INDUCED FIBRATIONS ON SPACES OF FIBER TRANSFERRING MAPS: THE C^r CASE

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Abstract. For C^r differentiable manifolds E and B of the same dimension, and $\pi: E \to B$ a local C^r diffeomorphism and onto, we examine, as with the topological case, the extent to which some basic properties of π are inherited by the induced map p on spaces of fiber transferring maps with respect to π.

If E and B are two connected, C^r differentiable manifolds ($r = 1, 2, \ldots, \infty$) of the same dimension, and $\pi: E \to B$ is a local C^r diffeomorphism and onto, then one can consider the C^r counterparts of the spaces $C_\pi(E)$, $H_\pi(E)$, $\mathrm{Top}_\pi(E)$, $C(B)$, $H(B)$ and $\mathrm{Top}(B)$ considered in [2]. The new spaces are denoted by $C^r_\pi(E)$, $H^r_\pi(E)$, $\mathrm{Diff}^r_\pi(E)$, $C^r(B)$, $H^r(B)$, and $\mathrm{Diff}^r(B)$, respectively. They are all given the weak or the strong C^r topology [1]. The two topologies coincide for compact manifolds. The proofs which follow, cover the case r is finite. However, all the results hold also for $r = \infty$, by the way weak and strong C^∞ topologies are defined [1].

Proposition 1. The obvious maps $p: C^r_\pi(E) \to C^r(B)$, $p: H^r_\pi(E) \to H^r(B)$, and $p: \mathrm{Diff}^r_\pi(E) \to \mathrm{Diff}^r(B)$ are well defined and continuous.

Proof. We work with the first map.

To prove that p is well-defined, means that if $\overline{f} \in C^r_\pi(E)$, then $p(\overline{f}) = f \in C^r(B)$; but, locally, $f = \pi \circ \overline{f} \circ \pi^{-1}$, and so it is C^r as a composition of C^r maps. Trivially, this implies that f is globally C^r and, thus, well defined.

For $\overline{f} \in C^r_\pi(E)$, let $N^r(\{U_i, \phi_i\}, \{K_i\}, \{V_i, \psi_i\}, \{\epsilon_i\}, i \in \Lambda)$ be a basic neighborhood of f [1].

Let us fix a U_i from the collection $\{U_i\}_{i\in\Lambda}$ (this collection is locally finite); let $k \in K_i$; then $f(k) \in V_i$. We take now $e, e_1 \in E$ such that $\overline{f}(e) = e_1 \in \pi^{-1}(f(k))$ and $e \in \pi^{-1}(k)$. There exists a chart $\tilde{V}^k_i$ of e_1 such that $\pi(\tilde{V}^k_i) \subset V_i$, and $\pi: \tilde{V}^k_i \to \pi(\tilde{V}^k_i)$ is a C^r diffeomorphism. We can find a chart $\tilde{W}^k_i$ around e such that $\overline{f}(\tilde{W}^k_i) \subset V^k_i$, and a chart U^k_i around k, with a compact neighborhood C^k_i of k such that $C^k_i \subset U^k_i \subset U_i$, and $\pi(\tilde{W}^k_i) = U^k_i$, with $\pi: \tilde{W}^k_i \to U^k_i$ a C^r diffeomorphism. Then $f(U^k_i) \subset \pi(\tilde{V}^k_i)$. We consider now $\pi^{-1}(C^k_i) \cap \tilde{W}^k_i$. If we repeat this for each $k \in K_i$, we can get a finite cover of K_i with $U^{k_l}_i$ and $C^{k_l}_i$, $l = 1, 2, \ldots, j_i$, ($C^{k_l}_i \subset U^{k_l}_i$). The collection $\{\tilde{W}^{k_l}_i\}_{i\in\Lambda, l=1,2,\ldots,j_i}$ is locally finite as is easily seen, and $\overline{f}(\pi^{-1}(C_i^{k_l}) \cap \tilde{W}_i^{k_l}) \subset \tilde{V}_i^{k_l}$. We consider the collection $\{\epsilon_i^{k_l}\}_{i\in\Lambda, l=1,2,\dots,j_i}$ of positive numbers, where $\epsilon_i^{k_l} = \epsilon_i$. Then, for the basic neighborhood $N^r(\{\tilde{W}_i^{k_l}, \phi_i \circ \pi\}, \{\pi^{-1}(C_i^{k_l}) \cap \tilde{W}_i^{k_l}\}, \{\tilde{V}_i^{k_l}, \psi_i \circ \pi\}, \{\epsilon_i^{k_l}\}, i \in \Lambda, l = 1, 2, \dots, j_i)$ of $\overline{f}$, we have that if $\overline{g}$ belongs to this neighborhood, then $g = p(\overline{g})$ satisfies: $g(K_i) \subset V_i$, $i \in \Lambda$. Let $x \in (\phi_i \circ \pi)(\pi^{-1}(C_i^{k_l}) \cap \tilde{W}_i^{k_l}) = \phi_i(C_i^{k_l})$; then, for $s = 0, \dots, r$, $\|D^s(\psi_i \circ \pi \circ \overline{f} \circ \pi^{-1} \circ \phi_i^{-1})(x) - D^s(\psi_i \circ \pi \circ \overline{g} \circ \pi^{-1} \circ \phi_i^{-1})(x)\| < \epsilon_i$. Since, locally, $\overline{f} = \pi^{-1} \circ f \circ \pi$, $\overline{g} = \pi^{-1} \circ g \circ \pi$, the above inequality leads to $\|D^s(\psi_i \circ f \circ \phi_i^{-1})(x) - D^s(\psi_i \circ g \circ \phi_i^{-1})(x)\| < \epsilon_i$, for $s = 0, \dots, r$, $x \in \phi_i(K_i)$; thus, $g \in N^r(\{U_i, \phi_i\}, \{K_i\}, \{V_i, \psi_i\}, \{\epsilon_i\}, i \in \Lambda)$, and p is continuous. The Proposition holds for the other two maps as an obvious consequence.

