STABLE RINGS AND SIDE DIVISORS

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Abstract. All rings are commutative rings with identity. $\tilde{R}$ denotes the set of all units in a ring R together with 0 and it is clear that $R\backslash\tilde{R} = \emptyset$ if and only if R is a field. In addition to some other results, it is shown that R is not stable if and only if there exists a unimodular sequence (y, u) in R with $y \in R$ and $u \in R\backslash\tilde{R}$ such that u is not a side divisor of y. For each $s \geq 1$, a sequence $a_1, a_2, \ldots, a_s, a_{s+1}$ of elements in a ring R is said to be stable, whenever the ideal is $(a_1, a_2, \ldots, a_s, a_{s+1}) = (a_1 + b_1a_{s+1}, \ldots, a_s + b_sa_{s+1})$ for some $b_1, b_2, \ldots, b_s$ in R. A sequence $a_1, a_2, \ldots, a_s, a_{s+1}$ of elements in R is called a unimodular sequence provided that $(a_1, a_2, \ldots, a_s, a_{s+1}) = R$. For any fixed positive integer n, we shall say R is n-stable (simply, stable for $n = 1$), whenever, for all $s \geq n$ any unimodular sequence, $a_1, a_2, \ldots, a_s, a_{s+1}$ in R is stable. $u \in R\backslash\tilde{R}$ is said to be a side divisor of $y \in R$, if $u|y - z$ for some $z \in \tilde{R}$. Besides two other different proofs, we apply the above result to show that $R[X]$ is not stable for any commutative ring R. At the end, it is shown that any Artinian ring is stable.

1. Introduction. All rings are commutative rings with identity. For each $s \geq 1$, a sequence $a_1, a_2, \ldots, a_s, a_{s+1}$ of elements in a ring R is said to be stable, whenever the ideal $(a_1, a_2, \ldots, a_s, a_{s+1}) = (a_1 + b_1a_{s+1}, \ldots, a_s + b_sa_{s+1})$ for some $b_1, b_2, \ldots, b_s \in R$. A sequence $a_1, a_2, \ldots, a_s, a_{s+1}$ of elements in R is called a unimodular sequence provided that $(a_1, a_2, \ldots, a_s, a_{s+1}) = R$. For each fixed positive integer n, we shall say n is in the stable range of R (simply, R is n-stable, or stable for $n = 1$), if for all $s \geq n$ any unimodular sequence $a_1, a_2, \ldots, a_s, a_{s+1}$ of elements in R is stable. It is obvious that any sequence $a_1, a_2, \ldots, a_s, a_{s+1}$ in R is stable, whenever a_i is a unit in R for some $i = 1, 2, \ldots, s, s+1$. For example, assume a_{s+1} is a unit in R, then $(a_1, a_2, \ldots, a_s, a_{s+1}) = (a_1 + \frac{a_{s+1} - a_1}{a_{s+1}}a_{s+1}, a_2 + 0a_{s+1}, \ldots, a_s + 0a_{s+1})$. For a detailed study of n-stable rings, see [1] and [3].

Let $\tilde{R}$ denote the set of all units in a ring R together with 0. It is clear that $R\backslash\tilde{R}$ is empty if and only if R is a field. Assume $R\backslash\tilde{R}$ is not empty. $u \in R\backslash\tilde{R}$ is a side divisor of an element y in R provided that $u|y - z$ for some $z \in \tilde{R}$. u is a pure side divisor of y, if $u|y - z$ for some $z \in \tilde{R}\backslash\{0\}$. u is a universal side divisor in R, whenever u is a side divisor of each $y \in R$. Furthermore, $u \in R\backslash\tilde{R}$ is said to be a pure universal side divisor in R, whenever it is a pure side divisor of each element y in R with $y \neq 0$ and $y \neq u$. See also the following remarks.

Remarks.

(a) If u is a pure side divisor of $y \in R$, then u does not divide y in R. Otherwise, $u|y-(y-z)$ for some $z \in \tilde{R}\backslash\{0\}$, which is a contradiction to the choice of u.

(b) The choice of $u \in R\backslash\tilde{R}$ makes it clear that if u is a pure side divisor of $y \in R$, then y is different from 0 and u. This means 0 does not have any pure side divisor in R and u cannot be a pure side divisor of itself.

(c) If u is a pure side divisor of y, then u is a side divisor of y. Conversely, if u is a side divisor of y and u does not divide y, then u is a pure side divisor of y. From this, it is clear that for any fixed element y in R, the set of all pure side divisors of y is contained in the set of all side divisors of y.

2. Preliminary Lemmas.

Lemma 1. If all unimodular sequences of size $n+1$ ($n \geq 1$ a fixed integer) are stable, then any unimodular sequence of size larger than n is stable.

Proof. The proof (by induction) is based on an argument communicated by D. Estes and R. Guralnick.

