PARABOLAS IN TAXICAB GEOMETRY

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1. Introduction. Reynolds [1] raised some open questions concerning the definition of taxicab parabolas. Moser, Kramer [2], and Iny [3] looked into these questions but did not answer all of them. We would like to provide an analysis to the solutions with more details in this paper.

Let's first take a look at the definition of taxicab metric.

Definition 1.1. Let $d_T : R^2 \times R^2 \to [0, \infty)$

be defined as

$$d_T((a_1, a_2), (b_1, b_2)) = |a_1 - b_1| + |a_2 - b_2|.$$

Then (d_T, R^2) forms a *metric space* and d_T is called the *taxicab metric* on R^2.

In R^2, there are several different but equivalent ways to define straight lines using Euclidean metric. However, Chen [4] showed that these ways are no longer equivalent when Euclidean metric is replaced by taxicab metric. Liu [5] used the idea of a line being the bisector of two points to define the taxicab bisector (or taxicab line).

Definition 1.2 Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be two points in R^2. Then the set

$$D = \{(x, y) | |x - x_1| + |y - y_1| = |x - x_2| + |y - y_2|\}$$

is called the *taxicab bisector* (or the *taxicab line*) of the points P_1 and P_2.

Since a parabola in Euclidean geometry is the set of the points which are equidistant to a fixed point; called the focus, and a fixed line; called the directrix, we need to discuss the distance between a point and a taxicab bisector before defining the taxicab parabola.

Theorem 1.1.[1] The shortest taxicab distance from a point to a Euclidean line is either the horizontal distance or the vertical distance whichever is smaller.

Liu [5] showed that there are three types of taxicab bisectors determined by the relation between the difference of x-coordinates and that of y-coordinates of P_1 and P_2. Two of them are Euclidean line or the union of a Euclidean line segment and two rays, while the third one is the union of a Euclidean line segment and regions enclosed by Euclidean rays. So Theorem 1.1 has helped us in defining $d_T(P, D)$ as the shortest taxicab distance from a point P to a bisector D.

Definition 1.3. Let D be a taxicab bisector and F be a point in R^2 such that $F \notin D$. The set $M = \{P \in R^2 | d_T(P, F) = d_T(P, D)\}$ is called a *taxicab parabola* with *focus* F and *directrix* D.

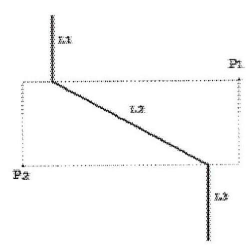
Throughout this paper, we denote a taxicab parabola by M with focus $F(f_1, f_2)$ and directrix D which is the taxicab bisector of two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$.

2. Tri-line Directrix. The first type of bisectors has three components. We are going to show the results with the relation between the two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ which determine the directrix of a parabola such that $x_2 < x_1$, $y_2 < y_1$, and $|x_1 - x_2| > |y_1 - y_2|$. Liu [5] showed that the bisector of $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ with such relation is $D = L_1 \cup L_2 \cup L_3$ where

$$L_1 = \{(x, y) | x = \frac{1}{2}(y_2 - y_1 + x_1 + x_2), y \geq y_1\},$$

$$L_2 = \{(x, y) | y = -x + \frac{1}{2}(y_2 + y_1 + x_1 + x_2), y_1 \leq y \leq y_2\}, \text{ and}$$

$$L_3 = \{(x, y) | x = \frac{1}{2}(y_1 - y_2 + x_1 + x_2), y \leq y_2\}.$$



Although we obtain the different graphs of M according to the location of its focus, F and directrix, D. The proofs among the cases are similar. So we will just provide one of them. Also, we will only show the results with F to the right of D.

Those cases with F to the left of D are not discussed here for the results are just the opposite of what we have here.

With the assumption, $x_2 < x_1$, $y_2 < y_1$ and $|x_1 - x_2| > |y_1 - y_2|$, we have the following four cases.

Case 2.1. Let $f_2 \geq y_1$. Then we have three subcases.

