

**ON THE INVESTIGATION OF A LINEAR EIGENVALUE
PROBLEM FOR MATRICES**

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Abstract. This paper investigates the properties of eigenvalues and eigenvectors of the problem $Jy + \lambda Ry = (1/\lambda)Cy$, where J is a “tridiagonal” real symmetric matrix and R and C are positive diagonal matrices. The results obtained are used to solve the corresponding system of differential equations with boundary and initial conditions.

1. Introduction. Let us consider the system of linear differential equations

$$a_{n-1} \frac{du_{n-1}(t)}{dt} + b_n \frac{du_n(t)}{dt} + r_n \frac{d^2u_n(t)}{dt^2} + a_n \frac{du_{n+1}(t)}{dt} = c_n u_n(t), \quad (1)$$

for $n = 0, 1, 2, \dots, N - 1$ with boundary conditions

$$u_{-1}(t) = 0, \quad u_N(t) + hu_{N-1}(t) = 0, \quad t \geq 0 \quad (2)$$

and initial conditions

$$u_n(0) = f_n, \quad \frac{du_n(0)}{dt} = g_n, \quad n = 0, 1, \dots, N - 1 \quad (3)$$

where $\{u_n(t)\}_{n=-1}^N$ is a desired solution; f_n, g_n ($n = 0, 1, \dots, N - 1$) are given complex numbers; the coefficients a_n, b_n, r_n , and c_n of the equation (1) and the number h in boundary condition (2) real besides

$$a_n \neq 0, \quad r_n > 0, \quad c_n > 0. \quad (4)$$

We seek the solution of equation (1), which has the form

$$u_n(t) = e^{\lambda t} y_n, \quad n = -1, 0, 1, \dots, N \quad (5)$$

where λ is a complex constant, and the y_n 's are complex numbers depending only upon λ and not upon t . We desire $\{y_n\}_{-1}^N$ to be nontrivial, that is not equal to 0, the zero vector. Substituting (5) into (1) and (2) we obtain $\lambda \neq 0$ and

$$a_{n-1}y_{n-1} + (b_n y_n + r_n \lambda)y_n + a_n y_{n+1} = \frac{1}{\lambda} c_n y_n, \quad n = 0, 1, 2, \dots, N-1 \quad (6)$$

$$y_{-1} = 0, \quad y_N + h y_{N-1} = 0 \quad (7)$$

Definition. The complex number λ is said to be an eigenvalue of boundary problem (6), (7) if for this value λ there exists a nonzero vector $\{y_n\}_{-1}^N$ satisfying equation (6) and boundary conditions (7). Further, the vector $y = \{y_n\}_0^{N-1}$ is called an eigenvector of problem (6), (7) corresponding to eigenvalue λ .

Thus, the functions in (5) are a nontrivial solution of problem (1), (2) if and only if λ is an eigenvalue and $y = \{y_n\}_0^{N-1}$ is the corresponding eigenvector of problem (6), (7).

Denote all the eigenvalues of problem (6), (7) by $\lambda_1, \dots, \lambda_m$ and the corresponding eigenvectors by $y^{(1)} = \{y_n^{(1)}\}_0^{N-1}, \dots, y^{(m)} = \{y_n^{(m)}\}_0^{N-1}$. Then by the linearity of problem (1), (2) the functions

$$u_n(t) = \sum_{j=1}^m \alpha_j e^{\lambda_j t} y_n^{(j)}, \quad n = -1, 0, 1, \dots, N \quad (8)$$

will form a solution of the problem (1), (2) where α_j ($j = 1, 2, \dots, m$) are arbitrary constants (independent of t and n). Now we must try to choose the constants α_j ($j = 1, 2, \dots, m$) so that (8) will also satisfy the initial conditions

$$\sum \alpha_j y_n^{(j)} = f_n, \quad \sum \alpha_j \lambda_j y_n^{(j)} = g_n, \quad n = 0, 1, \dots, N-1. \quad (9)$$

Definition. If for the arbitrary vectors $f = \{f_n\}_0^{N-1}$ and $g = \{g_n\}_0^{N-1}$ belonging to \mathbb{C}^N the unique expansions (9) hold with the same coefficients α_j ($j = 1, 2, \dots, m$) in both expansions, then we will say that the eigenvectors $y^{(1)}, \dots, y^{(m)}$ of problem (6), (7) form twofold basis in \mathbb{C}^N .

