BLACKHOLE ANALYSIS

David Choate

Abstract. Let $S = \{z = x + yj \mid x \text{ and } y \text{ real}, -\pi < y \leq \pi\}$ or equivalently, after an appropriate adjustment of the residue of y modulo 2π, $S = \{z = x + y \pmod{2\pi} j \mid x \text{ and } y \text{ real}\}$, a horizontal strip. Let $S_B = SU\{-\infty\}$. Also, let $z = x_1 + y_1 j$ and $w = x_2 + y_2 j \in S_B$. Define an operation $\oplus$, called blackhole-multiplication on S_B as

$$z \oplus w = (x_1 + x_2) + [(y_1 + y_2) \mod (2\pi)]j$$

if both z and w are in S; otherwise $z \oplus w = -\infty$.

Now define $z \otimes w = \log(e^z + e^w)$. Let C be the complex field. Then $(C, +, \cdot) \cong (S_B, \otimes, \oplus)$, a parallel universe where some defiant differential equations are taught humility.

Blackhole signal processing yields a new superposition.

And there exists a blackhole meta-algorithm which accelerates any program in which multiplication and exponentiation dominate addition and subtraction.

1. Blackhole Addition. Let $S = \{z = x + yj \mid x \text{ and } y \text{ real}, -\pi < y \leq \pi\}$ or equivalently, after an appropriate adjustment of the residue of y modulo 2π, $S = \{z = x + y \pmod{2\pi} j \mid x \text{ and } y \text{ real}\}$, a horizontal strip. Let $S_B = SU\{-\infty\}$.

Now let $z = x_1 + y_1 j$ and $w = x_2 + y_2 j \in S_B$. Define a blackhole or B-multiplication on S_B as

$$z \oplus w = (x_1 + x_2) + [(y_1 + y_2) \mod (2\pi)]j$$

provided both z and w are in S; otherwise, $z \oplus w = -\infty$.

And if $z = re^{j\theta}$, then as usual define $\log(z) = \ln|r| + [\theta \pmod{2\pi}]j$ again after the appropriate adjustment of the residue. Clearly

$$\log(zw) = \log(z) \oplus \log(w).$$

Now let $z, w \in S_B$. We define an operation called blackhole addition or B-addition on S_B, denoted by $\otimes$, as

$$z \otimes w = \log(e^z + e^w).$$

If we define $z \otimes -\infty = z = z \otimes -\infty$ for every z in S_B, then $\otimes$ is an operation on S_B.

And clearly

$$\log(z+w) = \log(z) \otimes \log(w).$$

2. The Field ($\mathbf{S_B}$, $\oplus$, $\otimes$). It is an easy exercise to show that $(S_B, \oplus, \otimes)$ is a field. But for the purpose of illustration observe that the B-additive identity is $-\infty$ since

$$z \otimes -\infty = \log(e^z + e^{-\infty}) = z.$$

Also observe that if $z \in S$, then its B-additive inverse is $\pi j \oplus z$ since

$$z \otimes (j\pi \oplus z) = \log(e^z + e^{j\pi \oplus z}) = \log[e^z(1+e^{j\pi})] = \log(0) = -\infty.$$

And certainly the B-additive inverse of $-\infty$ is $-\infty$. Therefore every element in S_B has a B-additive inverse. Furthermore, B-division, denoted by $\ominus$, may be defined as

$$z \ominus w = z \oplus (j\pi \oplus w).$$

Furthermore, the distributive law of B-multiplication over B-addition can be established with the following calculation. For $u, v, w \in S_B$,

$$\begin{aligned} u \oplus (v \otimes w) &= \log(e^u) \oplus \log(e^v + e^w) = \log[e^u(e^v + e^w)] \\ &= \log(e^{u+v} + e^{u+w}) = \log(e^{u \oplus v} + e^{u \oplus w}) \\ &= (u \oplus v) \otimes (u \oplus w). \end{aligned}$$

Theorem 2.1. $(C, \cdot, +) \cong (S_B, \oplus, \otimes)$.

