DESIGNING PAYOFFS FOR SOME PROBABILISTIC GAMBLING GAMES

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"A casino is a mathematical palace set up to separate players from their money. Every bet made in a casino has been calibrated within a fraction of its life to maximize profit while still giving players the illusion that they have a chance." These words were written by Nicholas Pileggi [4] as he tells a true story of the brutal reality of life in a casino.

When a new game is created, there is often a rush of excitement to play. However, the mathematician has already had much of the fun. The game had to be analyzed in advance to determine the average earnings so that the proper payoff may be established. When there is a fixed chance of winning in either a betting or lottery game, then there is usually a fixed payoff for winning. But when the public becomes aware of these odds, much of this initial excitement can wear off. By adjusting payoffs though, one might be able to rejuvenate interest.

In this article, we discuss two gambling games that are based on common discrete probability distributions. The first game is an application of the geometric distribution in which a contestant continues making play after play until having a success. The payoff will be a function of the number of plays needed to succeed. The second game is an application of the Poisson distribution. In this case, the payoff will be a function of the total number of successes in a certain time period. For each game, we shall explain a method of designing and adjusting the payoffs while always guaranteeing that the casino earns a profit.

1. A Geometric Gambling Game. An owner of a casino in Czarist St. Petersburg designed the following game. A player will toss a fair coin until it falls heads; if this occurs on the kth toss, the player receives 2^k rubles. The player receives nothing if no decision is reached in N tosses. The average earnings of a player are given by

$$2\left(\frac{1}{2}\right) + 2^2\left(\frac{1}{2}\right)^2 + 2^3\left(\frac{1}{2}\right)^3 + \cdots + 2^N\left(\frac{1}{2}\right)^N = N; \tag{1}$$

thus, if the game is to be fair, the player should pay N rubles as an "admission fee". But if the game were not stopped after N tosses, then the average earnings of the player approaches $+\infty$. Thus, the casino would be ruined since it obviously cannot charge an infinite admission fee. This is the well known St. Petersburg paradox. (See [2], for a further discussion).

This paradox arises from the natural idea of doubling the payoff as the probability of earning that payoff decreases by a factor of 2. How may the payoffs be modified to retain the fairness of the game yet remove the paradox of infinite average earnings?

The essential problem is that the casino offers larger payoffs for each additional play that it takes to succeed. Realistically, the largest payoff should go to the player who succeeds on the first play. If gamblers are allowed to play over and over until they succeed, then the amount of money they receive should not be "double or nothing" but instead should be a decreasing function of the number of plays needed to obtain the first success.

In order to define a workable game, we let M represent the amount the player will pay as an admission fee at the beginning of the game and let p be the probability of succeeding on any individual play. We let a_k be the amount the player will earn for succeeding for the first time on the kth play. We would like the payoffs a_k to have certain amenable properties: (i) the average earnings should be finite; (ii) for a fair game the average earnings should equal M, or otherwise should be less than M so the casino can make some profit; (iii) the payoffs should be a decreasing sequence; and (iv) the payoffs for winning on the first few plays should be greater than M in order to entice the gambler.

If we assume the individual plays to be independent, then the game creates a geometric distribution by counting the number of attempts needed for the player to succeed. If we let X denote the players earnings, then since $q = 1 - p$ is the probability of losing on any play, the player's average earnings are given by

$$\begin{aligned} E[X] &= a_1 p + a_2 q p + a_3 q^2 p + \cdots + a_k q^{k-1} p + \cdots \\ &= \frac{p}{q} \sum_{k=1}^{\infty} a_k q^k. \end{aligned} \tag{2}$$

When the above series converges, we obtain finite average earnings. By the ratio test, the condition

$$\lim_{k\to\infty} \left| \frac{a_{k+1} q^{k+1}}{a_k q^k} \right| < 1$$

insures the convergence. This expression is equivalent to

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} < \frac{1}{q}.$$

The sequence a_k can be made to satisfy this condition by letting

$$a_k = \frac{1}{(q+\epsilon)^k} \tag{3}$$

for some $\epsilon > 0$. To make the sequence decrease, we also require that $q + \epsilon > 1$. Using the sum of a geometric series we have

$$\begin{aligned} E[X] &= \frac{p}{q} \sum_{k=1}^{\infty} \left(\frac{q}{q+\epsilon} \right)^k \\ &= \frac{p}{\epsilon}. \end{aligned} \tag{4}$$

In actual practice then, the admission fee M is chosen after the average earnings are determined. Hence, we choose $M = p/\epsilon$ for a fair game or $M > p/\epsilon$ for an unfair game.