We will show that the boundary problem (6), (7) has precisely $2N$ simple real eigenvalues $\lambda_1, \dots, \lambda_{2N}$, half are negative and half are positive. The corresponding eigenvectors make up a twofold basis in \mathbb{C}^N , and we will give formulas for the coefficients α_j in (9). In addition, we will show that the eigenvectors corresponding only to the negative (positive) eigenvalues form an ordinary basis for \mathbb{C}^N .

2. Eigenvalue Problem. We consider the boundary value problem (6), (7) under condition (4). If $\{y_n\}_{-1}^N$ is a solution of problem (6), (7) then

$$\begin{aligned} (b_0 + \lambda r_0)y_0 + a_0y_1 &= (1/\lambda)c_0y_0 \\ a_{n-1}y_{n-1} + (b_n + \lambda r_n)y_n + a_ny_n + a_ny_{n+1} &= (1/\lambda)c_ny_n \\ a_{n-2}y_{n-2} + (b_{N-1} - ha_{N-1} + \lambda r_{N-1})y_{N-1} &= (1/\lambda)c_{N-1}y_{N-1} \end{aligned} \tag{10}$$

for $n = 1, 2, \dots, N$. Consequently, finding a nontrivial solution $\{y_n\}_{-1}^N$ of problem (6), (7) is equivalent to finding a nontrivial solution $\{y_n\}_0^{N-1}$ of system (10).

Setting

$$y = \begin{pmatrix} y_0 \\ y_1 \\ \dots \\ y_{N-1} \end{pmatrix}, J = \begin{pmatrix} b_0 & a_0 & 0 & 0 & \dots & 0 & 0 & 0 \\ a_0 & b_1 & a_1 & 0 & \dots & 0 & 0 & 0 \\ 0 & a_1 & b_2 & a_2 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & a_{N-3} & b_{N-2} & a_{N-2} \\ 0 & 0 & 0 & 0 & \dots & 0 & a_{N-2} & b_{N-1} - ha_{N-1} \end{pmatrix}, \tag{11}$$

$$R = \begin{pmatrix} r_0 & 0 & \dots & 0 \\ 0 & r_1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & r_{N-1} \end{pmatrix}, C = \begin{pmatrix} c_0 & 0 & 0 & \dots & 0 \\ 0 & c_1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & c_{N-1} \end{pmatrix},$$

we can write system (10) in the form

$$Jy + \lambda Ry = (1/\lambda)Cy. \tag{12}$$

We will investigate equation (12) in the space

$$\mathbb{C}^N = \{y = \{y_n\}_0^{N-1} : y_n \in \mathbb{C}, n = 0, 1, \dots, N - 1\}$$

with the inner product

$$(y, z) = \sum_{n=0}^{N-1} y_n \bar{z}_n \quad (13)$$

where the bar over a number denotes complex conjugation.

The matrices J , R , and C defined by (11) are selfadjoint, that is each of them satisfies the relation

$$(Ay, z) = (y, Az), \quad \text{for all } y, z \in \mathbb{C}^N. \quad (14)$$

At first assuming the existence of eigenvalues and eigenvectors of problem (12) we establish some of their properties.

Lemma 1. The eigenvalues of equation (12) are real.

Proof. Let the complex number λ be an eigenvalue of equation (12) and

$$y = \{y_n\}_0^{N-1} \neq 0$$

be a corresponding eigenvector. By forming the inner product of both sides of equation (12) by the vector y , we get

$$\lambda^2(Ry, y) + \lambda(Jy, y) - (Cy, y) = 0.$$

This equality is a quadratic equation with respect to λ , the discriminant of which is

$$(Jy, y)^2 + 4(Ry, y)(Cy, y) > 0$$

since the number (Jy, y) is real, in view of (14), and the numbers (Cy, y) and (Ry, y) are positive in view of (4). Consequently, the number λ will be real.

