UNIFORM CONVERGENCE OF THE T-DISTRIBUTIONS

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1. Introduction. The T-Distribution is primarily used by statisticians for studying the mean of a normally distributed population. However, the probability density functions of this distribution have an analytic property that should be of interest to mathematicians, in particular to students of advanced calculus and analysis. This distribution provides an example of commonly used functions which converge uniformly on compact sets to another commonly used function, namely the Standard Normal density function. In this article, we shall provide a proof of this uniform convergence.

2. Definition of the Densities. The T-Distribution with n degrees of freedom, denoted by $T(n)$, is defined by the probability density function

$$f_n(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\sqrt{n\pi}}\left(1+\frac{x^2}{n}\right)^{-\frac{n+1}{2}}, \tag{1}$$

where the Gamma function $\Gamma(z)$ is defined for $z > 0$ by

$$\Gamma(z) = \int_0^\infty y^{z-1}e^{-y}dy. \tag{2}$$

The Standard Normal Distribution, denoted by $Z \sim N(0,1)$, is defined by the probability density function

$$f(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}. \tag{3}$$

These density functions are defined for all x and create "Bell-Shaped" curves which are symmetric about the y-axis (Figures 1 and 2).

It is our goal to show that $\{f_n(x)\}$ converges uniformly to $f(x)$, denoted by $f_n \Rightarrow f$, on any interval $[a,b]$. Formally speaking: Given $\epsilon > 0$, there should exist an integer N such that $|f_n(x) - f(x)| < \epsilon$ for all x in $[a,b]$, provided $n \geq N$.

3. Separating the Functions. To begin simplifying the proof, we first note that it suffices to show $f_n \Rightarrow f$ on $[-t, t]$ for $t > 0$, by choosing $t \geq \max\{|a|, |b|\}$. But since f_n and f are even functions, it further suffices to prove the result only on the interval $[0, t]$.

Secondly, we shall separate the density functions into constant terms and function terms. To do so, we let

$$C_n = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\sqrt{n\pi}} \text{ and } C = \frac{1}{\sqrt{2\pi}}. \tag{4}$$

Further, we let

$$g_n(x) = \left(1 + \frac{x^2}{n}\right)^{-\frac{n+1}{2}} \text{ and } g(x) = e^{-\frac{x^2}{2}}.$$

Then our goal is to show that $C_n g_n \Rightarrow Cg$ on $[0, t]$. If we consider the difference, we can see how the proof will begin to break down into several parts. By the Triangle Inequality and the fact that $|g(x)| \leq 1$, we have for all x

$$\begin{aligned} |C_n g_n(x) - Cg(x)| &= |C_n g_n(x) - C_n g(x) + C_n g(x) - Cg(x)| \\ &\leq |C_n|\ |g_n(x) - g(x)| + |g(x)|\ |C_n - C| \\ &\leq |C_n|\ |g_n(x) - g(x)| + |C_n - C|. \end{aligned} \tag{5}$$

Thus, we will want to show that $C_n \to C$ and that $g_n \Rightarrow g$ on $[0, t]$.

4. The Limit of the Constants. To evaluate the limit of the constants C_n, we shall first re-write the Gamma function values in terms of factorials and exponentials. Three well known facts about the Gamma function will be needed: (i) $\Gamma(k) = (k-1)!$ for positive integers k; (ii) $\Gamma(z) = (z-1)\Gamma(z-1)$ for $z > 1$; and (iii) $\Gamma(1/2) = \sqrt{\pi}$. There properties are standard exercises in a mathematical statistics course [3].

Lemma 1. For a positive integer n, $\Gamma(n/2)$ can be written as follows:

$$\Gamma\left(\frac{n}{2}\right) = \begin{cases} \left(\frac{n}{2} - 1\right)!, & \text{for } n \text{ even;} \\ \frac{(n-1)!}{2^{n-1}\left(\frac{n-1}{2}\right)!}\sqrt{n}, & \text{for } n \text{ odd.} \end{cases}$$

Proof. Since $\Gamma(k) = (k-1)!$ for positive integers k, the result is immediate for even n. Since $\Gamma(1/2) = \sqrt{\pi}$, the result is also immediate for $n = 1$.

