EULER'S FORMULA AND DE MOIVRE'S FORMULA FOR QUATERNIONS

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Abstract. Natural generalizations of Euler's formula and De Moivre's formula for quaternions are derived.

1. Introduction. A quaternion q is a linear combination $a1 + bi + cj + dk$, where a, b, c, and d are real numbers and

$$\begin{aligned} 1 &= (1,0,0,0), \quad i = (0,1,0,0), \\ j &= (0,0,1,0), \quad k = (0,0,0,1). \end{aligned}$$

The sum of quaternions is the usual component-wise sum and the multiplication is defined so that $(1,0,0,0)$ is the identity and i, j, and k satisfy

$$i^2 = j^2 = k^2 = ijk = -1. \tag{1}$$

It follows from (1) that

$$ij = k, \quad jk = i, \quad ki = j, \quad \text{and} \quad ij = -ji, \quad jk = -kj, \quad ki = -ik.$$

A quaternion is usually written as $a + bi + cj + dk$ or as $\alpha + \beta j$, where α and β are complex numbers. The complex numbers do not commute with j, but satisfy $j\beta = \overline{\beta} j$. We can also write $q = a + \omega$, where $\omega = bi + cj + dk$, called the pure quaternion part of q. a is called the real part of q. The conjugate of q is $\overline{q} = a - \omega$. We can view the pure quaternion part $\omega = bi + cj + dk$ as a vector in $\mathbb{R}^3$. A simple computation shows

$$\omega_1\omega_2 = -\omega_1 \cdot \omega_2 + \omega_1 \times \omega_2, \tag{2}$$

where $\omega_1 \cdot \omega_2$ is the dot product and $\omega_1 \times \omega_2$ is the cross product in $\mathbb{R}^3$. It follows from (2) that $\omega_2\omega_1 = \overline{\omega_1\omega_2}$ for any pure quaternion ω_1 and ω_2. Let a_i be real numbers and β_i be pure quaternions. Then

$$(a_1 + \beta_1)(a_2 + \beta_2) = (a_1a_2 - \beta_1 \cdot \beta_2) + a_1\beta_2 + a_2\beta_1 + \beta_1 \times \beta_2. \tag{3}$$

It follows from (3) that $\overline{q_1q_2} = \overline{q_2}\ \overline{q_1}$ for any quaternion q_i. For more details on quaternions, we refer to [1].

2. Euler's Formula and De Moivre's Formula for Quaternions. We will use the notation

$$S^3 = \{q : |q| = 1\} \quad \text{and} \quad S^2 = \{\omega : |\omega| = 1, \overline{\omega} = -\omega\}.$$

S^3 is the set of all unit quaternions and S^2 is the set of all unit pure quaternions. S^3 is a group under quaternion multiplication and is isomorphic to $SU(2)$, the group of all 2 by 2 unitary matrices of determinant 1. The map

$$(a, b, c, d) \mapsto \begin{pmatrix} a+bi & -c+di \\ c+di & a-bi \end{pmatrix}$$

is a group isomorphism between S^3 and $SU(2)$.

Since $\omega \cdot \omega = 1$ and $\omega \times \omega = 0$ for any $\omega \in S^2$, we have the following proposition.

Proposition 1. $\omega^2 = -1$ for any $\omega \in S^2$, hence, any $\omega \in S^2$ has order 4.

We can express any $q = a + bi + cj + dk \in S^3$ as

$$q = \cos\theta + \omega \sin\theta, \tag{4}$$

where $\cos\theta = a$ and

$$\omega = \frac{1}{\sqrt{b^2+c^2+d^2}}(bi + cj + dk) = \frac{1}{\sqrt{1-a^2}}(bi + cj + dk).$$

This is similar to the polar coordinate expression of a complex number. We can view θ as the angle between the vector $q \in \mathbb{R}^4$ and the real axis (the subspace of real numbers) and $\omega \sin\theta$ as the projection of q onto the subspace $\mathbb{R}^3$ of pure quaternions. We will call (4) the polar expression of a unit quaternion q. Since $\omega^2 = -1$ for any $\omega \in S^2$, we have a natural generalization of Euler's formula for quaternions,

$$\begin{aligned} e^{\omega\theta} &= 1 + \omega\theta - \frac{\theta^2}{2!} - \omega\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \cdots \\ &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots + \omega\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) \\ &= \cos\theta + \omega\sin\theta \end{aligned}$$

for any real θ. If the power series definition

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \quad \text{and} \quad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

is used for quaternion x, then

$$\begin{aligned} \cos \omega &= \cos i = \cosh 1 \\ \text{and} \quad \sin \omega &= -\omega i \sin i = i\omega \sinh 1 \end{aligned}$$

for every $\omega \in S^2$. We note the cosine function is constant on the set S^2. For more on Euler's formula for complex numbers, we refer to [2].

A simple computation and the addition formula for cosine and sine, i.e.,

$$\cos(\theta + \psi) = \cos\theta \cos\psi - \sin\theta \sin\psi \quad \text{and} \quad \sin(\theta + \psi) = \cos\theta \sin\psi + \sin\theta \cos\psi$$

prove the following lemma.

Lemma. For any $\omega \in S^2$, we have

$$(\cos\theta + \omega \sin\theta)(\cos\psi + \omega \sin\psi) = \cos(\theta + \psi) + \omega \sin(\theta + \psi).$$

Remark. It follows from the lemma that $K_\omega = \{\cos\theta + \omega \sin\theta : 0 \leq \theta < 2\pi\}$ is a subgroup of S^3 and is isomorphic to S^1.

Proposition 2 (De Moivre's formula). Let $q = e^{\omega\theta} = \cos\theta + \omega \sin\theta \in S^3$, where θ is a real and $\omega \in S^2$. Then,

$$q^n = e^{\omega n\theta} = (\cos\theta + \omega \sin\theta)^n = \cos n\theta + \omega \sin n\theta \tag{5}$$

for every integer n.

Proof. The proof is by induction on the nonnegative integers n.

$$\begin{aligned} q^{n+1} &= (\cos\theta + \omega \sin\theta)^{n+1} \\ &= (\cos n\theta + \omega \sin n\theta)(\cos\theta + \omega \sin\theta) \\ &= \cos(n+1)\theta + \omega \sin(n+1)\theta. \end{aligned}$$

The formulas holds for all integers n, since

$$\begin{aligned} & q^{-1} = \cos\theta - \omega\sin\theta \\ \text{and}\quad & q^{-n} = \cos n\theta - \omega\sin n\theta = \cos(-n\theta) + \omega\sin(-n\theta). \end{aligned}$$

Corollary. There are infinitely many unit quaternions satisfying $x^n = 1$.

Proof. For every $\omega \in S^2$, we have a quaternion $q = \cos 2\pi/n + \omega \sin 2\pi/n$ of order n.

Example. $\frac{1}{2}(1 + i + j + k) = \cos\frac{\pi}{3} + \sin\frac{\pi}{3}(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ is of order 6 and $\frac{1}{2}(-1 + i + j + k) = \cos\frac{2\pi}{3} + \sin\frac{2\pi}{3}(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ is of order 3.

References

1. N. Jacobson, *Basic Algebra*, W. H. Freeman and Company, San Francisco, CA, 1974.
2. R. P. Boas, *Invitation to Complex Analysis*, Random House, New York, 1987.

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