Example 1. A player rolls a fair die until a six is rolled. The probability of succeeding on any play is $p = 1/6$ and thus, $q = 5/6$. So, if we let $\epsilon = 2/6$, then $q + \epsilon > 1$. The admission fee for a fair game is then

$$M = \frac{p}{\epsilon} = .50,$$

and the payoffs for succeeding on the kth play are $1/(q+\epsilon)^k = (6/7)^k$. So, a player pays 50 cents to enter the game, with the first four payoffs being

$$a_1 = 85¢, \ a_2 = 73¢, \ a_3 = 62¢, \ a_4 = 53¢.$$

These payoffs have been rounded down to the nearest cent to allow the casino a marginal profit. If one needs five or more rolls, then the payoffs are less than the player's admission fee; hence, the casino breaks even by profiting on these players. Of course, the casino can also profit by lowering these payoffs a bit more. We note that the average number of plays needed to obtain the first win is $1/p = 6$. (See [1]).

The condition $q+\epsilon > 1$ will guarantee that at least the first payoff a_1, is larger than the admission fee M. Indeed, since $\epsilon > 1 - q = p$, then $\epsilon q > pq$. That is, $\epsilon(1-p) > pq$ and thus $\epsilon > pq + p\epsilon = p(q+\epsilon)$. Hence, $a_1 = 1/(q+\epsilon) > p/\epsilon = M$. As illustrated in the above example, further payoffs may also be larger than M.

Since the variable ϵ only needs to satisfy the condition $\epsilon > 1 - q$, the casino is afforded some flexibility in the design of the game's payoffs. Therefore, ϵ may be chosen to suit one's taste in risk. Equating a_k to p/ϵ and solving for k, we see that the number of profitable payoffs satisfies the following relation.

$$k < \frac{-\ln M}{\ln(q+\epsilon)} = -\left(\frac{\ln p - \ln \epsilon}{\ln(q+\epsilon)}\right). \tag{5}$$

Using the values from Example 1, we have that $k < 4.5$. Thus, a player profits if succeeding by the fourth attempt. We may vary ϵ to allow the player more or less opportunities to profit. As ϵ is increased, the number of profitable payoffs k decreases and so does the fair admission fee. In fact, by applying L'Hopital's rule to the expression in (5), we see that the limit as ϵ approaches ∞ equals 1.

If ϵ decreases, the player will have more opportunity to make money, but the admission fee will increase. If we apply L'Hopital's rule again to (5), we see that

$$\lim_{\epsilon \to p^+} -\left(\frac{\ln p - \ln \epsilon}{\ln(q+\epsilon)}\right) = \frac{1}{p}. \tag{6}$$

However, since k is always rounded down to the previous integer, the limit for the number of profitable payoffs equals the greatest integer less than $1/p$. For $p = 1/6$ and $q = 5/6$, the limit of (5) as $\epsilon \to 1/6^+$ will be 5.

The above example illustrates the ideas here, but the payoffs and admission fee may seem low. These figures may be scaled by an appropriate constant C, without changing the above analysis. In Example 1, if the admission fee is increased by a factor of ten to five dollars, then the subsequent payoffs should also be increased tenfold. Suppose for a \$5 playing fee in Example 1, we set the first four payoffs as \$8, \$7, \$6, and \$5.25, with the remaining being approximately $10(6/7)^k$, for $k \geq 5$. Then the casino will make a profit while still enticing the player. Even non-degenerate gamblers would enjoy having four chances to make money! Meanwhile, the player's average earnings are

$$8\left(\frac{1}{6}\right)+7\left(\frac{5}{6}\right)\left(\frac{1}{6}\right)+6\left(\frac{5}{6}\right)^2\left(\frac{1}{6}\right)+5\left(\frac{5}{6}\right)^3\left(\frac{1}{6}\right)+10\sum_{k=5}^{\infty}\left(\frac{6}{7}\right)^k\left(\frac{5}{6}\right)^{k-1}\left(\frac{1}{6}\right)$$

$$\approx \$4.81;$$

thus, the casino averages a 19 cent take per \$5 fee.

Example 2. Every outcome of the dice can result in a new payoff scheme. The chance of rolling a 5 with two dice is 1/9. Suppose the casino desires a \$10 admission fee with the player having at most six chances to win. How may ϵ be chosen to satisfy these conditions?