Lemma 2. The eigenvectors y and z in equation (12) corresponding to distinct eigenvalues λ and μ , respectively satisfy the “orthogonality” relation

$$\lambda\mu(Ry, z) + (Cy, z) = 0. \quad (15)$$

Proof. Multiplying in the sense of inner product the first of the equalities

$$Jy + \lambda Ry = (1/\lambda)Cy, \quad Jz + \mu Rz = (1/\mu)Cz$$

from the right by z and the second one from the left by y , and remembering that λ and μ are real in view of Lemma 1, we get

$$(Jy, z) + \lambda(Ry, z) = \frac{1}{\lambda}(Cy, z)$$

$$(y, Jz) + \mu(y, Rz) = \frac{1}{\mu}(y, Cz).$$

Subtracting the second result from the first one, and using property (14) of the matrices J , R and C , we have

$$(\lambda - \mu)(Ry, z) = (1/\lambda - 1/\mu)(Cy, z).$$

Hence, the Lemma follows from the condition $\lambda \neq \mu$.

Now we investigate the existence of other properties of eigenvalues and eigenvectors of equation (12), which is equivalent to problem (6), (7). For this purpose we define the solution $\{\varphi_n(\lambda)\}_{-1}^N$ of equation (6) that satisfies the initial conditions

$$\varphi_{-1}(\lambda) = 0, \quad \varphi_0(\lambda) = 1. \quad (16)$$

Using (16), we can recursively find $\varphi_n(\lambda)$, $n = 1, 2, 3, \dots, N$ and will have the form

$$\varphi_n(\lambda) = (1/\lambda^n)P_{2n}(\lambda), \quad n = 0, 1, \dots, N, \quad (17)$$

where $P_{2n}(\lambda)$ is a polynomial in λ of degree $2n$ and

$$P_{2n}(\lambda) = \frac{r_0 r_1 \cdots r_{n-1}}{a_0 a_1 \cdots a_{n-1}} \lambda^{2n} + \cdots + (-1)^n \frac{c_0 c_1 \cdots c_{n-1}}{a_0 a_1 \cdots a_{n-1}}. \quad (18)$$

It is easy to see that every solution $\{y_n(\lambda)\}_{-1}^N$ of equation (10) satisfying the initial condition $y_{-1} = 0$ is equal to $\{\varphi_n(\lambda)\}_{-1}^N$ up to a constant factor

$$\varphi_n(\lambda) = \alpha \varphi_n(\lambda), \quad n = -1, 0, 1, \dots, N. \quad (19)$$

Consequently, we have the following lemma.

Lemma 3. To each eigenvalue λ_0 of problem (6), (7) corresponds up to a constant factor a single eigenvector, which can be taken to be the vector $\{\varphi_n(\lambda_0)\}_0^{N-1}$.

Set

$$\chi(\lambda) = \varphi_N(\lambda) + h\varphi_{N-1}(\lambda). \quad (20)$$

The function $\chi(\lambda)$ is called the characteristic function of problem (6), (7).

Lemma 4. The eigenvalues of problem (6), (7) coincide with the roots of the function $\chi(\lambda)$.

Proof. Let λ_0 be an eigenvalue of problem (6), (7) and $\{y_n(\lambda_0)\}_{-1}^N$ be a non-trivial solution of (6), (7) with $\lambda = \lambda_0$. Then (17) holds with $\alpha \neq 0$ and from boundary condition $y_N(\lambda_0) + hy_{N-1}(\lambda_0) = 0$ we get $\chi(\lambda_0) = 0$.

Conversely, if $\chi(\lambda_0) = 0$, then $\{\varphi_n(\lambda_0)\}_{-1}^N$ will be a nontrivial (remembering that by (16) we have $\varphi_0(\lambda) \neq 0$) solution of boundary problem (6), (7).

Lemma 5. The roots of the function $\chi(\lambda)$ are simple.

