left R/Re_2R-module and that $R(e_2+e_3)R/Re_3R$ is not a projective left R/Re_3R-module. Hence, the chain of idempotent ideals $0 \subset R(e_2+e_3)R \subset R$ cannot be refined to a heredity chain.

In Example 1, we see that if e is a primitive idempotent, then ReJ is a direct sum of local left ideals of R. However, the trace in J of the decomposable projective module $R(e_2+e_3)$ is neither local, nor a direct sum of local left ideals.

We will use the following characterization of tree subsets due to Burgess, Fuller, Green, and Zacharia [3].

Proposition 1. (Burgess, Fuller, Green, and Zacharia) Let $m = L(R)$. Let X be a subset of $R \setminus \{0\}$ such that $X = \cup_{i=1}^{n} e_i X$ and if $x, y \in X$ with $x \neq y$ then $Rx \neq Ry$. Then, X is a tree subset for R if and only if X can be written $X = Y_0 \cup \cdots \cup Y_{m-1}$ so that $R = \oplus_{y \in Y_0} Ry$; and for each l, $1 \leq l \leq m-1$, and $x \in Y_{l-1}$, there are subsets $Y_{lx} \subseteq Y_l$ so that $Y_l = \cup_{x \in Y_{l-1}} Y_{lx}$ and $Jx = \oplus_{y \in Y_{lx}} Ry$. Moreover, under these conditions, $J^l = \oplus_{y \in Y_l} Ry$ for $l = 1, \ldots, m-1$.

Theorem 1. The following are equivalent.

(a) J^k is a direct sum of local left ideals for all $k = 1, \ldots, L(R) - 1$.

(b) ReJ^k is a direct sum of local left ideals for all $k = 1, \ldots, L(R) - 1$ and for all idempotents $e \in R$.

(c) There exists a tree subset for the regular module ${}_RR$.

Proof.

(a $\Leftrightarrow$ c) This follows from Corollary 1.3 in [3].

(b $\Rightarrow$ a) By hypothesis, $R \cdot 1 \cdot J^k = J^k$ is a direct sum of local left ideals for each $k = 1, \ldots, L(R) - 1$.

(c $\Rightarrow$ b) Suppose there exists a tree subset $X = Y_0 \cup \cdots \cup Y_{L(R)-1}$ for ${}_RR$. Let e be an idempotent in R and let $x \in X$. We claim that $ReRx$ is either zero or a direct sum of local left ideals. To prove this, we induct on $l = L(Rx)$.

$l = 1$: In this case Rx is semisimple so that $ReRx$ is either zero or semisimple.

$l > 1$: We first note that $x \notin Y_{L(R)-1}$. Since $x = e_i x$ for some basic idempotent e_i, Rx is a local left ideal of R. Thus, if $ReRx = Rx$ we are done. Suppose $ReRx \neq Rx$. Then $ReRx \subset Jx$ and therefore $ReRx = ReJx$. But, $Jx = \oplus_{y \in Y_{(l+1)x}} Ry$ by Proposition 1. For $y \in Y_{(l+1)x}$, $Ry \subset Jx$ so that $L(Ry) < L(Rx)$. Thus, by induction, $ReRy$ is either zero or a direct sum of local left ideals. Thus, $ReRx = ReJx = \oplus_{y \in Y_{(l+1)x}} ReRy$ is either zero or a direct sum of local left ideals.

Let $1 \leq k \leq L(R) - 1$. Then by Proposition 1, $J^k = \oplus_{x \in Y_k} Rx$. Thus, $ReJ^k = \oplus_{x \in Y_k} ReRx$ is either zero or a direct sum of local left ideals.

Now, the existence of a tree subset for the regular module ${}_RR$ is a defining property of left monomial rings [3]. Thus, we have the following examples of rings satisfying the conditions of Theorem 1 [3].

Example 2. The following rings satisfy the conditions of Theorem 1.

(a) Left monomial rings.
(b) Left serial rings.
(c) Monomial algebras.
(d) Left hereditary left Artinian rings.
(e) Left Artinian rings with $J^2 = 0$.

Recall that a ring R is l-hereditary if for every two indecomposable projective R-modules P and Q, every non-zero homomorphism $h: P \to Q$ is monic [4]. We note that l-hereditary rings are quasi-hereditary [2]. As was shown by Burgess and Fuller, if ReR is a primitive heredity ideal, then every non-zero homomorphism $h: Re \to R$ is monic [2]. Hence, if every primitive ideal is heredity, R is l-hereditary. Conversely, suppose that R is l-hereditary and ReR a primitive ideal. Then any non-zero homomorphism $h: Re \to Re$ is monic. Thus, $eJe \subset ReJe = \mathrm{Tr}_{Je}(Re) = 0$. However, it is easy to see that not every primitive ideal in an l-hereditary ring is a heredity ideal. Consider the following example.

