INDUCED FIBRATIONS ON SPACES OF FIBER TRANSFERRING MAPS

Manolis Magiropoulos

Abstract. Starting with a continuous map $\pi\colon E \to B$, we define several categories of spaces of fiber transferring maps with respect to π and investigate the extent to which some basic properties of π are inherited by the induced map p on the above mentioned spaces.

Let π be a continuous map between two topological spaces E and B, where π is open and onto, and E and B are T_2 and locally compact. If $C(E)$ is the set of all continuous maps $E \to E$, a map $\overline{f} \in C(E)$ is a fiber transferring map with respect to π if and only if for each $b \in B$ there exists a $x_b \in B$ such that $\overline{f}(\pi^{-1}(b)) \subset \pi^{-1}(x_b)$. Let $C_\pi(E)$ denote the subspace of $C(E)$ consisting of all such maps (all function spaces are given the c.o. topology). Using the Theorem of Exponential Correspondence (cf. [3]), one proves that the induced correspondence

$$p\colon C_\pi(E) \to C(B),$$

given by

$$p(\overline{f})(b) = \pi\overline{f}(\pi^{-1}(b)),$$

is a well defined continuous map.

Employing the same argument outlined in the proof of Proposition 10 in [2], we obtain the following:

<u>Proposition 1</u>. If π is a fibration (with unique path lifting, regular), then so is p.

Now let $H_\pi(E)$ and $H(B)$ denote the subspaces of $C_\pi(E)$ and $C(B)$, respectively, consisting of homotopy equivalences and $\mathrm{Top}_\pi(E)$ and $\mathrm{Top}(B)$ denote the subspaces of $C_\pi(E)$ and $C(B)$ consisting of homeomorphisms which, in the case of $\mathrm{Top}_\pi(E)$, take fibers onto fibers. Once more, the reasoning exhibited at the appropriate part of the proof of Proposition 10 in [2] will establish the following propositions.

Proposition 2. If π is a fibration (with unique path lifting, regular), then the restriction of p on $H_\pi(E)$ gives a map $H_\pi(E) \to H(B)$ which is a fibration (with unique path lifting, regular).

Proposition 3. If π is a fibration with unique path lifting (regular), and E is, in addition, connected and locally path-connected, the restriction of p on $\mathrm{Top}_\pi(E)$ gives a map $\mathrm{Top}_\pi(E) \to \mathrm{Top}(B)$ which is a fibration with unique path lifting (regular).

At this point, it is quite reasonable to ask about the relation between the group $G(E/B)$ of deck transformations of a regular fibration $\pi: E \to B$ and the groups $G(C_\pi(E)/C(B))$, $G(H_\pi(E)/H(B))$, $G(\mathrm{Top}_\pi(E)/\mathrm{Top}(B))$ of deck transformations of the induced regular fibrations p. For this, we have the following.

Proposition 4. If E is connected and locally path-connected, we have the following commutative diagram of groups and group homomorphisms

$$\begin{array}{ccccc} G(C_\pi(E)/C(B)) & \longrightarrow & G(H_\pi(E)/H(B)) & \longrightarrow & G(\mathrm{Top}_\pi(E)/\mathrm{Top}(B)) \\ \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle i} \\ G(E/B) & & G(E/B) & & G(E/B) \end{array}$$

where the horizontal maps are obtained by restriction on the corresponding subspaces and the i-maps are group embeddings all given by $i(\overline{f})(\overline{h}) = \overline{f} \circ \overline{h}, \overline{f} \in G(E/B), \overline{h} \in C_\pi(E), H_\pi(E), \mathrm{Top}_\pi(E)$, respectively.