Proof. Differentiating the equation

$$a_{k-1}\varphi_{k-1}(\lambda) + (b_k + \lambda r_k)\varphi_k(\lambda) + a_k\varphi_{k+1}(\lambda) = \frac{1}{\lambda}c_k\varphi_k(\lambda)$$

with respect to λ , we get

$$a_{k-1}\dot{\varphi}_{k-1}(\lambda) + (b_k + \lambda r_k)\dot{\varphi}_k(\lambda) + a_k\dot{\varphi}_{k+1}(\lambda) - \frac{1}{\lambda}c_k\varphi_k(\lambda) = -r_k\varphi_k(\lambda) - \frac{1}{\lambda^2}c_k\varphi_k(\lambda)$$

where the dot over the function indicates the derivative with respect to λ . Multiplying the first equation by $\dot{\varphi}_k(\lambda)$ and the second one by $\varphi_k(\lambda)$, and subtracting the left and right members of the resulting equations, we get

$$\begin{aligned} & a_{k-1} \left[\varphi_{k-1}(\lambda)\dot{\varphi}_k(\lambda) - \dot{\varphi}_{k-1}(\lambda)\varphi_k(\lambda) \right] - a_k \left[\varphi_{k+1}(\lambda)\dot{\varphi}_k(\lambda) - \dot{\varphi}_{k+1}(\lambda)\varphi_k(\lambda) \right] \\ &= \left(r_k + \frac{1}{\lambda^2}c_k \right) \varphi_k^2. \end{aligned} \quad (21)$$

Summing up the last equation for the values $k = 0, 1, \dots, n$ ($n \leq N - 1$) and using the initial conditions (16), we get

$$a_n \left[\varphi_{n+1}(\lambda) \dot{\varphi}_n(\lambda) - \dot{\varphi}_{n+1}(\lambda) \varphi_n(\lambda) \right] = \sum_{k=0}^n \left(r_k + \frac{1}{\lambda_0^2} c_k \right) \varphi_k^2(\lambda_0). \quad (22)$$

Let $\chi(\lambda_0) = 0$. In particular, setting (22), $n = N - 1$ and $\lambda = \lambda_0$, and using the equality $\varphi_N(\lambda_0) = -h\varphi_{N-1}(\lambda_0)$ which follows from the boundary condition (7), we have

$$a_{N-1} \dot{\chi}(\lambda_0) \varphi_{N-1}(\lambda_0) = \sum_{k=0}^{N-1} \left(r_k + \frac{1}{\lambda_0^2} c_k \right) \varphi_k^2(\lambda_0). \quad (23)$$

The right-hand side of (23) is not zero since $r_k > 0$, $c_k > 0$, λ_0 is in view of Lemma 1, $\varphi_k(\lambda_0)$ ($k = 0, 1, \dots, N - 1$) are real and not all zero. Besides, in the left-hand side of (23), the value $\varphi_{N-1}(\lambda_0)$ can not be equal to zero. Indeed, if $\varphi_{N-1}(\lambda_0) = 0$, and hence, by the uniqueness property of the solution of equality (6), we get $\varphi_n(0) = 0$, for all n , which is a contradiction. Thus, from formula (23) it follows that $\chi(\lambda_0) \neq 0$, that is the root λ_0 of the function $\chi(\lambda)$ is simple.

Lemma 6. The function $\chi(\lambda)$ has precisely $2N$ distinct roots.

Proof. Since $\varphi_n(\lambda)$ for each n is a polynomial in λ of degree $2n$, the function $\chi(\lambda) = \varphi_N(\lambda) + h\varphi_{N-1}(\lambda)$ will be a polynomial in λ of degree $2N$. Therefore, the function $\chi(\lambda)$ has $2N$ roots, which are distinct by virtue of Lemma 5.

Lemma 7. Half of the roots of the function $\chi(\lambda)$ are negative and the other half are positive.

Proof. The eigenvalue (12) is equivalent to the eigenvalue problem

$$(\lambda^2 R + \lambda J - C)y = 0. \quad (24)$$

(Note that the value $\lambda = 0$ is not an eigenvalue of (24)). Therefore, the roots of the function $\chi(\lambda)$ coincide with the roots of the polynomial $\det(\lambda^2 R + \lambda J - C)$.

Now we consider the auxiliary eigenvalue problem

$$(\lambda^2 C - \lambda \epsilon R - C(\epsilon))y = 0 \quad (25)$$

depending on parameter $\epsilon \in [0, 1]$, where the matrix $C(\epsilon)$ is obtained from the matrix C by means of multiplying all its nondiagonal elements by ϵ . It is obvious that the analog of condition (4) is fulfilled for all $\epsilon \in (0, 1]$.