Suppose the result is true for some odd integer n and then consider the next odd integer $n+2$. Since $\Gamma(z) = (z-1)\Gamma(z-1)$, we have

$$\begin{aligned}
\Gamma\left(\frac{n+2}{2}\right) &= \frac{n}{2}\Gamma\left(\frac{n}{2}\right) \\
&= \frac{n}{2}\frac{(n-1)!}{2^{n-1}\left(\frac{n-1}{2}\right)!}\sqrt{\pi} \\
&= \frac{n!}{2^{n}\left(\frac{n-1}{2}\right)!}\sqrt{\pi}\left[\left(\frac{n+1}{n+1}\right)\left(\frac{1}{2/2}\right)\right] \\
&= \frac{(n+1)!}{2^{n+1}\left(\frac{n+1}{2}\right)!}\sqrt{\pi},
\end{aligned}$$

which is the desired result for $n+2$. The lemma then follows by induction on the odd integers.

By substituting the result of Lemma 1 into the definition of C_n, we obtain the following result.

Corollary 1. The constants C_n can be re-written as follows:

$$C_n = \begin{cases} \frac{n!}{2^n\left(\frac{n}{2}\right)!\sqrt{n}\left(\frac{n}{2}-1\right)!}, & \text{for } n \text{ even;} \\ \frac{\left[\left(\frac{n-1}{2}\right)!\right]^2 2^{n-1}}{\pi\sqrt{n}(n-1)!}, & \text{for } n \text{ odd.} \end{cases}$$

In order to evaluate $\lim_{n\to\infty} C_n$, we shall need Stirling's Formula [2] which allows us to replace $n!$ by $\sqrt{2\pi n}(n/e)^n$ when taking the limit. With this substitution, a little algebra, and the fact that $\lim_{n\to\infty}(1+a/n)^n = e^a$, we can now show that $C_n \to C$.

Theorem 1. $\lim_{n\to\infty} C_n = C$.

Proof. We apply Stirling's Formula by substituting for the factorial expressions in Corollary 1. For n even,

$$\lim_{n\to\infty} C_n = \lim_{n\to\infty} \frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}{2^n\sqrt{2\pi(n/2)}\left(\frac{n}{2e}\right)^{n/2}\sqrt{n}\sqrt{2\pi(n/2-1)}\left(\frac{n-2}{2e}\right)^{n/2-1}}$$

$$= \lim_{n\to\infty} \frac{n^{n/2}(n-2)}{e\sqrt{n}\sqrt{2\pi(n-2)}(n-2)^{n/2}}$$

$$= \frac{1}{\sqrt{2\pi}}\left(\lim_{n\to\infty}\frac{n-2}{\sqrt{n^2-2n}}\right)\times\frac{1}{e}\times\left(\lim_{n\to\infty}\frac{1}{(1-2/n)^{n/2}}\right)$$

$$= \frac{1}{\sqrt{2\pi}}\frac{1}{e}\frac{1}{(e^{-2})^{1/2}}$$

$$= \frac{1}{\sqrt{2\pi}}.$$

We leave the details for odd n as an exercise.

Since $\{C_n\}$ converges to C, we obtain the nice consequence that the sequence $\{C_n\}$ is bounded by some constant M [4]. So we can now improve Equation (5) as follows:

$$|f_n(x) - f(x)| = |C_n g_n(x) - Cg(x)| \leq M|g_n(x) - g(x)| + |C_n - C|. \tag{6}$$

5. The Uniform Convergence of $\mathbf{g_n(x)}$. We next further separate $g_n(x)$ into the product $h_n(x)k_n(x)$, where

$$h_n(x) = \left(1+\frac{x^2}{n}\right)^{-\frac{n}{2}} \text{ and } k_n(x) = \left(1+\frac{x^2}{n}\right)^{-\frac{1}{2}}. \tag{7}$$

We can see here that $\{h_n(x)\}$ converges pointwise to $e^{-x^2/2} = g(x)$ and $\{k_n(x)\}$ converges pointwise to 1. Hence, $\{f_n(x)\} = \{C_n h_n(x)k_n(x)\}$ converges pointwise to $Cg(x) = f(x)$. Of course, our goal is to demonstrate uniform convergence on $[0, t]$. We shall first show the separate uniform convergence of the sequences $\{k_n(x)\}$ and $\{h_n(x)\}$.