Since the desired admission fee is set at ten dollars, the value of M obtained using our analysis must necessarily be scaled by some constant C. We must choose an ϵ which satisfies the two conditions

$$\frac{8}{9}+\epsilon > 1 \quad \text{and} \quad -\left(\frac{\ln\frac{1}{9}-\ln\epsilon}{\ln(\frac{8}{9}+\epsilon)}\right) > 6.$$

Picking $\epsilon = 2/9$ will satisfy both inequalities, and since $M = p/\epsilon$ we have the unscaled admission fee is $M = 1/2$ so that the scale constant is $C = 20$.

The first six scaled payoffs are

$$a_1 = \$18.00,\ a_2 = \$16.20,\ a_3 = \$14.58,$$
$$a_4 = \$13.12,\ a_5 = \$11.81,\ a_6 = \$10.63.$$

Problem 1. Design a fair game based on the negative binomial distribution. That is, allow the game to be played until the player succeeds r times, for some integer $r > 1$. Define payoffs a_k, which are to be paid for succeeding for the rth time on the kth play, and determine the average earnings of the player.

2. A Poisson Parlor Game. There are many carnival type games for which the actual probability of success is unknown. For instance, what is the probability of tossing a ring onto a milk bottle or of hitting the passing ducks with a BB gun? So, in order to make a carnival booth more exciting, suppose the operator runs a game in which the contestant pays an admission fee M and then plays the game for a certain period of time. The player's earnings are based upon the number of successes obtained. In addition to creating finite average earnings which are less than or equal to M, the payoffs a_k, given for succeeding k times, should now satisfy the following properties: (i) the sequence a_k should be increasing with $a_0 = 0$; and (ii) beyond a certain point, a_k should be larger than M.

In order to define such payoffs, one need only observe the average number λ of successes (for example, 3.5 per minute). We assume, of course, that ringers have been excluded when computing this average value. Since the number of plays will be large, we can model this game with a Poisson random variable Y which counts the number of successful plays. Since the probability of obtaining k successes is $P(Y = k) = \lambda^k e^{-\lambda}/k!$ [1], the average value of the player's earnings X are given by

$$E[X] = \sum_{k=0}^{\infty} a_k \frac{\lambda^k e^{-\lambda}}{k!}. \tag{7}$$

One obvious choice for the payoffs is $a_k = k$ for $k \geq 0$. The average earnings are then simply the expected value of the Poisson distribution, which is the average number of successes λ:

$$E[X] = \sum_{k=0}^{\infty} kP(Y = k) = E[Y] = \lambda.$$

For example, if there are an average of $\lambda = 3.5$ successes per minute and the booth pays k dollars for every success, then the players' average earnings will be \$3.50. Suppose then that the booth charges a \$4 entry fee. Those players with 5 or more successes will profit while the booth profits on those with 3 or less successes. However, $P(Y \leq 3) \approx .5366$; so the booth profits just over half the time. Since the median of the Poisson distribution is the integer less than or equal to the average λ, this situation is common.

Ideally, the booth should profit on *most* of the plays. Thus, most payoffs should be low and then the payoffs should grow exponentially for those players whose total number of successes is well beyond the average λ. Is there a better choice for the payoffs a_k which increase the booth's probability of making a profit?

To make the series in (7) converge and give finite average earnings, we can let $a_k = k!/(\lambda+\epsilon)^k$ for some $\epsilon > 0$. However, this sequence may not be strictly increasing and $a_0 \neq 0$; thus, we shall multiply by k^n for some integer n. We then define our payoffs to be

$$a_k = \frac{k^n k!}{(\lambda+\epsilon)^k}, \tag{8}$$

for some $\epsilon > 0$ and where n is chosen to make the sequence strictly increasing. We need only find a reasonable lower bound for n.