The following Lemma has an easy proof which is omitted.

Lemma 2. The set theoretical inclusion $i: C^r(B) \to C(B)$ ($H^r(B) \to H(B)$, $\mathrm{Diff}^r(B) \to \mathrm{Top}(B)$, respectively) is a continuous map if $C^r(B)$ ($H^r(B)$, $\mathrm{Diff}^r(B)$, respectively) is given the weak or the strong C^r topology.

For the following results, the spaces are given only the weak C^r topology.

Proposition 3. $C_\pi^r(E)$ is the fibered product of $i: C^r(B) \to C(B)$, and $p: C_\pi(E) \to C(B)$. Similar statements hold for $H_\pi^r(E)$, $\mathrm{Diff}_\pi^r(E)$.

Proof. The fibered product of i and p is the subspace $\mathcal{U}$ of $C^r(B) \times C_\pi(E)$ consisting of all pairs $(f, \overline{f})$ for which $f = p(\overline{f})$. We now show that the correspondence $\tau: C_\pi^r(E) \to \mathcal{U}$, given by $\tau(\overline{f}) = (f, \overline{f})$ is a homeomorphism. That it is 1-1, is immediate; to show that it is onto, all we have to show is that if $f \in C^r(B)$ has a lifting $\overline{f} \in C_\pi(E)$, then $\overline{f} \in C_\pi^r(E)$. Since π is a local C^r diffeomorphism, locally $\overline{f} = \pi^{-1} \circ f \circ \pi$, thus $\overline{f}$ is C^r differentiable, and τ is onto. Now $\tau = p \times i$, where i is the inclusion $C_\pi^r(E) \to C_\pi(E)$ which, by an obvious modification of Lemma 2, is continuous. Since p is also continuous (Proposition 1), τ is continuous.

