NOTE ON BIRATIONAL EXTENSIONS IN D-DIMENSION

Mee-Kyoung Kim

Abstract. Let (R, m) be a d-dimensional regular local ring with quotient field K and (S, n) be a d-dimensional normal local domain birationally dominating R with $l(mS) = d$. In this paper, it is shown that the following three properties hold.

(1) S is dominated by the m-adic prime divisor of R;

(2) $n^i \cap R = m^i$, for all $i \geq 1$;

(3) $R/m = S/n$.

1. Introduction. Let (R, m) be a d-dimensional regular local ring with quotient field K of R. A valuation v of K which birationally dominates R is called a prime divisor of R if $\operatorname{tr.deg}_{R/m} V/m(V) = d - 1$, where V is the corresponding valuation ring of v and $m(V)$ denotes the maximal ideal of V. Then the order valuation v_m of K is called the m-adic prime divisor of R [7].

Let $v_1, \dots, v_r$ be elements of an ideal I in a local ring (T, p) and suppose that whenever $f(X_1, \dots, X_r)$ is a form of (arbitrary) degree s with coefficients in T such that $f(v_1, \dots, v_r) \equiv 0 \pmod{I^s p}$, then all the coefficients of f are in p. In these circumstances, the elements $v_1, \dots, v_r$ are said to be analytically independent in I. Elements $v_1, \dots, v_r$ are said to be analytically independent if they are analytically independent in the ideal they generate. The analytic spread, $l(I)$, of an ideal I in T is the dimension of the graded ring $\oplus_{\geq 0} I^n/pI^n$. In [6], Northcott and Rees proved that if T/p is an infinite field, then $l(I)$ is the maximum number of elements in I which are analytically independent in I.

Let (S, n) be a 2-dimensional normal local domain birational dominating a 2-dimensional regular local ring (R, m). In [2], Huneke and Sally showed that if R is maximally regular in S (i.e., if $R \subseteq R_0 \subseteq S$ and R_0 is a regular ring, then $R = R_0$), then S is dominated by the m-adic prime divisor of R, $n^i \cap R = m^i$ for all $i \geq 1$ and the residue fields of the two rings are the same. We extend these results to the case of a dimension $d \geq 3$ with an assumption $l(mS) = d$.

2. Main Theorems. In this section, (R, m) will denote a d-dimensional regular local ring with quotient field K and (S, n) will denote a d-dimensional normal local domain birationally dominating R (i.e., $R \subseteq S \subseteq K$ and $n \cap R = m$). Let $x_1, \dots, x_d$ be a regular system of parameters for R; i.e., $m = (x_1, \dots, x_d)$.

Lemma 1. If $l(mS) = d$, then $nS[x_2/x_1, \dots, x_d/x_1]$ is a prime ideal in $S[x_2/x_1, \dots, x_d/x_1]$.

Proof. Define the canonical homomorphism ϕ from $S[T_2, \ldots, T_d]$ onto $S[x_2/x_1, \ldots, x_d/x_1]/nS[x_2/x_1, \ldots, x_d/x_1]$ by $\phi(T_i) = x_i/x_1 + nS[x_2/x_1, \ldots, x_d/x_1]$ for $i = 2, 3, \ldots, d$, where $T_2, \ldots, T_d$ are indeterminates. Since $l(mS) = d$, $x_1, \ldots, x_d$ are analytically independent, and hence, $\text{Ker}\phi = n[T_2, \ldots, T_d]$. Thus, $nS[x_2/x_1, \ldots, x_d/x_1]$ is a prime ideal in $S[x_2/x_1, \ldots, x_d/x_1]$, by the First Isomorphism Theorem.

Theorem 1. If $l(mS) = d$, then S is dominated by the m-adic prime divisor of R.

Proof. Since $l(mS) = d$, $nS[x_2/x_1, \ldots, x_d/x_1]$ is a prime ideal in $S[x_2/x_1, \ldots, x_d/x_1]$, by Lemma 1. Moreover, since $n \cap R = m$, we get that

$$nS\left[\frac{x_2}{x_1}, \ldots, \frac{x_d}{x_1}\right] \cap R\left[\frac{x_2}{x_1}, \ldots, \frac{x_d}{x_1}\right] = mR\left[\frac{x_2}{x_1}, \ldots, \frac{x_d}{x_1}\right],$$

which is ht 1 prime and a principal ideal in $R[x_2/x_1, \ldots, x_d/x_1]$. Hence, we have the following local rings:

$$V = R\left[\frac{x_2}{x_1}, \ldots, \frac{x_d}{x_1}\right]_{mR[\frac{x_2}{x_1}, \ldots, \frac{x_d}{x_1}]} \subseteq W = S\left[\frac{x_2}{x_1}, \ldots, \frac{x_d}{x_1}\right]_{nS[\frac{x_2}{x_1}, \ldots, \frac{x_d}{x_1}]}.$$