Theorem 2. The sequence of functions $\{k_n(x)\}$ converges uniformly to 1 on the interval $[0, t]$.

Proof. Since $s(y) = y^{-1/2}$ is a decreasing function, we have for all x in $[0, t]$,

$$1 \geq \left(1 + \frac{x^2}{n}\right)^{-\frac{1}{2}} \geq \left(1 + \frac{t^2}{n}\right)^{-\frac{1}{2}}.$$

Since

$$\lim_{n\to\infty} k_n(t) = \lim_{n\to\infty} \left(1 + \frac{t^2}{n}\right)^{-\frac{1}{2}} = 1,$$

$\lim_{n\to\infty} k_n(x) = 1$ uniformly for x in $[0, t]$.

From Figure 2, it appears that the sequence of functions $\{f_n(x)\}$ increases to $f(x)$ towards the middle of the domain, but decreases to $f(x)$ on the tail ends. However, the functions $\{h_n(x)\}$ will form a decreasing sequence for each fixed x. This result is not immediately clear though. Consider the expression

$$\left(1 + \frac{x^2}{n}\right)^n.$$

As n increases, the base decreases but the exponent increases. Thus, it is not clear that this expression increases and that therefore, $\{h_n(x)\}$ is a decreasing sequence. However, the next two lemmas will demonstrate this property.

Lemma 2. For $0 \leq k \leq n$ and $a > 0$,

$$\binom{n}{k}\left(\frac{a}{n}\right)^k \leq \binom{n+1}{k}\left(\frac{a}{n+1}\right)^k.$$

Proof. We shall prove the result by induction on k. We can first observe that for $k = 0$, both sides of the inequality are equal to 1. For $k = 1$, both sides are equal to a. So we can now assume the result is true for an integer $k-1$. That is, by considering the ratio for $k-1$ we have

$$\frac{\binom{n}{k-1}\left(\frac{a}{n}\right)^{k-1}}{\binom{n+1}{k-1}\left(\frac{a}{n+1}\right)^{k-1}} \leq 1.$$

But this inequality simplifies to

$$\left(\frac{n+1}{n}\right)^{k-1} \frac{n-k+2}{n+1} \leq 1.$$

We then consider the ratio for k:

$$\begin{aligned}
\frac{\binom{n}{k}\left(\frac{a}{n}\right)^{k}}{\binom{n+1}{k}\left(\frac{a}{n+1}\right)^{k}} &= \left(\frac{n+1}{n}\right)^{k} \frac{n-k+1}{n+1} \\
&= \left(\frac{n+1}{n}\right)^{k-1} \frac{n-k+2}{n+1} \cdot \frac{n-k+1}{n-k+2} \cdot \frac{n+1}{n} \\
&\leq 1 \times \frac{n-k+1}{n-k+2} \cdot \frac{n+1}{n} \\
&= \frac{n^2-nk+2n+1-k}{n^2-nk+2n} \\
&\leq 1,
\end{aligned}$$

for $k \geq 1$. Hence, the result holds for $0 \leq k \leq n$.

Lemma 3. For $a \geq 0$, the sequence $(1+a/n)^n$ is increasing.