Proof. Define $\phi: C \to S_B$ by $\phi(z) = \log(z)$. Then

$$\phi(z_1 z_2) = \log(z_1 z_2) = \log(z_1) \oplus \log(z_2) = \phi(z_1) \oplus \phi(z_2).$$

Furthermore,

$$\phi(z_1 + z_2) = \log(z_1 + z_2) = \log(z_1) \otimes \log(z_2) = \phi(z_1) \otimes \phi(z_2).$$

If $w \in S$, then its preimage under ϕ, $w \in S$, is e^w. And since the preimage of $-\infty$ is 0, ϕ is onto.

If $\phi(z) = 0\otimes$, then $\log(z) = -\infty$, or $z = 0$; so ϕ is one-to-one.

3. Blackhole Calculus.

Definition 3.1. The function f is said to be B-differentiable at x if the limit

$$\lim_{h\to-\infty}\{[f(x\otimes h)\otimes(j\pi\oplus f(x))]\ominus h\},$$

denoted by $(f)'_B(x)$, exists.

Theorem 3.2. Suppose f is differentiable. Then f is B-differentiable and

$$(f)'_B(x)=[\log(dy/dx)]\oplus y\ominus x.$$

Proof. By the definition and l'Hopital's Rule,

$$(f)'_B(x)=\lim_{h\to-\infty}\langle\{f[\log(e^x+e^h)]\otimes[j\pi\oplus f(x)]\}\ominus h\rangle$$

$$=\log[\lim_{h\to-\infty}(\langle\exp\{f[\log(e^x+e^h)]\}-\exp(f(x))\rangle/e^h)]$$

$$=\log(\lim_{h\to-\infty}\{f'[\log(e^x+e^h)]\exp\{f[\log(e^x+e^h)]\}/(e^x+e^h)\})$$

$$=\log[f'(x)e^{f(x)\ominus x}]=\{\log[f'(x)]\}\oplus f(x)\ominus x.$$

Corollary 3.3. Let c and p be constants. Then

i. $(c)'_B=-\infty$.
ii. $(px)'_B(x)=\log(p)\oplus(p\ominus 1)x$.
iii. $[\log(x)]'_B=\ominus x$.
iv. $(e^x\oplus c)'_B=e^x$.
v. $[\log(px)]'_B=\log(p)\ominus x$.
vi. $[p\log(x)]'_B=\ominus x\oplus\log(px^{p-1})$.
vii. $(x^p)'_B=x^p\ominus x\oplus\log(px^{p-1})$.
viii. $(\sin x)'_B=\log(\cos x)\oplus\sin x\ominus x$.
ix. $(\cos x)'_B=\log(\ominus\sin x)\oplus\cos x\ominus x$.
x. $(f)''_B(x)=\log\{f''(x)+[f'(x)]^2-f'(x)\}\oplus f(x)\ominus 2x$.

Corollary 3.4. Let $f(x)$ be a real valued function in some interval I. Then $f(x)$ is increasing or decreasing in I if and only if $(f)'_B$ is real or purely imaginary in I.

Theorem 3.4.5. Let $f(x)$ be differentiable. Then $(f_B)'_B = (f')_B$.

Proof. By definition, $f_B(x) = \log[f(e^x)]$. Then,

$$[f_B(x)]'_B = \log[e^x f'(e^x)/f(e^x)] \oplus \log[f(e^x)] \ominus x = x \oplus \log[f'(e^x)] \ominus x = [f'(x)]_B.$$

The isomorphism in Theorem 2.1 is a portal into a parallel universe where we find the following.

Theorem 3.5. Suppose f and g are B-differentiable. Then

i. $(c \oplus f)'_B(x) = [c \oplus (f)]'_B(x)$.
ii. $(f \otimes g)'_B = (f)'_B \otimes (g)'_B$.
iii. $(f \oplus g)'_B = [f \oplus (g)'_B] \otimes [g \oplus (f)'_B]$.

To illustrate 3.5(iii) consider the following.

