SOLUTIONS

No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the problem editor.

81. [1995, 87] *Proposed by J. Sriskandarajah, University of Wisconsin Center-Richland, Richland Center, Wisconsin.*

Let ABC be a triangle with sides a, b, and c. Let K be the area of triangle ABC and s be the semi-perimeter of ABC.

(a) Prove that

$$\frac{K}{\tan\frac{A}{2}} + K\tan\frac{A}{2} = bc.$$

(b) Prove that

$$\frac{K}{s\tan\frac{A}{2}} + s = b + c.$$

Solution by Bob Prielipp, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin.

We begin by noting that

$$\tan\frac{A}{2} = \frac{r}{s-a} = \frac{rs}{s(s-a)} = \frac{K}{s(s-a)}$$

where r is the inradius of triangle ABC.

(a) Thus, by Heron's Formula and some algebra,

$$\begin{aligned}
\frac{K}{\tan\frac{A}{2}} + K\tan\frac{A}{2} &= s(s-a) + \frac{K^2}{s(s-a)} \\
&= s(s-a) + \frac{s(s-a)(s-b)(s-c)}{s(s-a)} \\
&= s(s-a) + (s-b)(s-c) = 2s^2 - s(a+b+c) + bc \\
&= 2s^2 - s(2s) + bc = bc.
\end{aligned}$$

(b) Also,

$$\frac{K}{s\tan\frac{A}{2}}+s=(s-a)+s=2s-a$$

$$=(a+b+c)-a=b+c.$$

Also solved by Russell Euler, Northwest Missouri State University, Maryville, Missouri; Herta T. Freitag, Roanoke, Virginia; Kandasamy Muthuvel, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin; Joseph B. Dence, University of Missouri-St. Louis, St. Louis, Missouri; Jayanthi Ganapathy, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin; Joe Howard, New Mexico Highlands University, Las Vegas, New Mexico; Donald P. Skow, University of Texas-Pan American, Edinburg, Texas; Joseph Wiener, University of Texas-Pan American, Edinburg, Texas; and the proposer.

82. [1995, 87] *Proposed by Curtis Cooper and Robert E. Kennedy, Central Missouri State University, Warrensburg, Missouri.*

Evaluate

$$\lim_{k\to\infty}\frac{\log\frac{10^{10^k}((10^{k-1})!)^{10}}{(10^k)!}}{k},$$

where $\log x$ denotes the base 10 logarithm of x.

Solution by Joseph B. Dence, University of Missouri-St. Louis, St. Louis, Missouri and the proposers.

Let $10^{k-1}=x$ and $c=1/\ln 10$. Then

$$\frac{\log\frac{10^{10^k}((10^{k-1})!)^{10}}{(10^k)!}}{k}=\frac{\log\frac{10^{10x}(x!)^{10}}{(10x)!}}{1+\log x}$$

$$=\frac{10x+10\log(x!)-\log((10x)!)}{1+\log x}$$

$$=\frac{10x+c(10\ln(x!)-\ln((10x)!))}{1+c\ln x}.$$

For large integral n we have [G. H. Hardy, *Divergent Series*, Oxford University Press, 1949, p. 334]

$$\ln(n!) = \left(n + \frac{1}{2}\right)\ln(n) - n + \frac{1}{2}\ln(2\pi) + O(n^{-1}).$$

Substitution in the above expression gives

$$\frac{10x + c\left(\frac{9}{2}\ln x + \frac{9}{2}\ln(2\pi) - (10x + \frac{1}{2})\ln 10\right) + O(x^{-1})}{1 + c\ln x}$$

and thus, by simplification and l'Hôpital's Theorem,

$$\begin{aligned}
\lim_{k\to\infty} \frac{\log \frac{10^{10^k}((10^{k-1})!)^{10}}{(10^k)!}}{k} &= \lim_{x\to\infty} \frac{\log \frac{10^{10x}(x!)^{10}}{(10x)!}}{1 + \log x} \\
&= \lim_{x\to\infty} \frac{10x + 10\log(x!) - \log((10x)!)}{1 + \log x} \\
&= \lim_{x\to\infty} \frac{\frac{9}{2}c\ln x + \frac{9}{2}c\ln 2\pi - \frac{1}{2}}{1 + c\ln x} \\
&= \frac{9}{2}c\lim_{x\to\infty}\left(x^{-1}/cx^{-1}\right) \\
&= \frac{9}{2}.
\end{aligned}$$

Also solved by Donald P. Skow, University of Texas-Pan American, Edinburg, Texas; Joe Howard, New Mexico Highlands University, Las Vegas, New Mexico; Joseph Wiener, University of Texas-Pan American, Edinburg, Texas; N. J. Kuenzi, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin; and Alan H. Rapoport, Ashford Medical Center, Santurce, Puerto Rico.