Proof. We prove first that the left horizontal map is a group homomorphism. But this is immediate once we prove that it is well defined. Let $\sigma \in G(C_\pi(E)/C(B))$ and let $\overline{f} \in \mathrm{Hom}_\pi(E)$. We show that $\sigma(\overline{f}) \in H_\pi(E)$. Let $e \in E$; since π is a regular fibration and E is connected and locally path-connected there is $\overline{h} \in G(E/B)$ such that $\sigma(\overline{f})(e) = \overline{h}(\overline{f}(e)) = (\overline{h} \circ \overline{f})(e)$. E is also path-connected and by uniqueness of liftings we have $\sigma(\overline{f}) = \overline{h} \circ \overline{f} \in H_\pi(E)$. Now by taking σ^{-1} we easily show that the restriction of σ on $H_\pi(E)$ is onto (if $\overline{f} \in H_\pi(E), \sigma^{-1}(\overline{f}) \in H_\pi(E)$ as we just saw and $\overline{f} = \sigma(\sigma^{-1}(\overline{f}))$.

The same argument proves also that the right horizontal map is well defined and it is a group homomorphism.

We now show that the maps i are group embeddings. First we show that they are well defined. We consider the Top case (the same argument works for the other two cases). If $\overline{h}_1, \overline{h}_2 \in \mathrm{Top}_\pi(E)$ and $\overline{h}_1 \neq \overline{h}_2$, then for $\overline{f} \in G(E/B)$ we have $\overline{f} \circ \overline{h}_1 \neq \overline{f} \circ \overline{h}_2$, thus, $i(\overline{f})$ is 1-1. Let $\overline{g} \in \mathrm{Top}_\pi(E)$ and $e \in E$; then $\overline{f}(\overline{f}^{-1}(\overline{g}(e))) = \overline{g}(e)$, that is, $i(\overline{f})(\overline{f}^{-1} \circ \overline{g}) = \overline{g}$, that is, $i(\overline{f})$ is onto. Now the map $\tau\colon \mathrm{Top}_\pi(E) \times E \to E$, given by $\tau(\overline{h}, e) = (\overline{f} \circ \overline{h})(e)$ is continuous, and the Theorem of Exponential Correspondence implies that $i(\overline{f})$ is continuous. $i(\overline{f})^{-1}$ is just $i(\overline{f}^{-1})$, so it is continuous, thus, $i(\overline{f})$ is a homeomorphism which obviously satisfies $p(i(\overline{f})) = p(\overline{f})$, that is, $i(\overline{f}) \in G(\mathrm{Top}_\pi(E)/\mathrm{Top}(B))$, that is, i is well defined. Evidently i is a group homomorphism. Now if $\overline{f}_1 \neq \overline{f}_2$, then $i(\overline{f}_1) \neq i(\overline{f}_2)$ since $\overline{f}_1 \circ 1_E \neq \overline{f}_2 \circ 1_E$. Thus, i is an embedding; commutativity of the diagram follows from the definition of the maps, and this proves that the other two i's are also group embeddings.

We are about ready to establish our basic result. In addition to the requirements of Proposition 3, each point of E and B is asked to possess a basis of open contractible neighborhoods whose topological closure is path-connected. This requirement is certainly met by manifolds. Then we have:

Theorem 5. If $\pi\colon E \to B$ is a regular covering map, the induced maps $p\colon C_\pi(E) \to p(C_\pi(E))$, $H_\pi(E) \to p(H_\pi(E))$, $\mathrm{Top}_\pi(E) \to p(\mathrm{Top}_\pi(E))$ are all regular covering maps.

Proof. The proof is given for the C-case; the same proof works for the other two cases.

For convenience, we use the term open ball for a basic open contractible neighborhood, and closed ball for the compact closure of an open ball (since the spaces are T_2, locally compact, we have at each point a basis of open contractible neighborhoods whose closures are path-connected and compact at the same time).

Before we continue with the proof we establish three claims.

Claim 1. If $S(C, A)$ denotes the standard subbasic element for the c.o. topology, then the collection $\mathcal{A} = \{S(C, A)/C$ is an evenly covered, closed ball of B, A is an evenly covered open ball of $B\}$, is a subbasis for the c.o. topology of $C(B)$.

