THE ARITHMETIC-MEAN – GEOMETRIC-MEAN INEQUALITY DERIVED FROM CLOSED POLYNOMIAL FUNCTIONS

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An activity which is encouraged in the teaching and study of mathematics is that of exploring how classical results may be derived from other concepts. The Arithmetic-Mean – Geometric-Mean Inequality (AMGM) states

$$\prod_{m=1}^{n} x_m \leq \left(\frac{\sum_{m=1}^{n} x_m}{n}\right)^n$$

for all positive integers n and nonnegative reals $x_1, \ldots, x_n$, with equality if and only if $x_k = x_j$ for all k and j, where $\prod_{m=1}^{n} x_m$ and $\sum_{m=1}^{n} x_m$ denote the product and sum of the numbers $x_1, \ldots, x_n$, respectively. The purpose of this note is to show that this classical inequality is an easy consequence of the fact that the function P defined on $\mathbb{R}^n$, Euclidean n-space, by $P(x_1, \ldots, x_n) = \sum_{m=1}^{n} |x_m|$ is a closed function, i.e. if A is a closed subset of $\mathbb{R}^n$, then $P(A) = \{P(x_1, \ldots, x_n) : (x_1, \ldots, x_n) \in A\}$, the image of A under P, is a closed subset of $\mathbb{R}$. When restricted to $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_m \geq 0 \text{ for each } m\}$, P is a polynomial function. We found this proof while studying closed functions between Euclidean spaces. Although we do not know if the proof is new, it does represent an excellent opportunity for students to see continuous functions, closed functions, and greatest lower bound working together. We also give a proof using compactness and continuity of the function Q defined on $\mathbb{R}^n$ by $Q(x_1, \ldots, x_n) = \prod_{m=1}^{n} x_m$ (see [1, 2]).

The following result will be applied to establish the AMGM.

Lemma. The function P defined on $\mathbb{R}^n$ by $P(x_1, \ldots, x_n) = \sum_{m=1}^{n} |x_m|$ is a closed function.

Proof. Let $A \subset \mathbb{R}^n$ be closed, let $r \in \mathbb{R}$, and let $\{(x_1^k, \ldots, x_n^k)\}_{k=1}^{\infty}$ be a sequence in A satisfying $P(x_1^k, \ldots, x_n^k) \to r$. Then $\{P(x_1^k, \ldots, x_n^k) : k = 1, \ldots\}$ is bounded and hence, the sequence $\{(x_1^k, \ldots, x_n^k)\}_{k=1}^{\infty}$ is a bounded sequence in A. By the Bolzano-Weierstrass Theorem, this sequence has a subsequence, which we again call $\{(x_1^k, \ldots, x_n^k)\}_{k=1}^{\infty}$, such that $(x_1^k, \ldots, x_n^k) \to (y_1, \ldots, y_n)$ and $(y_1, \ldots, y_n) \in A$, since A is closed. By the continuity of P, we arrive at $P(y_1, \dots, y_n) = r$, i.e. $r \in P(A)$.

Example 1. The function P defined on $\mathbb{R}^2$ by $P(x,y) = x + y$ is not a closed function, since $A = \{(n + \frac{1}{n}, -n) : n \text{ is a positive integer }\}$ is a closed subset of $\mathbb{R}^2$, $P(n + \frac{1}{n}, -n) \to 0$, $0 \notin P(A)$.

Theorem. The inequality

$$\prod_{m=1}^{n} x_m \leq \left(\frac{\sum_{m=1}^{n} x_m}{n} \right)^n$$

holds for all positive integers n and nonnegative reals $x_1, \dots, x_n$, with equality if and only if $x_k = x_j$ for all k and j.

Proof. Let $c \geq 0$ and let $A = \{(x_1, \dots, x_n) \in \mathbb{R}_+^n : \prod_{m=1}^{n} x_m = c\}$. Then A is a closed subset of $\mathbb{R}^n$, since the real-valued function Q defined on $\mathbb{R}^n$ by $Q(x_1, \dots, x_n) = \prod_{m=1}^{n} x_m$ is well-known to be continuous. From the Lemma, the function P defined on $\mathbb{R}^n$ by $P(x_1, \dots, x_n) = \sum_{m=1}^{n} |x_m|$ is a closed function, $(c, 1, \dots, 1) \in A$, and $P(A) \subset [0, \infty)$, so $\inf P(A)$ exists and $\inf P(A) \in P(A)$. Choose $(v_1, \dots, v_n) \in A$ with $P(v_1, \dots, v_n) = \inf P(A)$. If j and k are integers and $1 \leq j < k \leq n$, define $z_m = v_m$ when m is neither j nor k and $z_j = z_k = \sqrt{v_j v_k}$. Then $(z_1, \dots, z_n) \in A$ so $\sum_{m=1}^{n} v_m \leq \sum_{m=1}^{n} z_m$ and $(\sqrt{v_j} - \sqrt{v_k})^2 \leq 0$, giving $v_j = v_k$. It follows that $c = v_1^n$ and that $\inf P(A) = n v_1$. Now if $(x_1, \dots, x_n) \in A$, we obtain

$$\prod_{m=1}^{n} x_m = c = v_1^n \leq \left(\frac{\sum_{m=1}^{n} x_m}{n} \right)^n.$$