83. [1995, 88] *Proposed by Donald P. Skow, University of Texas-Pan American, Edinburg, Texas.*

(a) Let O_n denote the nth octagonal number. Prove that

$$O_nO_{n+2} + 2O_{n+1} - 1$$

is a perfect square.

(b) Let N_n denote the nth nonagonal number. Prove that

$$N_nN_{n+2} + N_{n+1} + 3$$

is a perfect square.

(c) Determine a nontrivial function of three consecutive heptagonal numbers which always produces a perfect square.

Solution to (a) and (b) by Bob Prielipp, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin; Russell Euler, Northwest Missouri State University, Maryville, Missouri; Joe Howard, New Mexico Highlands University, Las Vegas, New Mexico; Joseph B. Dence, University of Missouri-St. Louis, St. Louis, Missouri; Lawrence Somer, The Catholic University of America, Washington, D.C.; Gayla Singleton (student), Southeast Missouri State University, Cape Girardeau, Missouri; J. Sriskandarajah, University of Wisconsin Center-Richland, Richland Center, Wisconsin; Herta T. Freitag, Roanoke, Virginia; and the proposer.

It is known that the nth k-gonal number is given by

$$\frac{n}{2}\big(2 + (n-1)(k-2)\big).$$

Thus,

$$O_n = n(3n-2),$$

$$\text{and} \quad N_n = \frac{n(7n-5)}{2}.$$

Therefore,

$$O_nO_{n+2} + 2O_{n+1} - 1 = [n(3n-2)][(n+2)(3n+4)] + 2(n+1)(3n+1) - 1$$
$$= 9n^4 + 24n^3 + 10n^2 - 8n + 1 = (3n^2 + 4n - 1)^2$$

and

$$N_nN_{n+2} + N_{n+1} + 3 = \frac{n(7n-5)}{2}\frac{(n+2)(7n+9)}{2} + \frac{(n+1)(7n+2)}{2} + 3$$

$$= \frac{49n^4 + 126n^3 + 25n^2 - 72n + 16}{4} = \left(\frac{7n^2 + 9n - 4}{2}\right)^2.$$

If H_n denotes the nth heptagonal number, then

$$H_n = \frac{n(5n-3)}{2}.$$

Solution to (c) by Russell Euler, Northwest Missouri State University, Maryville, Missouri; Joseph B. Dence, University of Missouri-St. Louis, St. Louis, Missouri; Lawrence Somer, The Catholic University of America, Washington, D.C.; J. Sriskandarajah, University of Wisconsin Center, Richland Center, Wisconsin; Herta T. Freitag, Roanoke, Virginia; and the proposer.

$$H_nH_{n+2} + H_{n+1} = \left(\frac{5n^2 + 7n - 2}{2}\right)^2.$$

Note that $(5n^2 + 7n - 2)/2$ is an integer since

$$5n^2 + 7n - 2 \equiv n^2 + n \equiv n(n+1) \equiv 0 \pmod 2.$$

Solution to (c) by Joe Howard, New Mexico Highlands University, Las Vegas, New Mexico and Herta T. Freitag, Roanoke, Virginia.

$$H_nH_{n+2} + 3H_{n+1} - 3 = \left(\frac{5n^2 + 7n}{2}\right)^2.$$

Solution to (c) by Bob Prielipp, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin and Herta T. Freitag, Roanoke, Virginia.

$$H_nH_{n+2} - H_{n+1} + 5 = \left(\frac{5n^2 + 7n - 4}{2}\right)^2.$$

One incorrect solution to part (c) was also received.

Herta T. Freitag has generalized this problem. Her generalization can be found in this issue of the *Missouri Journal of Mathematical Sciences* in her article entitled "From the Legacy of Pythagoras."

84. [1995, 88] *Proposed by W. F. Wheatley and James Ethridge, Jackson, Mississippi.*

Let n be a positive integer.

(a) How many n-digit base 10 numbers are there whose digits from left-to-right are nondecreasing?