Proof of Claim 1. Obviously, the open balls A of the form described above form a basis for the topology of B. Now let K be a compact set of B and U an open set, such that $K \subset U$; for each $k \in K$, choose an evenly covered closed ball $C_k \subset U$, with $k \in \mathrm{int}C_k$; the collection $\{\mathrm{int}C_k, k \in K\}$, is an open cover of K. Let $\{\mathrm{int}C_{k_i}\}_{i=1}^n$ be a finite subcover of K; then we have $K \subset U_{i=1}^n C_{k_i} \subset U$; this completes the proof of Claim 1, because of statement 5.1 in [1].

Claim 2. The collection $B = \{S(\tilde{C},\tilde{A})/\tilde{C}$ is a slice of an evenly covered, closed ball C of $B, \tilde{A}$ is a slice of an open ball A which is evenly covered $\}$ is a subbasis for the c.o. topology in $C_\pi(E)$.

Proof of Claim 2. Replica of the proof of Claim 1.

Claim 3. Let $f \in S(C,A)$, where C, A are taken as in Claim 1. Let $c \in C$ and let $\overline{c}$ be a pre-image of c contained in some slice $\tilde{C}$ of C. If $\overline{f} \in C_\pi(E)$ is a lifting of f with $\overline{f}(\overline{c}) \in \tilde{A}$, where $\tilde{A}$ is some slice of A, then $\overline{f}(\tilde{C}) \subset \tilde{A}$.

Proof of Claim 3. Let $\overline{c}_1$ be another element of $\tilde{C}$; we set $c_1 = \pi(\overline{c}_1)$. Join c to c_1 by a path φ lying in C; then $f \circ \varphi$ is a path lying in A. The path $(\pi/\tilde{C})^{-1} \circ \varphi$ is the unique lifting of φ starting at $\overline{c}$, and lies entirely in $\tilde{C}$. The path $\overline{f}(\pi/\tilde{C})^{-1} \circ \varphi$ is a lifting of $f \circ \varphi$ starting at $\overline{f}(\overline{c})$. The path $(\pi/\tilde{A})^{-1} \circ (f \circ \varphi)$ is also a lifting of $f \circ \varphi$ starting at $\overline{f}(\overline{c})$. By uniqueness of path lifting we have that $\overline{f} \circ (\pi/\tilde{C})^{-1} \circ \varphi = (\pi/\tilde{A})^{-1} \circ (f \circ \varphi)$ so the ends are the same, and, consequently, $\overline{f}(\overline{c}_1) \in \tilde{A}$. Since $\overline{c}_1$ was chosen arbitrarily, $\overline{f}(\tilde{C}) \subset \tilde{A}$.

Back now to the proof of the theorem. From now on if $\tilde{A}$ is a subset of E, then A is its projection, that is, $A = \pi(\tilde{A})$.

Let $f \in p(C_\pi(E))$, let $b \in B$ and let $f(b) \in A$, where A is an evenly covered open ball; then $f \in S(b,A)$. Let $e \in \pi^{-1}(b)$ and let $\overline{f}$ be a lifting of f in $C_\pi(E)$; then $\overline{f} \in S(e,\tilde{A})$, where $\tilde{A}$ is an appropriate slice of A. Our goal is to prove that the restriction of $p, p'\colon S(e,\tilde{A}) \to S(b,A)$ is a homeomorphism. We show first that it is 1-1. If $\overline{f}, \overline{g} \in S(e,\tilde{A})$ and $\overline{f} \neq \overline{g}$, then $\overline{f}, \overline{g}$ cannot be liftings of the same map $B \to B$, for if this were the case, then $\overline{f}(e) = \overline{g}(e)$, since $\tilde{A}$ is a slice of A. But E is connected and uniqueness of the lifting implies $\overline{f} = \overline{g}$ which is a contradiction. Next we show that p' is onto. Let $h \in S(b,A)$; since E is locally contractible and π is regular, the group $G(E/B)$ of deck transformations acts transitively on the fibers. Let $\overline{h}_1 \in C_\pi(E)$ is a lifting of h (subbasic elements $S(C,A)$ are always meant with respect to appropriate subspaces; hence, $h \in S(b,A)$ means $h \in S(b,A) \cap p(C_\pi(E)))$ and set $a = \tilde{A} \cap \pi^{-1}(h(b))$. If $\overline{r} \in G(E/B)$ satisfies $\overline{r}(\overline{h}_1(e)) = a$, then $\overline{h} = \overline{r} \circ \overline{h}_1$ is the lifting of h we need, and we are done.