It is clear that equality holds if and only if $x_k = x_j$ for all k and j.

It is interesting that the set A in the Proof of the Theorem is not necessarily bounded in $\mathbb{R}^n$ if $n > 1$, since $(m, c/m, 1, \dots, 1) \in A$ for every positive integer m. Otherwise, we could have relied on the continuity of P to produce the proof, since A would have been compact. We now give a proof using compactness and continuity. Let $c > 0$ and let $A = \{(x_1, \dots, x_n) \in \mathbb{R}_+^n : \sum_{m=1}^{n} x_m = c\}$ ($c = 0$ forces $x_j = x_k = 0$ for all j and k). Then A is a closed subset of $\mathbb{R}^n$, since the real-valued function Q defined on $\mathbb{R}^n$ by $Q(x_1, \dots, x_n) = \sum_{m=1}^{n} x_m$ is continuous. It is obvious that $(c, 0, \dots, 0) \in A$ and A is bounded by nc. Therefore A is compact and nonempty.

The function P defined on $\mathbb{R}^n$ by $P(x_1, \dots, x_n) = \prod_{m=1}^n x_m$ is continuous so $P(A)$ is compact and nonempty. Hence, $\sup P(A)$ exists and $\sup P(A) \in P(A)$. Choose $(v_1, \dots, v_n) \in A$ with $P(v_1, \dots, v_n) = \sup P(A)$. If j and k are integers and $1 \leq j < k \leq n$, define $z_m = v_m$ if m is neither j nor k and $z_j = z_k = (v_j + v_k)/2$. Then $(z_1, \dots, z_n) \in A$ so $\prod_{m=1}^n v_m \geq \prod_{m=1}^n z_m$ and $(v_j - v_k)^2 \leq 0$, giving $v_j = v_k$. It follows that $c = nv_1$ and that $\sup P(A) = v_1^n$. If $(x_1, \dots, x_n) \in A$, we obtain

$$\prod_{m=1}^n x_m \leq v_1^n = (c/n)^n = \left(\frac{\sum_{m=1}^n x_m}{n} \right)^n .$$

Again, it is clear that equality holds if and only if $x_k = x_j$ for all k and j.

Example 2 shows that the function P used in the last proof might fail to be a closed function.

Example 2. The polynomial function P defined on $\mathbb{R}^2$ by $P(x, y) = xy$ is not closed, since the subset $A = \{(n, 1/n^2) : n \text{ if a positive integer } \}$ is closed in $\mathbb{R}^2$ and $P(n, 1/n^2) \to 0$, while $0 \notin P(A)$.

In the proof of the Lemma, we used the fact that the function P is continuous and that A is bounded if $P(A)$ is bounded. We close with more general statements in a proposition and corollary. For notational purposes, if $a, x \in \mathbb{R}^n$, we represent the inner product of a and x by $a \cdot x$, the norm of x by $\|x\|$, and the angle between a and x by $\varphi(a, x)$.

Proposition. Let $\epsilon > 0$ and let $\mathbb{R}^n_{a,\epsilon} = \{x \in \mathbb{R}^n : |\varphi(a, x) - \frac{\pi}{2}| \geq \epsilon\}$. If $A \subset \mathbb{R}^n_{a,\epsilon}$ and $a \neq 0$, then

(a) A is bounded if and only if $\{a \cdot x : x \in A\}$ is bounded, and

(b) P defined on $\mathbb{R}^n_{a,\epsilon}$ by $P(x) = a \cdot x$ is a closed function.

Proof. Condition (a) follows immediately from $a \cdot x = \|x\| \|a\| \cos(\varphi(a, x))$. In view of (a), the proof of (b) is essentially the same as that of the Lemma.

Corollary. Let $A \subset \mathbb{R}^n_+$, let $a \in \mathbb{R}^n_+$, $a \neq 0$, and let P be defined on $\mathbb{R}^n$ by $P(x) = a \cdot x$. Then A is bounded if and only if $P(A)$ is bounded.

References

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