Proof. We simply use the Binomial Theorem to expand the expression for n and for $n+1$, and then compare terms using Lemma 2:

$$\left(1+\frac{a}{n}\right)^n = \sum_{k=0}^{n}\binom{n}{k}\left(\frac{a}{n}\right)^k \le \sum_{k=0}^{n}\binom{n+1}{k}\left(\frac{a}{n+1}\right)^k$$

$$\le \sum_{k=0}^{n+1}\binom{n+1}{k}\left(\frac{a}{n+1}\right)^k = \left(1+\frac{a}{n+1}\right)^{n+1}.$$

In particular, the sequence of functions

$$\left(1+\frac{x^2}{n}\right)^n$$

increases to e^{x^2}. Thus, the sequence

$$h_n(x) = \left(\left(1+\frac{x^2}{n}\right)^n\right)^{-\frac{1}{2}}$$

decreases to

$$g(x) = \left(e^{x^2}\right)^{-\frac{1}{2}}.$$

Since $\{h_n(x)\}$ is a sequence of continuous functions which decreases on the compact interval $[0,t]$ to the continuous function $g(x)$, Dini's Theorem [5] states that $\{h_n(x)\}$ in fact, converges uniformly to $g(x)$ to $[0,t]$. We therefore have our next result.

Theorem 3. The sequence of functions $\{h_n(x)\}$ converges uniformly on $[0,t]$ to

$$g(x) = e^{-\frac{x^2}{2}}.$$

We now know that $k_n(x) \Rightarrow 1$ and $h_n(x) \Rightarrow g(x)$ on $[0,t]$. Does this imply that the product $h_n(x)k_n(x)$ also converges uniformly to the product of limits? In general, the answer is no. A good advanced calculus exercise is to define two sequences of functions which converge uniformly for which the product does not converge uniformly. Fortunately for us here though, we have some additional boundedness conditions at our disposal. We know that $|g(x)| \leq 1$ for all x and $|k_n(x)| \leq 1$ for all x and all $n \geq 1$. Thus,

$$\begin{aligned} |h_n(x)k_n(x) - g(x)| &= |h_n(x)k_n(x) - k_n(x)g(x) + k_n(x)g(x) - g(x)| \\ &\leq |k_n(x)|\ |h_n(x) - g(x)| + |g(x)|\ |k_n(x) - 1| \\ &\leq |h_n(x) - g(x)| + |k_n(x) - 1|. \end{aligned} \tag{8}$$

Since $h_n(x) \Rightarrow g(x)$ and $k_n(x) \Rightarrow 1$ on $[0,t]$, equation (8) implies that $h_n(x)k_n(x) \Rightarrow g(x)$ on $[0,t]$. Thus we state:

Theorem 4. The sequence of functions $\{g_n(x)\}$ converges uniformly on $[0,t]$ to $g(x)$.

From equation (8), we can get an idea of some conditions that cause the product of two uniformly convergent sequences of functions also to converge uniformly. Suppose $h_n \Rightarrow g$ and $k_n \Rightarrow k$. (In (8), $k \equiv 1$.) If g is bounded and $\{k_n\}$ is uniformly bounded, then $h_n k_n$ will converge uniformly. For continuous sequences of functions on a closed interval $[a,b]$, this will always be the case. Indeed, g will be continuous if it is the uniform limit of the continuous functions $\{h_n\}$ [6]. Thus, g will be bounded on $[a,b]$. Moreover, each continuous function k_n is also bounded on $[a,b]$. But then a uniformly convergent sequence of bounded functions must be uniformly bounded (another good advanced calculus exercise).

6. Uniform Convergence of the T-Distribution. It would be a shame to complete this entire discussion without a single ϵ-δ proof. So we now put all the results together to obtain the desired uniform convergence.

Theorem 5. Let $f_n(x)$ be the probability density function of the T-Distribution with n degrees of freedom and let $f(x)$ be the probability density function of the Standard Normal Distribution. Then $\{f_n(x)\}$ converges uniformly to $f(x)$ on any compact interval $[a,b]$.

Proof. It suffices to prove the result on an interval $[-t,t]$; but since all functions are even, it is sufficient to prove the result on $[0,t]$.

Let $\epsilon > 0$ be given. By Theorem 1, there exist integers N_1 and N_2 such that if n is even and $n \geq N_1$, then $|C_n - C| < \epsilon/2$, and if n is odd and $n \geq N_2$, then $|C_n - C| < \epsilon/2$. By Corollary 2, there exists an integer N_3 such that if $n \geq N_3$, then $|g_n(x) - g(x)| < \epsilon/(2M)$ for all x in $[0,t]$, where the convergent sequence $\{C_n\}$ is bounded by M. Let $N = \max\{N_1, N_2, N_3\}$. If $n \geq N$, then by Equation (6),

$$|f_n(x) - f(x)| \leq M|g_n(x) - g(x)| + |C_n - C| < \epsilon,$$

for all x in $[0,t]$.