We now prove that ${p'}^{-1}$ is continuous. This will finish the proof since p' is continuous.

Let $\overline{h} \in S(e,\tilde{A})$; a typical basic neighborhood of $\overline{h}$ in $S(e,\tilde{A})$ looks like $S(e,\tilde{A}) \cap (\cap_{i=1}^{n} S(\tilde{D}_i,\tilde{A}_i))$, where $\tilde{D}_i$ and $\tilde{A}_i$ are balls of the type described in Claim 2. Hence, it is sufficient to find a neighborhood W_i of $h = p'(\overline{h})$, $i = 1, 2, \ldots, n$, such that $p'^{-1}(W_i) \subset S(e, \tilde{A}) \cap S(\tilde{D}_i, \tilde{A}_i)$, $i = 1, 2, \ldots, n$.

So, to simplify notation, we take a neighborhood $S(e, \tilde{A}) \cap S(\tilde{D}, \tilde{V})$ of $\overline{h}$, where $\tilde{D}$, $\tilde{V}$ are as in Claim 2. Let $\overline{d} \in \tilde{D}$; we join e and $\overline{d}$ by a path $\overline{\varphi}$. Then $\pi \circ \overline{\varphi}$ is a path from b to $d = \pi(\overline{d})$ (E, B are locally path-connected; since they are also connected, then are path-connected). The composition $\overline{h} \circ \overline{\varphi}$ defines a path starting at some point in $\tilde{A}$ and ending at some point in $\tilde{V}$. We cover $(\overline{h} \circ \overline{\varphi})([0, 1])$ by a finite number of open balls of the type of Claim 2, say $\tilde{A}_0, \tilde{A}_1, \ldots, \tilde{A}_k$; then $M = \{(\overline{h} \circ \overline{\varphi})^{-1}(\tilde{A}_i)\}_{i=0}^{k}$ is an open cover of $[0, 1]$ and as such it has a Lebesgue number, that is, there is $n \in \mathbb{N}$ such that $[j/n, (j + 1)/n]$ is a subset of some element in M, where $j = 0, 1, \ldots, n - 1$. By rearranging and allowing repetitions if necessary, we may say that $[j/n, (j + 1)/n] \subset (\overline{h} \circ \overline{\varphi})^{-1}(\tilde{A}_j)$, $j = 0, 1, \ldots, n - 1$, and so $(\overline{h} \circ \overline{\varphi})([j/n, (j + 1)/n]) \subset \tilde{A}_j$, $j = 0, 1, \ldots, n - 1$. Let $\tilde{A}_{00}$ be the path-component of $\tilde{A} \cap \tilde{A}_0$ containing $(\overline{h} \circ \overline{\varphi})(0)$, $\tilde{A}_{j,j+1}$ the path component of $\tilde{A}_j \cap \tilde{A}_{j+1}$ containing $(\overline{h} \circ \overline{\varphi})((j + 1)/n)$, $j = 0, 1, \ldots, n - 2$, $\tilde{A}_{n-1,n}$ the path component of $\tilde{A}_{n-1} \cap \tilde{V}$ containing $(\overline{h} \circ \overline{\varphi})(1)$. All these path-components are open sets, since they are path-components of locally path-connected sets. Now taking projections, we have that the set