Example 3.6. Let $f(x) = px$ and $g(x) = qx$. Then

$$\begin{aligned}
[f \oplus (g)'_B] \otimes [g \oplus (f)'_B] &= \{px \oplus [(\log q \oplus (q \ominus 1)x]\} \otimes \{qx \oplus [\log p \oplus (p \ominus 1)x]\} \\
&= \log\{qe^{[px+(q-1)x]} + pe^{[qx+(p-1)x]}\} \\
&= \log\{qe^{[px\oplus(q\ominus 1)x]} + pe^{[qx\oplus(p\ominus 1)x]}\} \\
&= \log(p \oplus q) \oplus [(p \oplus q) \ominus 1]x \\
&= (f \oplus g)'_B.
\end{aligned}$$

Definition 3.7. Let

$$\otimes \sum_{i=1}^{n} a_i = a_1 \otimes a_2 \otimes \cdots \otimes a_n.$$

The blackhole definite integral from a to b is given by

$$\lim_{(\triangle x)_B \to -\infty} \left\langle \otimes \sum_{i=1}^{n} [(f(x_i))_B \oplus (\triangle x)_B] \right\rangle,$$

where $(\triangle x)_B = [b \otimes (j\pi \oplus a)] \ominus n$ and x_i is in the ith subinterval. For this limit we use the notation

$$\otimes \int_a^b [(f(x)]_B \oplus (dx)_B].$$

Theorem 3.8.

$$\otimes \int_a^b [f(x)]_B \oplus [(dx)_B] = \log \left[\int_a^b e^{[f(x)]_B + x} dx \right].$$

Proof.

$$\otimes \int_a^b [f(x)]_B \oplus [(dx)_B] = \otimes \int_a^b [f(x)_{B^a} \oplus (dx)_B]$$

$$= \lim_{(\triangle x)_B \to -\infty} \log[(e^{f(x_1)})(e^{x_1} - e^{x_0}) + \cdots + (e^{f(x_n)})(e^{x_n} - e^{x_{n+1}})]$$

$$= \lim_{(\triangle x)_B \to -\infty} \log[e^{f(x_1)+x_1} + \cdots + e^{f(x_n)+x_n} + e^{f(x_1)+x_0} + \cdots + e^{f(x_n)+x_{n-1}}].$$

Set $x_0 = a$. Without loss of generality we may assume that $(\triangle x)_B = x_i \otimes (j\pi \oplus x_{i-1}) = \log[\exp(x_i) - \exp(x_{i-1})]$. Set $\triangle x = (b-a)/n$. After multiplying the last n terms of the argument by $e^{\triangle x}/e^{\triangle x}$ and all terms by $\triangle x/\triangle x$ we have by l'Hopital's Rule that

$$\otimes \int_a^b [(f(x))_B \oplus (dx)_B]$$

$$= \log\left[\lim_{(\triangle x)_B \to -\infty} \frac{(e^{\triangle x} - 1)}{(\triangle x e^{\triangle x})}\right] \oplus \log \int_a^b [e^{f(x)+x} dx] = \log \int_a^b e^{f(x)+x} dx$$

$$= \log\left[\int_a^b e^{f(x)+x} dx\right].$$

In order to get a feel for indefinite blackhole integrals consider the following.

Example 3.9. Recall that $(\log x)'_B = \ominus x$.

$$\otimes \int [(\ominus x)_B \oplus (dx)_B] = \log\left[\int e^0 dx\right] = \log\{x + e^c\} = (\log x) \otimes (c).$$

Example 3.10. Recall that $(px)'_B = \log(p) \oplus (p \ominus 1)x$.

$$\otimes \int [\log(p) \oplus (p \ominus 1)x] \oplus (dx)_B] = \log\left(\int pe^{px} dx\right) = \log(e^{px} + e^c) = (px) \otimes (c).$$

Example 3.11. Recall that $(e^x)'_B = e^x$.

$$\otimes \int [e^x \oplus (dx)_B] = \log\left\{\int [\exp(e^x + x)] dx\right\} = \log[\exp(e^x) + e^c] = (e^x) \otimes (c).$$

Note 3.12. These examples indicate that

$$\otimes \int [(f(x))_* \oplus (dx)_*] = [F_*(x) \otimes c],$$

where $F_*(x)$ is the B-antiderivative of $f(x)$.