This mode of convergence is perhaps the strongest form of convergence (other than uniform convergence on $(-\infty, \infty)$). In particular, it implies convergence in distribution. That is, for all t

$$\lim_{n\to\infty} P(T(n) \leq t) = \lim_{n\to\infty} P(Z \leq t), \tag{9}$$

where $Z \sim N(0,1)$. The density functions are used to compute these probabilities. Thus, equation (9) is equivalent to

$$\lim_{n\to\infty} \int_{-\infty}^{t} f_n(x)dx = \int_{-\infty}^{t} f(x)dx. \tag{10}$$

This result can be proved alternately with basic integration theory. One can show that $\{f_n(x)\}$ converges pointwise to $f(x)$ and that $\{f_n(x)\}$ are uniformly bounded; hence, equation (10) follows from Arzela's Theorem [4] or by the Dominated Convergence Theorem [6]. Or one can show that $\{f_n(x)\}$ converges in measure (or probability) to $f(x)$ and show that this condition implies convergence in distribution [1].

However, by symmetry and since the total probability under the curves is 1,

$$P(T(n) \leq t) = \frac{1}{2} + \int_0^t f_n(x)dx \text{ and } P(Z \leq t) = \frac{1}{2} + \int_0^t f(x)dx.$$

Thus, to prove convergence in distribution, it suffices to show that for each t

$$\lim_{n\to\infty} \int_0^t f_n(x)dx = \lim_{n\to\infty} \int_0^t f(x)dx.$$

However, this result is an easy consequence of uniform convergence on $[0, t]$ and we leave it as an exercise.

References

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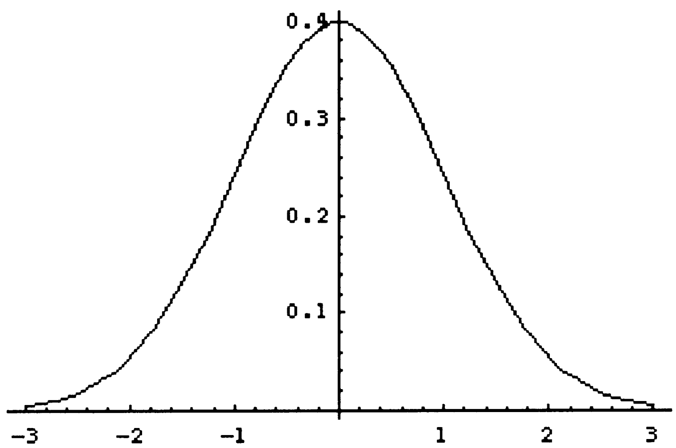

Figure 1. The Standard Normal Distribution.

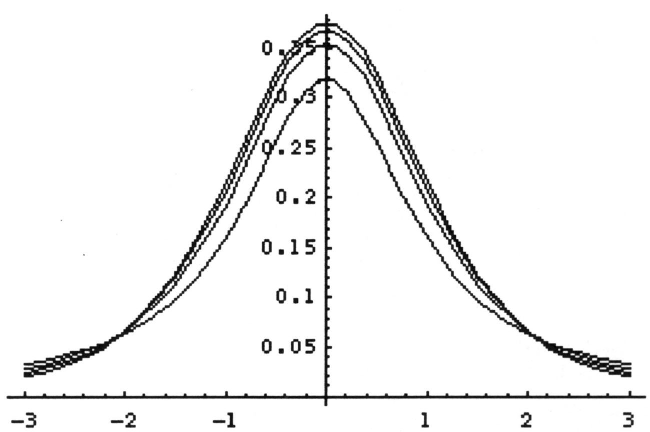

Figure 2. The $T(1)$-$T(4)$ Distributions.