Let $(a_1, a_2, \ldots, a_n, a_{n+2}) = R$, with $1 = \sum_{i=1}^{n+2} a_i x_i = \sum_{i=1}^{n} a_i x_i + y$, where $y = a_{n+1}x_{n+1} + a_{n+2}x_{n+2}, a_i, x_i$ in R. Hence, $(a_1, a_2, \ldots, a_n, y) = R$ and by the induction hypothesis we have $(a_1+b_1y, a_2+b_2y, \ldots, a_n+b_ny) = R$ for appropriate $b_1, b_2, \ldots, b_n$ in R. Then $(a_1+b_1x_{n+2}a_{n+2}, \ldots, a_n+b_nx_{n+2}a_{n+2}, a_{n+1}+0\cdot a_{n+2}) = R$ and the proof is complete.

Lemma 2. Let A be a proper ideal of a ring R. Then R/A is n-stable, whenever R is n-stable.

Proof. Let $(a_1+A, a_2+A, \ldots, a_s+A, a_{s+1}+A) = R/A$, where $s \geq n$. Then $1+A = \sum_{i=1}^{s+1}(a_ix_i+A), x_i \in R$, so that $1 = (\sum_{i=1}^{s+1} a_ix_i) + a$ for some $a \in A$. Thus, $(a_1, a_2, \ldots, a_s, a_{s+1}x_{s+1}+a) = R$. Now by hypothesis, there exist $b_1, b_2, \ldots, b_s \in R$ such that $(a_1+b_1y, a_2+b_2y, \ldots, a_s+b_sy) = R$, where $y = a_{s+1}x_{s+1}+a$. Hence, $(a_1+b_1y+A, \ldots, a_s+b_sy+A) = R/A$. Consequently, $(a_1+(b_1x_{s+1})a_{s+1}+A, \ldots, a_s+(b_sx_{s+1})a_{s+1}+A) = R/A$ and the result follows.

Lemma 3. The direct product of n-stable rings is n-stable if and only if each factor of the product is n-stable.

Proof. Necessity. Let $\pi_k\colon \prod_{i\in I} R_i \to R_k$ be the kth canonical projection. Then the conclusion is immediate from Lemma 2 above.

Sufficiency. Assume R_i is n-stable for each $i \in I$. Let

$$(\{a_{1,i}\}, \{a_{2,i}\}, \dots, \{a_{s,i}\}, \{a_{s+1,i}\}) = R = \prod_{i\in I} R_i.$$

$\{1_{R_i}\} \in R$ implies $\{1_{R_i}\} = \sum_{j=1}^{s+1}\{a_{j,i}\}\{x_{j,i}\} = \sum_{j=1}^{s+1}\{a_{j,i}x_{j,i}\}, \{x_{j,i}\} \in R$, so that $1_{R_i} = \sum_{j=1}^{s+1} a_{j,i}x_{j,i}$, i.e., $(a_{1,i}, a_{2,i}, \dots, a_{s,i}, a_{s+1,i}) = R_i$ for each $i \in I$. Since R_i is n-stable, there exist $b_{1,i}, b_{2,i}, \dots, b_{s,i} \in R_i$ such that $1_{R_i} \in (a_{1,i} + b_{1,i}a_{s+1,i}, \dots, a_{s,i} + b_{s,i}a_{s+1,i})$, which implies $1_{R_i} = \sum_{j=1}^{s}(a_{j,i} + b_{j,i}a_{s+1,i})(y_{j,i}), y_{j,i} \in R_i$. Thus, $\{1_{R_i}\} = \{\sum_{j=1}^{s}(a_{j,i} + b_{j,i}a_{s+1,i})(y_{j,i})\} = \sum_{j=1}^{s}(\{a_{j,i}\} + \{b_{j,i}\}\{a_{s+1,i}\})\{y_{j,i}\}$. Therefore, we have $\{1_{R_i}\} \in (\{a_{1,i}\} + \{b_{1,i}\}\{a_{s+1,i}\}, \dots, \{a_{s,i}\} + \{b_{s,i}\}\{a_{s+1,i}\})$ and the result follows.

Lemma 4. If R is a local ring, then it is stable.

Proof. Let R be a local ring with the maximal ideal M. Hence, any unimodular sequence $a_1, a_2, \dots, a_s, a_{s+1}$ in R must have a unit element. Otherwise, each ideal $(a_i) \subseteq M$ for each $i = 1, 2, \dots, s, s+1$ and this make $R \subseteq M$, which is a contradiction. Now, the result is immediate from the fact that any unimodular sequence is stable, if it contains a unit element. See the argument in the introduction above.

3. Main Results with Some Examples.

Example 1. Let $\mathbb{Z}$ be the ring of rational integers. Then $\tilde{\mathbb{Z}} = \{1, 0, -1\}$ and 2 is a universal side divisor in $\mathbb{Z}$, since for each $y \in \mathbb{Z}$ we have $y \equiv 0 \pmod 2$ or $y \equiv 1 \pmod 2$.

It is shown in [4] that an integral domain with no universal side divisor cannot be a Euclidean domain, and in [1] it is shown that a stable principal ideal domain is always Euclidean. Hence, if R is a PID which has no universal side divisor then R is not stable. The converse, however, need not be true. Again, let $R = \mathbb{Z}$ and note that $\mathbb{Z}$ is not stable [1], but 2 is a universal side divisor in $\mathbb{Z}$ (Example 1 above).