The eigenvalues of equation (25) are nonzero for all $\epsilon \in [0, 1]$ and coincide with the roots of the polynomial

$$\det(\lambda^2 R + \lambda \epsilon J - C(\epsilon)). \quad (26)$$

For each $\epsilon \in (0, 1]$ the roots of the polynomial (26) are distinct by virtue of Lemma 5 being applicable to equation (25). Denote them by

$$\lambda_1(\epsilon) < \lambda_2(\epsilon) < \dots < \lambda_{2N}(\epsilon).$$

Since $\lambda_j(\epsilon)$ ($j = 1, 2, \dots, 2N$) are the eigenvalues of a matrix of order $2N$ being continuous in $\epsilon \in [0, 1]$ (see section 3) they will be continuous functions of ϵ [3]. Note that at the point $\epsilon = 0$ we do not state that $\lambda_1(\epsilon), \lambda_2(\epsilon), \dots, \lambda_{2N}(\epsilon)$ are distinct.

Now we show that for all values of $\epsilon \in (0, 1]$ half of the $\lambda_j(\epsilon)$ ($j = 1, 2, \dots, 2N$) are negative and the other half positive:

$$\lambda_j(\epsilon) < 0 \quad (j = 1, \dots, N), \quad \lambda_j(\epsilon) > 0 \quad (j = N + 1, \dots, 2N).$$

Hence, in particular, for $\epsilon = 1$, the statement of the lemma will follow.

Assume the contrary. For some value of $\epsilon \in (0, 1]$ let

$$\lambda_j(\epsilon) < 0 \quad (j = 1, \dots, K), \quad \lambda_j(\epsilon) > 0 \quad (j = K + 1, \dots, 2N) \quad (27)$$

where $0 \leq K \leq 2N$ and $K \neq N$ (for $K = 0$ all the eigenvalues $\lambda_j(\epsilon)$ are understood to be positive, and for $K = 2N$ negative). Since $\lambda_j(\epsilon)$ ($j = 1, \dots, 2N$) are different from zero and are distinct and continuous functions for all values of $\epsilon \in (0, 1]$, taking inequalities (27) to the limit as $\epsilon \rightarrow 0$, we get

$$\lambda_j(0) \leq 0 \quad (j = 1, \dots, K), \quad \lambda_j(0) \geq 0 \quad (j = K + 1, \dots, 2N).$$

But this is a contradiction, since for $\epsilon = 0$ the roots of the polynomial (26) are the numbers

$$\pm \sqrt{\frac{c_j}{r_j}} \quad (j = 0, 1, \dots, N - 2)$$

half of which are negative and the other half positive. Thus, the lemma is proved.

Now we can summarize the results obtained above in the following theorem.

Theorem 1. The boundary value problem (6), (7) has precisely $2N$ real distinct eigenvalues λ_j ($j = 1, 2, \dots, 2N$). These eigenvalues are different from zero, half of them are negative and the other half positive. To each eigenvalue λ_j corresponds up to constant factor a single eigenvector which can be the vector $\varphi^{(j)} = \{\varphi_n(\lambda_j)\}_{n=0}^{N-1}$ where $\{\varphi_n(\lambda)\}_{n=0}^{N-1}$ is a solution of equation (6) satisfying the initial condition (16).

Theorem 2. The eigenvector $\varphi^{(j)} = \{\varphi_n(\lambda_j)\}_{n=0}^{N-1}$, $j = 1, \dots, 2N$ of problem (6), (7) form a twofold basis in \mathbb{C}^N , that is for arbitrary vectors $f = \{f_n\}_0^{N-1}$ and $g = \{g_n\}_0^{N-1}$ belonging to \mathbb{C}^N the expansion formula hold

$$f_n = \sum_{j=1}^{2N} \alpha_j \varphi_n(\lambda_j), \quad g_n = \sum_{j=1}^{2N} \alpha_j \lambda_j \varphi_n(\lambda_j), \quad n = 0, 1, \dots, N-1 \quad (28)$$

where the coefficients α_j of expansion are defined by the formula

$$\alpha_j = (1/\rho_j) \sum_{k=0}^{N-1} (c_k f_k + \lambda_j r_k g_k) \varphi_k(\lambda_j), \quad j = 1, 2, \dots, 2N \quad (29)$$

in which

$$\rho_j = \sum_{k=0}^{N-1} (c_k + \lambda_j^2 r_k) \varphi_k^2(\lambda_j), \quad j = 1, 2, \dots, 2N. \quad (30)$$