(b)* Consider a $2 \times n$ array with base 10 digits in each entry of the array. Suppose that the 2 rows form n-digit base 10 numbers whose digits from left-to-right are nondecreasing and that the n columns form 2-digit base 10 numbers whose digits from bottom-to-top are nondecreasing. How many such arrays are there?

Solution I to part (a) by Ronald K. Smith, Graceland College, Lamoni, Iowa.

The answer is

$$\binom{n+8}{n}.$$

There is a 1-1 correspondence between the sets A, B, and C where

$$\begin{aligned} A &= \{n - \text{digit base 10 numbers whose digits} \\ &\qquad \text{from left-to-right are nondecreasing}\}, \\ B &= \{(x_1, x_2, \dots, x_n) \mid 1 \le x_1 \le x_2 \le \cdots \le x_n \le 9\}, \\ C &= \{(y_1, y_2, \dots, y_{n+1}) \mid y_1 + y_2 + \cdots + y_{n+1} = 8, y_i \ge 0\}. \end{aligned}$$

To show that B is equivalent to C, set $y_1 = x_1 - 1$, $y_i = x_i - x_{i-1}$ for $i = 2, \dots, n$ and $y_{n+1} = 9 - x_n$. Then

$$\sum_{i=1}^{n+1} y_i = 8, \quad \text{and} \quad y_i \ge 0 \ \text{ for } i = 1, 2, \dots, n+1.$$

This is clearly reversible: $x_1 = 1 + y_1$, $x_i = x_{i-1} + y_i$ for $i = 2, 3, \ldots, n$. The number of elements in C is

$$\binom{n+8}{n}.$$

To see this, take a row of $n+8$ 1's in parentheses. Choose n of them and convert to ,'s. Replace the string in each of the resulting $n+1$ slots with the number of 1's there (with any empty strings being replaced by 0). Since there were 8 1's, the sum is clearly 8, and each slot is non-negative.

To count the number of elements in A directly, follow this algorithm. We will do an example with $n = 5$.

1. Put $n+8$ 1's in parentheses: (1111111111111)
2. Choose n of them to convert to ,'s: (,111,,111,11,)
3. Treat as $n+1$-tuple, replacing each string with the number of 1's: (0,3,0,3,2,0)
4. Convert to n digits: $d_1 = y_1 + 1$, $d_i = d_{i-1} + y_i$ for $i = 2, \ldots n$: (1,4,4,7,9)
5. Treat as an integer in A: 14479

To reverse the procedure

4'. Convert an element of A to digits: (1,4,4,7,9)
3'. Convert to $n+1$-tuple: $y_1 = d_1 - 1$, $y_i = d_i - d_{i-1}$ for $i = 2, \ldots, n$, $y_{n+1} = 9 - d_n$: (0,3,0,3,2,0)
2'. Replace numbers with strings of 1's: (,111,,111,11,)
1'. Replace n commas with 1's: (1111111111111)

Solution II to part (a) by N. J. Kuenzi, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin.

The solution is based upon the following identity.

$$\sum_{j=0}^{m} \binom{n-1+j}{n-1} = \binom{n+m}{n}.$$

For $d = 1, 2, \ldots, 9$, let $A(n,d)$ be the number of n-digit base 10 numbers satisfying the left-to-right nondecreasing condition and having right digit d. Note for $n \geq 2$ the right digit of such a number cannot be 0.

For $n = 1$,

$$A(1,d) = 1 = \binom{d}{0}.$$

For $n = 2$,

$$A(2, d) = d = \binom{d}{1}.$$

For $n \geq 2$, consider $A(n+1, d)$. With right digit d the n leftmost digits can be any of the n-digit numbers satisfying the left-to-right condition and having rightmost digit 1 through d. So,

$$A(n+1, d) = \sum_{j=1}^{d} A(n, j).$$

Using this summation, we have

$$A(3, d) = \sum_{j=1}^{d} A(2, j) = \sum_{j=1}^{d} \binom{j}{1} = \binom{d+1}{2}.$$

Suppose

$$A(n, d) = \binom{n-2+d}{n-1}$$

for $d = 1, 2, \ldots, 9$. Then

$$A(n+1, d) = \sum_{j=1}^{d} A(n, j) = \sum_{j=1}^{d} \binom{n-2+j}{n-1}$$

$$= \sum_{k=0}^{d-1} \binom{n-1+k}{n-1} = \binom{n-1+d}{n}.$$

So by the principle of mathematical induction

$$A(n, d) = \binom{n-2+d}{n-1}$$

for $n \geq 2$ and $d = 1, 2, \ldots, 9$. Now the number of n-digit base 10 numbers which satisfy the left-to-right nondecreasing condition is the same as the number of $(n+1)$-digit base 10 numbers satisfying the left-to-right nondecreasing condition and having right digit 9. Thus, the solution is given by

$$A(n+1, 9) = \binom{n+8}{n}.$$

Solution III to part (a) by N. J. Kuenzi, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin.