Finally, we have to show that τ^{-1} is continuous. Let $N^r(\{U_i, \phi_i\}, \{K_i\}, \{V_i, \psi_i\}, \{\epsilon_i\}, i \in I)$ be a basic neighborhood of $\overline{f} \in C_\pi^r(E)$ (here I is finite). We can always assume that $\pi: U_i \to \pi(U_i)$, $\pi: V_i \to \pi(V_i)$ are C^r diffeomorphisms, for each $i \in I$. To see why this is so, let us say that U_i, V_i, for some i, do not have this property. Then, for each $k \in K_i$, consider a chart (W_k, ψ_i) around $\overline{f}(k)$, with $W_k \subset V_i$ and $\pi: W_k \to \pi(W_k)$ being a C^r diffeomorphism. By continuity of $\overline{f}$ we may find a compact neighborhood C_k of k and a chart (U'_k, ϕ_i) around k such that $C_k \subset U'_k \subset U_i$, $\overline{f}(C_k) \subset W_k$, and $\pi: U'_k \to \pi(U'_k)$ is a C^r diffeomorphism. Now K_i is covered by a finite number of such C_k's say $C_{k_1}, C_{k_2}, \dots, C_{k_j}$. We replace in our neighborhood,

U_i with $\{U'_{k_l}\}_{l=1}^{j}$, K_i with $\{C_{k_l}\}_{l=1}^{j}$, V_i with $\{W_{k_l}\}_{l=1}^{j}$, and the positive numbers corresponding to the new sets can all be taken equal to ϵ_i. The new neighborhood is obviously a subneighborhood of the old one and the charts satisfy the desired property.

We consider now the following neighborhood of $(f,\overline{f})$ in $C^r(B)\times C_\pi(E)$: $N^r(\{\pi(U_i),\phi_i\circ\pi^{-1}\},\{\pi(K_i)\},\{\pi(V_i),\psi_i\circ\pi^{-1}\},\{\epsilon_i\}, i\in I)\times(\bigcap_{i\in I}S(K_i,V_i))$.

If $(g,\overline{g})$ is a member of $\mathcal{U}$ which belongs to the above neighborhood, then on one hand $\overline{g}(K_i)\subset V_i$, and on the other hand, for $s=0,1,\dots,r$ and $x\in\phi_i(K_i)$, $\|D^s(\psi_i\circ\overline{f}\circ\phi_i^{-1})(x)-D^s(\psi_i\circ\overline{g}\circ\phi_i^{-1})(x)\|=\|D^s(\psi_i\circ\pi^{-1}\circ f\circ\pi\circ\phi_i^{-1})(x)-D^s(\psi_i\circ\pi^{-1}\circ g\circ\pi\circ\phi_i^{-1})(x)\|<\epsilon_i$, since $x\in\phi_i(K_i)=(\phi_i\circ\pi^{-1})(\pi(K_i))$, and locally (on U_i or V_i) π is a C^r diffeomorphism, so that π^{-1} makes sense. That is, $\overline{g}$ belongs to the chosen neighborhood of $\overline{f}$, and τ^{-1} is continuous.

One should notice that since π has taken a local C^r diffeomorphism, it has the unique path lifting property. This, together with Propositions 1, 2, and 3 in [2], and Proposition 3 above gives:

Proposition 4. If $\pi\colon E\to B$ is a fibration (regular), then the maps $p\colon C^r_\pi(E)\to C^r(B), H^r_\pi(E)\to H^r(B), \mathrm{Diff}^r_\pi(E)\to\mathrm{Diff}^r(B)$ are all fibrations (regular).

The next theorem follows the C^r counterpart of Theorem 5 in [2].

Theorem 5. If $\pi\colon E\to B$ is a regular covering map, then so are the obvious maps $p\colon C^r_\pi(E)\to p(C^r_\pi(E)), H^r_\pi(E)\to p(H^r_\pi(E)), \mathrm{Diff}^r_\pi(E)\to p(\mathrm{Diff}^r_\pi(E))$.

Proof. By Theorem 5 in [2], the corresponding map $p\colon C_\pi(E)\to p(C_\pi(E))$ is a regular covering map. From Proposition 3 above, one immediately sees that $C^r_\pi(E)$ is the fibered product of $C_\pi(E)\to p(C_\pi(E))$ and the inclusion $p(C^r_\pi(E))\to p(C_\pi(E))$; this proves the theorem for the C^r_π case. The other two cases are handled similarly.