Other blackhole theorems are also immediate from Theorem 2.1.

Theorem 3.13.

(i)
$$\otimes \int_a^a [(f(x))_B + (dx)_B] = -\infty.$$

(ii)
$$\left\langle \otimes \int_a^b [(f(x))_B \oplus (dx)_B] \right\rangle \otimes \left\langle \otimes \int_b^c [(f(x))_B \oplus (dx)_B] \right\rangle = \left\langle \otimes \int_a^c [(f(x))_B \oplus (dx)_B] \right\rangle.$$

And clearly the blackhole version of the First Fundamental Theorem of Calculus is given by

$$(iii) \qquad \otimes \int_a^b [(f(x))_B \oplus (dx)_B] = [F_B(b)] \otimes [(j\pi) \oplus F_B(a)].$$

To illustrate Theorem 3.13 (iii) consider the following.

Example 3.14. Let $f(x) = \log(p) \oplus (p \ominus 1)x$. Then, as we have seen $F_B(x) = (px) \otimes (c)$. Consequently,

$$\otimes \int_a^b [(f(x))_B \oplus (dx)_B] = \log \left\langle \int_a^b e^{[\log(p)+(p-1)x+x]} dx \right\rangle$$

$$= \log\left[\int_a^b pe^{px} dx\right] = \log[e^{pb} + e^c - e^{pa} - e^c] = [F_B(b)] \otimes [(j\pi) + F_B(a)].$$

To argue the Second Fundamental Theorem of Blackhole Calculus let a be in an interval over which $f(x)$ is continuous. Then certainly

$$(d/dx)\left[\int_a^x f(t)dt\right] = f(x).$$

Now observe that

$$(d/dx)_B \left\langle \otimes \int_a^x [f(t)_B \oplus (dt)_B] \right\rangle = (d/dx)_B \left\langle \log\left[\int_a^x e^{f(t)+t} dt\right]\right\rangle$$

$$= \log \frac{e^{[f(x)+x]}}{\int_a^x e^{f(t)+t} dt} \oplus \left\langle \log\left[\int_a^x e^{f(t)+t} dt\right]\right\rangle \oplus x$$

$$= f(x) \oplus x \ominus x = f(x).$$

Theorem 3.16.

$$\left(\int\int f(x,y)dydx\right)_B = \log\left(\int\int \exp\{[f(x)]_B + x + y\}dydx\right).$$

Proof.

$$\left[\int\int f(x,y)dydx\right]_B = \left\{\int\left[\int f(x,y)dy\right]_B dx\right\}_B$$

$$= \left(\int \log\left\langle\int \exp\{[f(x)]_B + y\}dy\right\rangle dx\right)_B$$

$$= \left(\log\int e^x\left\langle\int \exp\{[f(x)]_B + y\}dy\right\rangle dx\right)_B$$

$$= \log\left\langle\int\int \exp\{[f(x)]_B + x + y]dy\right\rangle dx\Big)_B.$$

Example 3.17. Clearly $(\ln x + c_1)(\ln x + c_2) = \int\int dxdy/xy$. We now descend to obtain

$$\int\int (1/xy)_B \oplus (dx)_B \oplus (dy)_B = \int\int (\ominus x \ominus y) \oplus (dx)_B \oplus (dy)_B$$

$$= \log\left\{\int\int \exp(-x-y+x+y)dxdy\right\}$$

$$= \log([(x+c_1)(x+c_2)].$$

We now ascend to obtain

$$\exp\{\log[(\ln x + c_1)(\ln y + c_2)]\} = (\ln x + c_1)(\ln y + c_2).$$

4. Blackhole Differential Equations. To understand how to apply blackhole calculus to ordinary space consider the following.