Proof. Consider the space $\mathbb{C}^N \times \mathbb{C}^N$ of vectors denoted by $[y, z]$, where $y, z \in \mathbb{C}^N$. Define in this space the inner product by the formula

$$\langle [y, z], [u, v] \rangle = (Ry, u) + (Cz, v)$$

where (\cdot, \cdot) in the right-hand side denotes the inner product in \mathbb{C}^N defined by the formula (13). In view of Lemma 2 the vectors

$$\Phi_j = [\varphi^{(j)}, \lambda \varphi^{(j)}], \quad j = 1, 2, \dots, 2N$$

are orthogonal:

$$\langle \Phi_j, \Phi_i \rangle = 0, \quad j \neq i.$$

Consequently, the vectors ϕ_1, \dots, ϕ_{2N} are linearly independent in space $\mathbb{C}^N \times \mathbb{C}^N$. Since the number of them is equal to $2N$ and $\dim(\mathbb{C}^N \times \mathbb{C}^N) = 2N$, they form a basis in this space. Therefore, for the arbitrary vector $[f, g]$ belonging to $\mathbb{C}^N \times \mathbb{C}^N$ we have the unique expansion

$$[f, g] = \sum_{j=1}^{2N} \alpha_j \phi_j,$$

and

$$\begin{aligned} \alpha_j &= (1/\rho_j)\langle [f, g], \phi_j \rangle = (1/\rho_j)\{(C_f, \varphi^{(j)}) + \lambda_j(R_g, \varphi^{(j)})\} \\ \rho_j &= \langle \phi_j, \phi_j \rangle = (C\varphi^{(j)}, \varphi^{(j)}) + \lambda_j^2(R\varphi^{(j)}, \varphi^{(j)}). \end{aligned}$$

Hence, we have the following theorem.

Theorem 3. All the eigenvectors of the problem (6), (7) form an ordinary basis in the space \mathbb{C}^N .

Proof. We may assume that

$$\lambda_1 < \dots < \lambda_N < 0 < \lambda_{N+1} < \dots < \lambda_{2N}.$$

Let $x = \{x_n\}_{j_0}^{N-1} \in \mathbb{C}^N$ and

$$(x, \varphi^{(j)}) = 0, \quad j = 1, \dots, N. \tag{31}$$

We must show that $x = 0$. Applying Theorem 2 to the vectors $f = C^{-1}x$ and $g = 0$ we have

$$C^{-1}x = \sum_{j=1}^{2N} \alpha_j \varphi^{(j)}, \quad 0 = \sum_{j=1}^{2N} \alpha_j \lambda_j \varphi^{(j)} \tag{32}$$

where

$$\alpha_j = (1/\rho_j)(x, \varphi^{(j)}), \quad j = 1, 2, \dots, 2N. \tag{33}$$

From (33), in view of (31), we have $\alpha_j = 0$, $j = 1, \dots, N$ and therefore, (32) takes the form:

$$C^{-1}x = \sum_{j=N+1}^{2N} \alpha_j \varphi^{(j)}, \quad 0 = \sum_{j=N+1}^{2N} \alpha_j \lambda_j \varphi^{(j)}.$$

Multiplying the last equalities by x in the sense of the inner product in \mathbb{C}^N , we get

$$(C^{-1}x, x) = \sum_{j=N+1}^{2N} \alpha_j (\varphi^{(j)}, x) = \sum_{j=N+1}^{2N} \rho_j |\alpha_j|^2, \quad (34)$$

$$0 = \sum_{j=N+1}^{2N} \alpha_j \lambda_j (\varphi^{(j)}, x) = \sum_{j=N+1}^{2N} \lambda_j \rho_j |\alpha_j|^2. \quad (35)$$

Since $\lambda_j > 0$, $\rho_j > 0$, $j = N + 1, \dots, 2N$, from (31) it follows that $\alpha_j = 0$, $j = N + 1, \dots, 2N$. Consequently, from (30) we have $(R^{-1}x, x) = 0$.