Any n-digit base 10 number whose digits from left-to-right are nondecreasing can be thought of as a string of x_1 1's, followed by a string of x_2 2's, etc. with

$$x_1 + x_2 + \cdots + x_9 = n$$

and $x_i \geq 0$, $i = 1, 2, \ldots, 9$. For example, the 7-digit number 2355778 has

$$x_1 = x_4 = x_6 = x_9 = 0, \quad x_2 = x_3 = x_8 = 1, \quad \text{and} \quad x_5 = x_7 = 2.$$

If

$$x_1 = 2, \quad x_2 = 3, \quad x_5 = x_7 = 1, \quad \text{and} \quad x_3 = x_4 = x_6 = x_8 = x_9 = 0,$$

the 7-digit number is 1122257.

The number of solutions in nonnegative integers to the equation

$$x_1 + x_2 + \cdots + x_9 = n$$

is

$$\binom{n+8}{n} = \binom{n+8}{8}.$$

(See Theorem 2, page 74, *Introduction to Combinatorics*, Berman and Fryer, Academic Press, 1972.) Hence, for $n \geq 2$, the number of n-digit base 10 numbers whose digits from left-to-right are nondecreasing is

$$\binom{n+8}{n}.$$

Similarly, the number of n-digit base b numbers whose digits from left-to-right are nondecreasing is

$$\binom{n+b-2}{n}.$$

Also solved by Alan H. Rapoport, Ashford Medical Center, Santurce, Puerto Rico and the proposers.

Comment by N. J. Kuenzi, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin.

Suppose the nondecreasing condition is changed to nonincreasing. Then any n-digit base 10 number whose digits from left-to-right are nonincreasing can be thought of as a string of x_9 9's, followed by a string of x_8 8's, etc. with

$$x_0 + x_1 + \cdots + x_9 = n$$

and $x_i \geq 0$, $i = 0, 1, \ldots, 9$. The number of nonnegative integer solutions to the equation

$$x_0 + x_1 + \cdots + x_9 = n$$

is

$$\binom{n+9}{n} = \binom{n+9}{9}.$$

For $n \geq 2$, only the solution $x_0 = n$ does not correspond to an n-digit number satisfying the nonincreasing condition. Hence, for $n \geq 2$, the number of n-digit base 10 numbers whose digits from left-to-right satisfy the nonincreasing condition is

$$\binom{n+9}{n} - 1.$$

Similarly, for $n \geq 2$ the number of n-digit base b numbers whose digits from left-to-right satisfy the nonincreasing condition is

$$\binom{n+b-1}{n} - 1.$$

Comment on part (b) by Alan H. Rapoport, Ashford Medical Center, Santurce, Puerto Rico. I have found by brute force that the number of 2×1 arrays satisfying the condition in part (b) is 45, the number of 2×2 arrays is 825, and the number of 2×3 arrays is 9075. This suggests the following conjecture. The total number of $2 \times n$ arrays satisfying the condition in part (b) is

$$\frac{(8+n)!}{8!n!} \cdot \frac{(9+n)!}{9!(n+1)!}.$$

Furthermore, I would like to offer the following generalization which is a pure guess. The number of $2 \times n$ arrays satisfying the condition in part (b) is

$$\frac{(8+n)!}{8!n!} \cdot \frac{(9+n)!}{9!(n+1)!} \cdot 1!.$$

The number of $3 \times n$ arrays satisfying a similar condition to part (b) is

$$\frac{(8+n)!}{8!n!} \cdot \frac{(9+n)!}{9!(n+1)!} \cdot \frac{(10+n)!}{10!(n+2)!} \cdot 1! \cdot 2!.$$

The number of $4 \times n$ arrays satisfying a similar condition to part (b) is

$$\frac{(8+n)!}{8!n!} \cdot \frac{(9+n)!}{9!(n+1)!} \cdot \frac{(10+n)!}{10!(n+2)!} \cdot \frac{(11+n)!}{11!(n+3)!} \cdot 1! \cdot 2! \cdot 3!.$$

And so on.