Example 4.1. Let $dy/dx = y$.

Solution. We now descend into the blackhole to obtain

$$(dy/dx)_B = (y)_B.$$

The left side can be calculated using Theorem 3.2. The right term can be determined by "e-ing" each variable and then "logging" the resulting expression. So we have

$$\log(dy/dx) \oplus y \ominus x = y, \text{ (ii)}$$

or

$$dy/dx = e^x \text{ which implies } y = e^x + \ln(c).$$

To ascend to ordinary space by "logging" each variable and then "e-ing" the entire expression, or

$$\exp[\log(y)] = \exp[e^{\log(x)} + \log(c)] \text{ which gives } y = ce^x.$$

Example 4.2. $dy/dx = -x/y$.

Solution. We now descend to obtain

$$\log(dy/dx) \oplus y \ominus x = (j\pi) \oplus x \ominus y,$$

or

$$\log(dy/dx) = (j\pi) \oplus 2x \ominus 2y \text{ which implies } e^{2x}/2 + e^{2y}/2 = c.$$

We now ascend to obtain

$$\exp(x^2/2 + y^2/2) = e^c \text{ which implies } x^2/2 + y^2/2 = c.$$

Example 4.3. $dy/dx = (y/x)\{1 - [\ln(y)]/[\ln(x)]\}$.

Solution. We descend to obtain

$$\log(dy/dx) \oplus y \ominus x = y \ominus x \oplus \log[1 - (y/x)],$$

or

$$dy/dx + y/x = 1$$

whose solution is

$$y = x/2 + c/x.$$

We now ascend to obtain

$$y = \exp\{[\ln(x)/2] + [c/\ln(x)]\},$$

or the solution

$$y = (\sqrt{x})e^{[c/\ln(x)]}.$$

Example 4.4. $dy/dx = (y/x)(\ln y/\ln x)[1 - (\ln x)(\ln y)]$.

Solution. We descend to

$$\log(dy/dx) \oplus y \ominus x = y \ominus x \oplus \log[(y/x)(1 - xy)],$$

or

$$dy/dx + (-1/x)y = (-1)y^2$$

which is recognized at once as a Bernoulli differential equation whose solution is

$$y = [2x/(x^2 + 2c)].$$

We now ascend to obtain the solution

$$y = \exp\{2\ln x/[(\ln x)^2 + 2c]\}.$$

Example 4.5. $d^2y/dx^2 = (dy/dx)^2/y$.

Solution. Descend to obtain

$$\{\log[y'' + (y')^2 - y']\} \oplus y \ominus 2x = 2[\log(y') \oplus y \ominus x] \ominus y,$$

or

$$y'' - y' = 0$$

whose solution is

$$y = c_1 + c_2e^x.$$

We now ascend to obtain the solution

$$y = c_1 \exp(c_2x).$$

5. Blackhole Signal Processing. All undefined and underdefined terms and symbols used in this section can be found in chapter 5 of [1]. In fact in [1] we are given a definition of a superposition H, a generalization of a system transformation, which must satisfy the following.

1. $H[x_1(n)\triangle x_2(n)] = H[x_1(n) \circ x_2(n)]$.
2. $H[c : x(n)] = c \odot H[x(n)]$.

Here, $\triangle$ is an input operation, $\circ$ is an output operation and $\odot$ represents scalar multiplication.

Now define $H\colon C \to S_B$ by $H(z) = \log(z)$.

If we let

i. $\triangle$ be ordinary addition, $+$, in C,
ii. $\circ$ be subaddition, $\otimes$, in S_B,
iii. $:$ be scalar multiplication in C, and
iv. $*$ be a scalar operation in S_B over C defined by

$$c * H[x] = \log(c) \oplus H(x),$$

then we have a generalized superposition H (where H stands for homomorphism.)

But in [1] we can show that this homomorphic system can be written as a cascade of three systems provided that $\otimes$ is commutative and associative and that we can prove the following.