Example 3. Let Γ be the digraph

$$1 \overset{a}{\underset{b}{\rightrightarrows}} 2 \overset{c}{\underset{d}{\rightrightarrows}} 3.$$

Let k be a field and let $R = k\Gamma/I$, where $I = (cb - da)$. Then the indecomposable projective left R-modules have diagrams as follows.

$$\begin{array}{ccccc} & & 1 & & \\ & 2 & & 2 & \\ 3 & & 3 & & 3 \end{array} \qquad\qquad \begin{array}{ccc} & 2 & \\ 3 & & 3 \\ & & \end{array} \qquad\qquad \begin{array}{c} 3 \\ \\ \\ \end{array}$$

Notice that ${}_R Re_2 R = Re_2 \oplus Je_1$, and $Je_1 = Re_2 Je_1 = \mathrm{Tr}_{Je_1}(Re_2)$. Since $c(Je_1) = 5$ and $c(Re_2) = 3$, we see that Je_1 (and hence, ${}_R Re_2 R$) is not projective. Thus, R is not a left hereditary ring. We claim that R is l-hereditary. To see this, it will suffice to show that any non-zero homomorphism $h\colon Re_2 \to Je_1$ is monic. But any such homomorphism h is given by right multiplication by some $x = e_2 x e_1 \in e_2 J e_1 = \langle a, b\rangle$. Thus, $x = \alpha a + \beta b$ for some $\alpha, \beta \in k$.

Now, $Re_2 = \langle e_2, c, d\rangle$. Suppose $\lambda_1 e_2 + \lambda_2 c + \lambda_3 d \in \mathrm{Ker}\ (h)$, with $\lambda_1, \lambda_2, \lambda_3 \in k$. Then, $0 = h(\lambda_1 e_2 + \lambda_2 c + \lambda_3 d) = \lambda_1 \alpha a + \lambda_1 \beta b + \lambda_2 \alpha ca + (\lambda_3 \alpha + \lambda_2 \beta) da + \lambda_3 \beta db$. If either $\alpha \neq 0$ or $\beta \neq 0$, we see that $\lambda_i = 0$ for $i = 1, 2, 3$. Thus, h is monic and R is l-hereditary.

The rings in Examples 1 and 3 are l-hereditary rings which are not left hereditary. We will show that under conditions weaker than those of Theorem 1, the notions of l-hereditary and left hereditary are the same.

Theorem 2. Suppose that J is a direct sum of local left ideals. Then the following are equivalent.
(a) R is left hereditary.
(b) R is l-hereditary.
(c) Every primitive ideal of R is a heredity ideal.

Proof.

(a $\Rightarrow$ b) This is clear.

(b $\Rightarrow$ c) Let R be l-hereditary and let $I = Re_i R$ be a primitive ideal of R. As we have seen, if R is l-hereditary then $eJe = 0$ for every primitive idempotent e.

Thus, it will suffice to show that ${}_RI$ is projective. For $j \neq i$, we have $Re_iRe_j = \mathrm{Tr}_{Re_j}(Re_i) = Re_iJe_j$. Thus,

$$Re_iR = \bigoplus_{j=1}^{n} Re_iRe_j = Re_i \oplus \bigoplus_{j \neq i} Re_iJe_j.$$

Now, each non-zero Re_iJe_j is a direct sum of local left ideals of the form Im (ρ_x), where the homomorphism $\rho_x\colon Re_i \to Je_j$ is given by right multiplication by x. By assumption, ρ_x is monic and thus, Re_iR is projective.

(c $\Rightarrow$ a) Assume that every primitive ideal is heredity. Recall that R is left hereditary if and only if Je_k is projective for each k, $1 \leq k \leq n$ [5].

Let $1 \leq k \leq n$. Since $J = \oplus_{i=1}^{n} Je_i$ is a direct sum of local left ideals, $Je_k = \oplus_{j=1}^{l} L_j$ for some collection $L_1, \dots, L_l$ of local left ideals. Let $1 \leq i \leq l$ and consider the local left ideal L_i. Assume L_i has projective cover Re_t. Notice that $t \neq k$ since $Re_kJe_k = 0$. Since Re_tR is heredity, $Re_tRe_k = Re_tJe_k$ is a direct sum of copies of Re_t. But $Re_tJe_k = \oplus_{j=1}^{l} Re_tL_j$ and hence, $Re_tL_i = \mathrm{Tr}_{L_i}(Re_t) = L_i$ is isomorphic (by Krull-Schmidt) to a copy of Re_t. Thus, L_i is projective and R is left hereditary.

Since the radical of a left hereditary ring is necessarily a direct sum of local left ideals, we also have the following result.

Corollary 3. Suppose R is l-hereditary. Then, R is left hereditary if and only if J is a direct sum of local left ideals.

Suppose J is a direct sum of local left ideals. We then have the following dichotomy of the left global dimensions of such rings R with the property that each primitive ideal is a projective left R-module.

Corollary 4. If J is direct sum of local left ideals and each primitive ideal is projective as a left R-module, then either R is left hereditary or lgldim $R = \infty$.

Proof. If R is not left hereditary, then by Theorem 2 there exists an i, $1 \leq i \leq n$, such that Re_iR is not a heredity ideal. Hence, $e_iJe_i \neq 0$ and by Corollary 1.5 of [6], we have that lgldim $R = \infty$.

As noted above, if ReR is a heredity ideal, then both ${}_RReR$ and ReR_R are projective. Thus, there exists a right-hand version of Theorem 2 and we have the following result.

Corollary 5. Suppose ${}_RJ$ is a direct sum of local left ideals, and J_R is a direct sum of local right ideals. Then R is hereditary if and only if every primitive ideal of R is a heredity ideal.

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