Now let $\pi\colon E\to B$ be a regular fibration. If $G(C^r_\pi(E)/C^r(B))$, $G(H^r_\pi(E)/H^r(B))$, $G(\mathrm{Diff}^r_\pi(E)/\mathrm{Diff}^r(B))$ denote the groups of deck transformations of the induced fibrations p of Proposition 4, then we have:

Proposition 6. The following diagram is a commutative diagram of groups and group homomorphisms:

$$
\begin{array}{ccccc}
G(E/B) & & & & \\
\downarrow{\scriptstyle i} & & & & \\
G(C_\pi(E)/C(B)) & \xrightarrow{r} & G(H_\pi(E)/H(B)) & \xrightarrow{r} & G(\mathrm{Top}_\pi(E)/\mathrm{Top}(B)) \\
\downarrow{\scriptstyle r} & & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle r} \\
G(C^r_\pi(E)/C^r(B)) & \xrightarrow{r} & G(H^r_\pi(E)/H^r(B)) & \xrightarrow{r} & G(\mathrm{Diff}^r_\pi(E)/\mathrm{Diff}^r(B)) \\
\uparrow{\scriptstyle i} & & & & \\
G(E/B) & & & &
\end{array}
$$

where r is defined in all cases by restriction to the corresponding subset, and the i's are group embeddings given by $i(\overline{f})(\overline{h}) = \overline{f} \circ \overline{h}$.

Proof. The upper half of the diagram is just Proposition 4 in [2]. The horizontal r's are well-defined homomorphisms by an obvious C^r modification of the argument given in the above mentioned proposition. We now show that the vertical r's are well-defined homomorphisms. If $\overline{f} \in C^r_\pi(E)$, and $\sigma \in G(C_\pi(E)/C(B))$, then $\sigma(\overline{f}) \in C^r_\pi(E)$ (since $p(\overline{f}) \in C^r(B)$, all its liftings are members of $C^r_\pi(E)$ as we saw in the proof of Proposition 3; but $\sigma(\overline{f})$ is such a lifting of $f = p(\overline{f})$, proving the assertion). That $r(\sigma)$ is 1-1 and onto is straightforward (repeat similar argument in Proposition 4 in [2]). Continuity of $r(\sigma)$ follows from the fact that this map is the restriction of the map $1 \times \sigma \colon C^r(B) \times C_\pi(E) \to C^r(B) \times C_\pi(E)$ to the fibered product $C^r_\pi(E)$. Finally, $(r(\sigma))^{-1} = r(\sigma^{-1})$, which is continuous; thus, r is well defined. That it is a homomorphism is straightforward. The same proof works for the other two vertical r's. We now show that the obvious map $i \colon G(E/B) \to G(\mathrm{Diff}^r_\pi(E)/\mathrm{Diff}^r(B))$ is a well-defined group embedding. We first notice that the elements of $G(E/B)$ are all the liftings of the identity $1 \colon B \to B$; consequently, they are all C^r diffeomorphisms. This means that the map $\overline{f} \circ \overline{h}$ is C^r differentiable for $\overline{f} \in G(E/B)$, $\overline{h} \in \mathrm{Diff}^r_\pi(E)$. By a similar argument given in Proposition 4 in [2], $i(\overline{f})$ is 1-1, onto. Continuity of $i(\overline{f})$ follows from the fact that it is the restriction of $1 \times i(\overline{f}) \colon \mathrm{Diff}^r(B) \times \mathrm{Top}_\pi(E) \to \mathrm{Diff}^r(B) \times \mathrm{Top}_\pi(E)$ to the fibered product $\mathrm{Diff}^r_\pi(E)$. Since $i(\overline{f})^{-1} = i(\overline{f}^{-1})$, $i(\overline{f})$ is bicontinuous, thus i is well defined (the same proof works for $i\colon G(E/B) \to G(C^r_\pi(E)/C^r(B))$ and $i\colon G(E/B) \to G(H^r_\pi(E)/H^r(B))$. The same argument employed in Proposition 4 in [2], proves that i is a group embedding. Commutativity of the whole diagram is straightforward.

References

1. M. W. Hirsch, *Differential Topology*, Springer-Verlag, 1976.
2. M. Magiropoulos, "Induced Fibrations on Spaces of Fiber Transferring Maps," *Missouri Journal of Mathematical Sciences*, 12 (2000), 36–41.

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