Theorem 5.1. The additive group S_B space under $\otimes$ is a vector space over C with scalar multiplication $*$.

Proof. Let $\alpha, \beta \in C$ and $v, w \in S_B$. We can now easily establish the four properties of a vector space.

i.

$$\begin{aligned} \alpha * (v \otimes w) &= \log(\alpha) \oplus (v \otimes w) \\ &= [\log(\alpha) \oplus v] \otimes [\log(\alpha) \oplus w] \\ &= (\alpha * v) \otimes (\alpha * w). \end{aligned}$$

ii.

$$\begin{aligned} (\alpha \oplus \beta) * v &= \log(\alpha + \beta) \oplus v \\ &= [\log(\alpha) \otimes \log(\beta)] \oplus v \\ &= [\log(\alpha) \oplus v] \otimes [\log(\beta) \oplus w] \\ &= (\alpha * v) \otimes (\beta * w). \end{aligned}$$

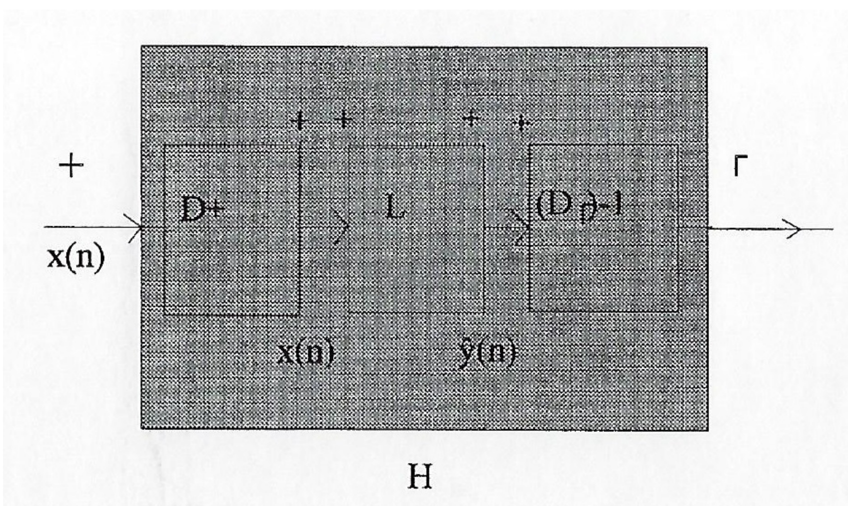
iii.

$$\begin{aligned} \alpha * (\beta * v) &= \log(\alpha) \oplus [\log(\beta) \oplus v] \\ &= \log(\alpha\beta) \oplus v \\ &= (\alpha\beta) * v. \end{aligned}$$

iv.

$$\begin{aligned} 1 * v &= \log(1) \oplus v \\ &= v. \end{aligned}$$

Again using [1], we know that since the system inputs constitute a vector space of complex numbers under addition and ordinary scalar multiplication and that the homomorphic system H outputs constitute a vector space under $\otimes$, the blackhole addition, and $*$, the scalar multiplication, all systems of this class can be represented as a cascade of three systems where the existence of D and L, a linear system, is guaranteed.



6. Whitehole Analysis. Set $S_W = SU\{+\infty\}$. We define an operation $\oslash$ on S_W by

$$z \oslash w = \log\{1/[(1/e^z) + (1/e^w)]\}$$

if $z, w \in S$ and $+\infty$ otherwise. It is now easy to show, by similar arguments as before, the following.

Theorem 6.1. $(C, +, \cdot) \cong (S_W, \oslash, \oplus)$.

Theorem 6.2. $(f)'_W(x) = f(x) \ominus x \ominus \log[f'(x)]$.