Theorem 1. If $u \in R\backslash\tilde{R}$ is a pure side divisor of an element $y \in R$, then (y, u) is a stable unimodular sequence.

Proof. If u is a pure side divisor of y, then there exists $z \in \tilde{R}\backslash\{0\}$ such that $u|y-z$. Then $y-z=ru$ for some $r \in R$. Since $z = y - ru$, then $z \in (y,u)$ and consequently $(y,u) = R = (z) = (y-ru)$.

Corollary 1. If $u \in R\backslash\tilde{R}$ is a pure universal side divisor in R, then (y,u) is a stable unimodular sequence in R for all $y \in R\backslash\{0,u\}$.

Theorem 2. Let u be a fixed element in $R\backslash\tilde{R}$ such that (y,u) is a unimodular sequence for some y in R. If (y,u) is stable then u is a pure side divisor of y.

Proof. If (y,u) is stable, then $(y,u) = (y+ru)$ for some $r \in R$. Since (y,u) is unimodular, we have $R = (y,u) = (y+ru)$, thus $y + ru = z$ is a unit in R. Hence, $u|y-z$ implies u is a pure side divisor of y.

Corollary 2. Let u be a fixed element in $R\backslash\tilde{R}$ and $(y,u) = R$ for each $y \in R\backslash\{0,u\}$. If (y,u) is stable, then u is a pure universal side divisor in R.

Theorem 3. Assume $R\backslash\tilde{R}$ is not empty. R is not stable if and only if there exists a unimodular sequence (y,u) in R with $y \in R$ and $u \in R\backslash\tilde{R}$ such that u is not a side divisor of y.

Proof. Necessity. If R is not stable, then by Lemma 1 above there exists a non-stable unimodular sequence (y,u), where neither y nor u is a unit of R. Thus, neither y nor u can be zero, so u is an element of $R\backslash\tilde{R}$. If we assume that u is a side divisor of y, then for some $z \in \tilde{R}$ we have $u|y-z$. Since (y,u) is not stable z cannot be a unit of R, thus $z = 0$. Hence, $u|y$, i.e., $y = qu$ for some $q \in R$. Let $1 = yy' + uu', y', u' \in R$. Then $1 = (qy' + u')u$ which contradicts the fact that u is not a unit in R.

Sufficiency. Assume R is stable. Then $R = (y,u) = (y+ru)$ for some $r \in R$. Hence, for some unit $z \in \tilde{R}\backslash\{0\}$ we have $y + ru = z$. Thus, $u|y-z$, i.e., u is a side divisor of y.

Example 2. By applying the sufficient part of Theorem 3 and the fact that $5 \in \mathbb{Z} = (3,5)$ is not a side divisor of 3 in $\mathbb{Z}$, then it is clear that $\mathbb{Z}$ is not stable.

Example 3. Let $R[X]$ be the ring of polynomials over R and assume $\widetilde{R[X]} = \tilde{R}$. Again by applying the sufficient part of Theorem 3 and the fact that $1 - X^2 \in R[X] = (X, 1-X^2)$ is not a side divisor of X, we can conclude that $R[X]$ is not stable. From this, it is clear that neither $\mathbb{Z}[X]$ nor $F[X]$ is a stable ring, where $\mathbb{Z}$ is the ring of rational integers and F is a field.

Theorem 4. Let $R[X]$ be the ring of polynomials with an indeterminate over a commutative ring R with identity. Then $R[X]$ is not a stable ring.

Proof. First approach. Let N be the nilradical of R. Then the result follows from Example 3 and Lemma 2 above, since $R[X]/N[X] \cong (R/N)[X]$.

Second approach. Following Lemma 6.1 in [1], it is shown that $K[X]$ is not stable whenever K is a field. Now using this fact and $R[X]/M[X] \cong (R/M)[X]$, where M is a maximal ideal of R, together with Lemma 2 above, we conclude that $R[X]$ is not stable. Compare this result with [2].

Third approach. Suppose $R[X]$ is stable, then $R[X] = (X, 1 - X^2) = (X + f(X)(1-X^2))$ for some $f(X) = \sum_{i=0}^{n} f_i x^i$, where f_i is in R for each $i = 0, 1, \dots, n$. Thus, $X + f(X)(1-X^2)$ is a unit in $R[X]$, which forces f_0 and each of $(f_1+1), (f_2 - f_0), (f_3 - f_1), \dots, (f_n - f_{n-2}), f_{n-1}, f_n$ to be a unit and a nilpotent in R, respectively. Now from this and the fact that the difference of two nilpotent elements is again a nilpotent, we come to a contrary situation that forces f_0 to be both a unit and a nilpotent element in R.

Theorem 5. Any commutative Artinian ring with identity is stable.

Proof. By applying Lemma 3, Lemma 4 above, and the fact that any Artinian ring is the direct product of local rings [5], the result is immediate.

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