Let $m(V)$ and $m(W)$ be the maximal ideals of V and W, respectively. Then, $V \subseteq W \subseteq K$ and $m(W) \cap V = m(V)$. But, V is the discrete valuation of v_m, the m-adic prime divisor of R. Thus, $V = W$ and

$$\begin{aligned} m(V) \cap S &= m(W) \cap S \\ &= nS\left[\frac{x_2}{x_1}, \ldots, \frac{x_d}{x_1}\right]_{nS[\frac{x_2}{x_1}, \ldots, \frac{x_d}{x_1}]} \cap S \\ &= n. \end{aligned}$$

We remark that if $l(mS) = d$, then $nS[x_2/x_1, \ldots, x_d/x_1]$ is a ht 1 prime ideal, since $V = W$ in the proof of Theorem 1.

Theorem 2. If $l(mS) = d$, then $n^i \cap R = m^i$, for all $i \geq 1$.

Proof. Let $V = R[x_2/x_1, \ldots, x_d/x_1]_{mR[x_2/x_1,\ldots,x_d/x_1]}$ be the discrete valuation of v_m, the m-adic prime divisor of R. Since $l(mS) = d$, we get that $m(V) \cap S = n$ by Theorem 1. Thus, for any $\alpha \in n$,

$$(*) \qquad v_m(\alpha) \geq 1.$$

For each $i \geq 1$, let $z \in n^i \cap R$. Let us express

$$z = \sum_{j=1}^{t} a_{j_1}\alpha_{j_2}\cdots\alpha_{j_i},$$

where $\alpha_{j_1}, \ldots, \alpha_{j_i} \in n$. Then, by (*),

$$\begin{aligned} v_m(z) &\geq \min_{j=1,\ldots,t}\{v_m(\alpha_{j_1}\alpha_{j_2}\cdots\alpha_{j_i})\} \\ &= \min_{j=1,\ldots,t}\{v_m(\alpha_{j_1}) + \cdots + v_m(\alpha_{j_i})\} \\ &\geq i. \end{aligned}$$

By the definition of order valuation v_m, $z \in m^i$. To see the other inclusion we will use an inductive argument. $n \cap R = m$ by the hypothesis. Assume inductively that $m^i \subseteq n^i \cap R$. Then,

$$\begin{aligned} m^{i+1} &= mm^i \\ &\subseteq (n \cap R)(n^i \cap R) \\ &\subseteq n(n^i \cap R) \cap R(n^i \cap R) \\ &\subseteq nn^i \cap R \\ &= n^{i+1} \cap R. \end{aligned}$$

Thus, $m^i \subseteq n^i \cap R$, for all $i \geq 1$.

Corollary 1. If $l(mS) = d$, then the natural map from $gr_m(R)$, the associated graded ring of R, to $gr_n(S)$, the associated graded ring of S, is an injection.

Proof. It is clear, since $n^i \cap R = m^i$, for all $i \geq 1$.

Theorem 3. If $l(mS) = d$, then $R/m = S/n$.

Proof. By the Dimension Inequality [3], we have that

$$ht(n) \geq ht(m) + tr.\deg_R S - tr.\deg_{R/m} S/n,$$

where $tr.\deg_R S$ is the transcendence degree of the field of fractions of S over that of R. Since S birationally dominates R and $\dim(S) = \dim(R)$, $tr.\deg_{R/m} S/n = 0$. Let $(V, m(V))$ be the m-adic divisor of R. Then $V/m(V)$ is a pure transcendental extension of R/m of transcendental degree $d-1$, i.e., $tr.\deg_{R/m} V/m(V) = d-1$. By Theorem 1, we have the following injections:

$$\frac{R}{m} \hookrightarrow \frac{S}{n} \hookrightarrow \frac{V}{m(V)}.$$

Hence, we have

$$tr.\deg_{R/m} V/m(V) = tr.\deg_{R/m} S/n + tr.\deg_{S/n} V/m(V).$$

Therefore,

$$tr.\deg_{S/n} V/m(V) = d-1.$$

Hence, we get

$$\frac{R}{m}(Y_1, \ldots, Y_{d-1}) \cong \frac{V}{m(V)} \cong \frac{S}{n}(Z_1, \ldots, Z_{d-1}),$$

where Y_i, Z_i are indeterminates. Since $R/m \subseteq S/n$, we have $R/m = S/n$.