Corollary 6.3

i. $(c)'_W = +\infty$.
ii. $(x)'_W = 0$.
iii. $(px)'_W = (p \ominus 1)x \ominus \log(p)$.
iv. $(e^x)'_W = e^x \ominus 2x$.
v. $(-e^{-x})'_W = -e^{-x}$.
vi. $[\log(px)]'_W = [\log(x)] - x$.
vii. $[p\log(x)]'_W = [\log(x/p)] - x$.
viii. $[\sin(x)]'_W = \sin(x) \ominus x \ominus \log[\cos(x)]$.
ix. $[p\log(x)]'_W = \cos(x) \ominus x \ominus \log[-\sin(x)]$.
x. $[f'']_W = y \ominus 2x \ominus \log[(y')^2 - y' - y'']$.

Theorem 6.4. $[\int f(x)dx]_W = \ominus \log\{\ominus[\int e^{-[f(x)+x]}dx]\}$.

Theorem 6.5. $(y)'_B \oplus (y)'_W = 2(y \ominus x)$.

7. Blackhole Vectors. Again using Theorem 2.1 we see at once that the blackhole distance D_B between any two points (a, b) and (c, d) in Blackhole space is given by

7.1 $$D_B[(a, b), (c, d)] = \{\log[(e^c - e^a)^2 + (e^d - e^b)^2]\}/2.$$

As we have seen before a positive number in C is transformed into a real in B (and negative into complex.) And so it is not surprising then that a Blackhole distance can be negative but never complex. Now let $\langle a, b\rangle_B$ be a vector in B. Denote the norm of this vector, the Blackhole distance between the point (a, b) in B and $-\infty$, by $\|\langle a, b\rangle_B\|_B$. Furthermore, blackhole vector addition is defined by $\langle a, b\rangle_B \otimes \langle c, d\rangle_B = \langle a \otimes c, b \otimes d\rangle_B$. A particular case of 7.1 is given by

7.2 $$\|\langle a, b\rangle_B\|_B = D_B[(a, b), (-\infty, -\infty)] = [\log(e^{2a} + e^{2d})]/2.$$

The triangle inequality may be restated as

7.3 Let v and w be two vectors in B. Then

$$\|v_B \otimes w_B\|_B \leq \|v_B\|_B \otimes \|w_B\|_B.$$

Proof. By the triangle inequality

$$\begin{aligned} \|v_B\|_B \otimes \|w_B\|_B &= \{[\log(e^{2a} + e^{2b})]/2\} \otimes \{[\log(e^{2c} + e^{2d})]/2\} \\ &= \log\left[\sqrt{(e^{2a} + e^{2b})} + \sqrt{(e^{2c} + e^{2d})}\right] \\ &= \log\left\{\sqrt{[(e^a + e^c)^2 + (e^b + e^d)^2]}\right\}. \end{aligned}$$

But,

$$\begin{aligned} &\{\log[(e^a + e^c)^2 + (e^b + e^d)^2]\}/2 \\ &= \|\langle \log(e^a + e^c), \log(e^b + e^d)\rangle\|_B \\ &= \|\langle a \otimes c, b \otimes d\rangle\| \\ &= \|v_B \otimes w_B\|_B. \end{aligned}$$

8. Blackhole Programming. We know, by Theorem 2.1, that each operation and function has a unique blackhole image. For example

1. $f(x) \to \log[f(e^x)]$.
2. $d^2y/dx^2 \to \{\log[(d^2y/dx^2) \oplus (dy/dx)^2 \oplus dy/dx]\} \oplus y \ominus 2x$.

Consequently there exists a meta-blackhole algorithm which, though possibly of interest in itself, will accelerate any program in which multiplication and exponentiation dominate addition and subtraction. But we save this for a later paper.

Acknowledgement. This research was sponsored by the Air Force Office of Scientific Research/AFSC, United States Air Force, under Contract F49620-93-C-0063. The Air Force is authorized to reproduce and distribute reprints for governmental purposes notwithstanding any copyright notification hereon.

Reference

1. A. V. Oppenheim and R. W. Schafer, *Digital Signal Processing*, Prentice-Hall, Inc., Englewood Cliffs, NJ, 1975.

David Choate
300 N. Broadway
Department of Mathematics
Transylvania University
Lexington, KY 40508-1797
email: dchoate@mail.transy.edu