$$W = S(\varphi(0), A_{00}) \cap \\ [\cap_{j=0}^{n-1} [S(\varphi[j/n, (j + 1)/n]), A_j) \cap S[\varphi((j + 1)/n), A_{j,j+1})] \cap S(D, V)$$

is a neighborhood of h. Let $\sigma \in W$. We need to prove that the lifting $\overline{\sigma}$ of σ, such that $\overline{\sigma}(e) \in \tilde{A}$, satisfies $\overline{\sigma}(\overline{d}) \subset \tilde{V}$; then Claim 3 will give $\overline{\sigma}(\tilde{D}) \subset \tilde{V}$ and we will be done. We join first $(\sigma \circ \varphi)(0)$ and $(h \circ \varphi)(0)$ by a path β lying entirely in A_{00}. Then we join $(\sigma \circ \varphi)(1/n)$ and $(h \circ \varphi)(1/n)$ by a path γ lying entirely in A_{01}. After reparametrization, we obtain paths $\sigma \circ \varphi$ and $\beta \star (h \circ \varphi) \star \gamma^{-1}$ which start and end at the same point (by $\star$ is meant the obvious composition of paths). But A_0 is contractible, so $\sigma \circ \varphi$ is homotopic to $\beta \star (h \circ \varphi) \star \gamma^{-1}$. Let $\overline{\beta}$ and $\overline{\gamma}$ be the liftings of β and γ which lie in $\tilde{A}_{00}$ and $\tilde{A}_{01}$, respectively; then $\overline{\beta} \star (\overline{h} \circ \overline{\varphi}) \star \overline{\gamma}^{-1}$ is a lifting of $\beta \star (h \circ \varphi) \star \gamma^{-1}$ (reparametrize). Reparametrizing $(\overline{\sigma} \circ \overline{\varphi})/[0, 1/n]$, we are getting a lifting of $\sigma \circ \varphi$ which starts at the same point with $\overline{\beta} \star (\overline{h} \circ \overline{\varphi}) \star \overline{\gamma}^{-1}$, and since $\sigma \circ \varphi \sim \beta \star (h \circ \varphi) \star \gamma^{-1}$ it must end at the same point, thus, $(\overline{\sigma} \circ \overline{\varphi})(1/n) \in \tilde{A}_{01}$. In a similar way we show that $(\overline{\sigma} \circ \overline{\varphi})(2/n) \in \tilde{A}_{12}$ (using $[1/n, 2/n]$ and contractibility of A_1 and path-connectivity of A_{12}) and finally following the path all the way down, we will get that $(\overline{\sigma} \circ \overline{\varphi})(1) \in \tilde{A}_{n-1,n} \subset \tilde{V}$, so $\overline{\sigma}(\overline{d}) \in \tilde{V}$. Thus, the continuity of ${p'}^{-1}$ has been established and $S(e, \tilde{A})$ is homeomorphic to $S(b, A)$. Let $\{\tilde{A}_j\}$, $j \in J$ be the family of all slices of A. As we saw above, since π is regular, for each $j \in J$ there is a lifting $\overline{f}_j$ of f satisfying $\overline{f}_j(e) \in \tilde{A}_j$; then for each $j \in J$, $S(e, \tilde{A}_j)$ is homeomorphic to $S(b, A)$ (we just proved it). On the other hand $p^{-1}(S(b, A)) = \amalg_{j \in J} S(e, \tilde{A}_j)$.

Thus, $S(b, A)$ is an evenly covered neighborhood of f, and p is a covering map. Regularity follows from Propositions 1, 2, and 3.

References

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Manolis Magiropoulos
General Department of School of Technological Applications
Technological Educational Institute of Heraklion
Stavromenos, Heraklion 71500
Crete, Greece
email: mageir@stef.teiher.gr