Corollary 2. If $l(mS) = d$, then $x_1, \ldots, x_d$ form a subset of a minimal basis for every ideal J of S containing mS.

Proof. By Theorem 2, we have the following commutative diagram:

$$\begin{array}{ccc} \frac{m}{m^2} & \hookrightarrow & \frac{n}{n^2} \\ \| & & \uparrow \\ \frac{J\cap R}{(J\cap R)m} & \hookrightarrow & \frac{J}{nJ}. \end{array}$$

By Theorem 3, $x_1, \ldots, x_d$ are linearly independent over $S/n(= R/m)$. Hence, $x_1, \ldots, x_d$ form a subset of a minimal basis for J.

Lemma 2. [2] Let (A, p) be a 2-dimensional regular local ring with quotient field k and (B, q) be a 2-dimensional normal local domain birationally dominating A. Suppose that A is maximally regular in S. Then

(1) $ht(pB) = 1$;

(2) pB is not a principal ideal;

(3) $l(pB) = 2$.

Proof.

(1) We may assume that $A \neq B$ and that A/p is infinite. By Zariski's Main Theorem [5], $ht(pB) = 1$.

(2) Suppose that pB is principal. Express

$$pB = \alpha B, \quad \text{for some } \alpha \in p = (\alpha, \beta).$$

Then, $\beta/\alpha \in B$. Since $A[\beta/\alpha]$ is a 2-dimension, $q \cap A[\beta/\alpha]$ is either (i) a ht 2 maximal ideal of $A[\beta/\alpha]$ containing $pA[\beta/\alpha]$ or (ii) a ht 1 prime $pA[\beta/\alpha]$.

Case i. If $q \cap A[\beta/\alpha]$ is a ht 2 maximal ideal of $A[\beta/\alpha]$ containing $pA[\beta/\alpha]$, then $C = A[\beta/\alpha]_{q \cap A[\beta/\alpha]}$ is the quadratic transformation of A, i.e., C is a 2-dimensional regular local ring such that $A \subset C \subset B$, which is a contradiction to the maximal regularity of A in B.

Case ii. If $q \cap A[\beta/\alpha] = pA[\beta/\alpha]$ is a ht 1 prime ideal of $A[\beta/\alpha]$, then $D = A[\beta/\alpha]_{pA[\beta/\alpha]}$ is a discrete valuation ring, i.e., D is a 1-dimensional regular local ring such that $A \subset D \subset B$, which is a contradiction to the maximal regularity of A in B.

(3) It is true that

$$ht(pB) \leq l(pB) \leq \dim(B).$$

The first inequality is Lemma 4 in [6] and the second inequality is a result of Burch [1]. Since $\dim(B) = 2$ and $ht(pB) = 1$, $l(pB) = 1$ or 2. Suppose that $l(pB) = 1$. By Lemma 4.5 in [4], pB is principal, which is a contradiction to (2). Hence, $l(pB) = 2$.

Corollary 3. [2] Let (A, p) be a 2-dimensional regular local ring with quotient field k and (B, q) be a 2-dimensional normal local domain birationally dominating A. Suppose that A is maximally regular in B. Then

(1) B is dominated by the p-adic prime divisor of A;

(2) $q^i \cap A = p^i$, for all $i \geq 1$;

(3) $A/p = B/q$.

Proof. By Lemma 2, $l(pB) = 2$. Hence, (1), (2), and (3) are clear by Theorems 1, 2, and 3.

Acknowledgment. The author was partially supported by BSRI-96-1435.

References

1. L. Burch, "Codimension and Analytic Spread," *Proc. Cambridge Phil. Soc.*, 72 (1972), 369–373.
2. C. Huneke and J. Sally, "Birational Extensions in Dimension Two and Integrally Closed Ideals," *J. of Algebra*, 115 (1988), 418–500.
3. H. Matsumura, *Commutative Ring Theory*, Cambridge Studies in Advanced Math. 8, Cambridge Univ. Press, Cambridge, 1986.
4. S. McAdam, *Asymptotic Prime Divisors*, Lecture Notes in Math., Springer-Verlag, 1983.
5. M. Nagata, *Local Ring*, Interscience, New York, 1962.
6. D. G. Northcott and D. Rees, "Reduction of Ideals in Local Rings," *Proc. Cambridge Phil. Soc.*, 50 (1954), 145–158.
7. O. Zariski and P. Samuel, *Commutative Algebra*, Vol. II, Von Nostrand, Princeton, 1960.

Mee-Kyoung Kim
Department of Mathematics
Sung Kyun Kwan University
Suwan 440-746, Korea
email: mkkim@yurim.skku.ac.kr

æ