Hence, $x = 0$ since

$$(R^{-1}x, x) = \sum_0^{N-1} (1/r_n) |x_n|^2.$$

3. Application. We now give an application of the results that were obtained above.

Theorem 4. Problem (1), (2), (3) has a unique solution $\{u_n(t)\}_{-1}^N$ which can be represented in the form

$$u_n(t) = \sum_1^{2N} \alpha_j e^{\lambda_j t} \varphi_n(\lambda_j), \quad n = -1, 0, 1, \dots, N \quad (36)$$

where $\lambda_1, \dots, \lambda_{2N}$ are the eigenvalues of problem (6), (7), and $\varphi^{(1)} = \{\varphi_n(\lambda_1)\}_0^{N-1}$, \dots , $\varphi^{(2N)} = \{\varphi_n(\lambda_{2N})\}_0^{N-1}$ are the corresponding eigenvectors. Furthermore, the coefficients $\alpha_1, \dots, \alpha_{2N}$ are defined by the formula (29), (30).

Proof. From the explanation given in the Introduction from Theorem 2 it follows that formula (36) gives a solution of problem (1), (2), (3). We now prove the uniqueness of this solution. To do this, write down problem (1), (2), (3) in the form

$$J(du(t)/dt) + r(d^2u(t)/dt^2) = Cu(t), \quad 0 \leq t < \infty \tag{37}$$

$$u(0) = f, \quad du(0)/dt = g \tag{38}$$

where J , R , and C are matrices defined by (11), and $u(t) = \{u_n(t)\}_0^{N-1}$, $f = \{f_n\}_0^{N-1}$, $g = \{g_n\}_0^{N-1}$. Furthermore, we can rewrite problem (33), (34) in the following equivalent form

$$\begin{aligned} du(t)/dt &= v(t), & dv(t)/dt &= R^{-1}Cu(t) - R^{-1}Jv(t) \\ u(0) &= 0, & v(0) &= g \end{aligned}$$

or

$$\frac{d}{dt} \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} 0 & I \\ R^{-1}C & -R^{-1}J \end{pmatrix} \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}, \quad \begin{pmatrix} u(0) \\ v(0) \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}.$$

The uniqueness of the solution of this last problem is well known [2].

Theorem 5. For arbitrary vector $f = \{f_n\}_0^{N-1} \in \mathbb{C}^N$ the problem (1), (2) has a unique solution $\{u_n(t)\}_{-1}^N$ that satisfies the conditions

$$u(n) = f_n, \quad \lim_{t \rightarrow \infty} u_n(t) = 0, \quad n = 0, 1, \dots, N - 1. \tag{39}$$

Proof. In view of Theorem 3 the functions

$$u_n(t) = \sum_{j=1}^N \beta_j e^{\lambda_j t} \varphi_n(\lambda_j), \quad n = -1, 0, 1, \dots, N$$

form a solution of problem (1), (2), (39), where $\lambda_1, \dots, \lambda_N$ are the negative eigenvalues of problem (6), (7), and β_1, \dots, β_N are defined by the help of the expansion

$$f = \sum_{j=1}^N \beta_j \varphi^{(j)}.$$

For the proof of uniqueness we note that in view of Theorem 4 the general solution of problem (1), (2) has the representation

$$u_n(t) = \sum_{j=1}^{2N} \alpha_j e^{\lambda_j t} \varphi_n(\lambda_j), \quad n = -1, 0, 1, \dots, N.$$

From the last equality and second condition of (39) it follows that $\alpha_{N+1} = \dots = \alpha_{2N} = 0$. Furthermore, setting $t = 0$ we get $\alpha_j = \beta_j$, $j = 1, \dots, N$.

We remark that from the proof of Theorem 5, it follows that for the solution of problem (1), (2), (39) we have

$$u_n(t) = O(e^{-\delta t}), \quad n = -1, 0, 1, \dots, N,$$

as $t \rightarrow \infty$, where $\delta > 0$ is the modulus of the greatest negative eigenvalue of boundary problem (6), (7).

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