In order for the sequence to increase, we need $a_{k+1}/a_k \geq 1$ for all k, which simplifies to $(k+1)^{n+1}/k^n \geq \lambda+\epsilon$. It suffices then to find an integer n which makes the minimum of this sequence larger than $\lambda+\epsilon$. We consider instead the function $f(x) = (x+1)^{n+1}/x^n$, which has first derivative

$$f'(x) = \frac{(x+1)^n(x-n)}{x^{n+1}}.$$

A critical point exists at $x = n$ for which $f(x)$ achieves a minimum value. The minimum of f and of the sequence is then $(n+1)^{n+1}/n^n$. But then,

$$\frac{(n+1)^{n+1}}{n^n} = (n+1)\left(1+\frac{1}{n}\right)^n \geq ne.$$

Hence, in order to make $(n+1)^{n+1}/n^n \geq \lambda+\epsilon$, we can make $ne \geq \lambda+\epsilon$ or

$$n \geq \frac{\lambda+\epsilon}{e}. \tag{9}$$

In fact, for larger values of $\lambda + \epsilon$, n will consequently be larger and then $(n+1)^{n+1}/n^n \approx (n+1)e$. Therefore, it may suffice to take $n \geq (\lambda+\epsilon)/e - 1$.

Finally, with this chosen value of n, the average earnings are

$$E[X] = e^{-\lambda} \sum_{k=0}^{\infty} k^n \left(\frac{\lambda}{\lambda+\epsilon}\right)^k. \tag{10}$$

Example 3. A booth operator charges a flat fee for one-minute of ring tossing. Suppose there is an average of 3.5 successful tosses per minute. For $\lambda = 3.5$, if we let $\epsilon = 1$, then we must use $n \geq 4.5/e$. Thus, for $n = 2$, the contestant's average earnings (and hence, the admission fee for a fair game) are

$$e^{-3.5} \sum_{k=0}^{\infty} k^2 \left(\frac{3.5}{4.5}\right)^k = e^{-3.5} \left(\frac{\frac{7}{9} + \left(\frac{7}{9}\right)^2}{\left(1-\frac{7}{9}\right)^3}\right)$$

$$= \$3.80.$$

The payoffs for k successful tosses are $k^2 k!/(4.5)^k$ which yields $a_0 = 0$, $a_1 = .22$, $a_2 = .40$, $a_3 = .59$, $a_4 = .94$, $a_5 = 1.62$, $a_6 = 3.12$, $a_7 = 6.61$, $a_8 = 15.35$, $a_9 = 38.84$, $a_{10} = 106.57$, $\ldots$, $a_{20} = 83,933,306.59$, $\ldots$. Hopefully, the booth operator will quickly spot any ringers!

For increasingly larger values of λ, the choice of n in equation (9) will also need to be larger. However, the series in equation (10) can be summed with the following formula [3].

$$\sum_{k=1}^{\infty} k^n x^k = \frac{1}{(1-x)^{n+1}} \sum_{i=1}^{n} a_i x^i, \tag{11}$$

for $-1 < x < 1$, where

$$a_i = \sum_{j=0}^{i-1} (-1)^j \binom{n+1}{j} (i-j)^n. \tag{12}$$

Example 4. Suppose there is an average of 10 successes for a period of target shooting. If we let $\epsilon = 5$, then we must use $n = 6$ in equation (8). The average earnings are then

$$e^{-10} \sum_{k=1}^{\infty} k^6 \left(\frac{2}{3}\right)^k$$

$$= \frac{e^{-10}}{\left(\frac{1}{3}\right)^7} \left[\frac{2}{3} + 57\left(\frac{2}{3}\right)^2 + 302\left(\frac{2}{3}\right)^3 + 302\left(\frac{2}{3}\right)^4 + 57\left(\frac{2}{3}\right)^5 + \left(\frac{2}{3}\right)^6\right]$$

$$= \$18.14.$$

In this case, we might wish to scale the admission fee and the payoffs down. If we wish a fee of \$10, then we multiply each by 10/18.14. The fair payoff for k successes is now

$$a_k = \frac{10}{18.14} \frac{k^6 k!}{15^k}.$$

Here is a list of the first 20 payoffs.

k	Fair Payoff	k	Fair Payoff
1	.04	11	4.51
2	.31	12	6.08
3	.71	13	8.51
4	1.07	14	12.40
5	1.36	15	18.75
6	1.62	16	29.46
7	1.91	17	48.04
8	2.27	18	81.23
9	2.76	19	142.31
10	3.47	20	258.13

As with the Geometric Gambling Game, the admission fee or the payoffs can be scaled further to favor the booth. In particular, we can prevent the payoffs from growing so rapidly by adjusting the value of ϵ. It remains an open problem to determine the relationship between ϵ and the number of successes needed so that the payoff exceeds the average earnings.

Problem 2. Design a Binomial Betting game, for a finite number of attempts n and known probability of success p, which makes payoffs a_k for having k successes in n attempts. For an integer $m \geq 1$, what will the average earnings be if $a_k = k^m$?

References

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