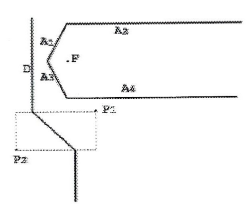
(A) If $y_1 \leq f_2 - f_1 + \frac{1}{2}(y_2 - y_1 + x_1 + x_2)$ and let $\alpha = \frac{1}{2}(f_1 + \frac{1}{2}(y_2 - y_1 + x_1 + x_2))$, then $M = \bigcup_{i=1}^{4} A_i$ where

$$A_1 = \{(x,y)|y = 2x + f_2 - f_1 - \frac{1}{2}(y_2 - y_1 + x_1 + x_2), \alpha \leq x \leq f_1\},$$

$$A_2 = \{(x,y)|y = f_2 + f_1 - \frac{1}{2}(y_2 - y_1 + x_1 + x_2), x \geq f_1\},$$

$$A_3 = \{(x,y)|y = -2x + f_2 + f_1 + \frac{1}{2}(y_2 - y_1 + x_1 + x_2), \alpha \leq x \leq f_1\}, \text{ and}$$

$$A_4 = \{(x,y)|y = f_2 - f_1 + \frac{1}{2}(y_2 - y_1 + x_1 + x_2), x \geq f_1\}.$$



Proof of (A). Let $x_2 < x_1$, $y_2 < y_1$, $|x_1 - x_2| > |y_1 - y_2|$, and $f_2 \geq y_1$. Since F is to the right of D, $f_1 > \frac{1}{2}(y_2 - y_1 + x_1 + x_2)$ implies $f_2 - f_1 + \frac{1}{2}(y_2 - y_1 + x_1 + x_2) < f_2$. For any $(x,y) \in R^2$, if it is on the left side of D, then $d_T((x,y), D) < d_T((x,y), F)$. Hence, all we need to check are those points on the right side of D. From the assumption $y_1 \leq f_2 - f_1 + \frac{1}{2}(y_2 - y_1 + x_1 + x_2) < f_2$, any $(x, y) \in R^2$ on the right side of D along with $y < f_2 - f_1 + \frac{1}{2}(y_2 - y_1 + x_1 + x_2)$ has the following result:

$$d_T((x,y),F) > |x - f_1| + f_2 - (f_2 - f_1 + \frac{1}{2}(y_2 - y_1 + x_1 + x_2))$$

$$= |x - f_1| + f_1 - \frac{1}{2}(y_2 - y_1 + x_1 + x_2) > d_T((x,y),D).$$

This implies that $(x, y) \notin M$.

Also if $y \geq y_1$, then $d_T((x, y), D) = |x - \frac{1}{2}(y_2 - y_1 + x_1 + x_2)|$.

(i) If $\frac{1}{2}(y_2 - y_1 + x_1 + x_2) \leq x \leq f_1$ and $y \geq f_2$, then

$$|x - f_1| + |y - f_2| = |x - \frac{1}{2}(y_2 - y_1 + x_1 + x_2)|$$

$$\Longrightarrow f_1 - x + y - f_2 = x - \frac{1}{2}(y_2 - y_1 + x_1 + x_2)$$

$$\Longrightarrow y = 2x - f_1 + f_2 - \frac{1}{2}(y_2 - y_1 + x_1 + x_2).$$

In particular, when $y = f_2$, we have $f_2 = 2x - f_1 + f_2 - \frac{1}{2}(y_2 - y_1 + x_1 + x_2)$ which implies $x = \frac{1}{2}(f_1 + \frac{1}{2}(y_2 - y_1 + x_1 + x_2)) = \alpha$.

(ii) If $x \geq f_1$ and $y \geq f_2$, then

$$|x - f_1| + |y - f_2| = |x - \frac{1}{2}(y_2 - y_1 + x_1 + x_2)|$$

$$\Longrightarrow x - f_1 + y - f_2 = x - \frac{1}{2}(y_2 - y_1 + x_1 + x_2)$$

$$\Longrightarrow y = f_1 + f_2 - \frac{1}{2}(y_2 - y_1 + x_1 + x_2).$$

(iii) If $\frac{1}{2}(y_2-y_1+x_1+x_2) \leq x \leq f_1$ and $f_2-f_1+\frac{1}{2}(y_2-y_1+x_1+x_2) \leq y \leq f_2$, then

$$|x-f_1|+|y-f_2| = |x-\frac{1}{2}(y_2-y_1+x_1+x_2)|$$

$$\Longrightarrow f_1-x+f_2-y = x-\frac{1}{2}(y_2-y_1+x_1+x_2)$$

$$\Longrightarrow y = -2x+f_1+f_2+\frac{1}{2}(y_2-y_1+x_1+x_2).$$

Again when $y=f_2$, $x=\alpha$.

(iv) If $x \geq f_1$ and $f_2-f_1+\frac{1}{2}(y_2-y_1+x_1+x_2) \leq y \leq f_2$, then

$$|x-f_1|+|y-f_2| = |x-\frac{1}{2}(y_2-y_1+x_1+x_2)|$$

$$\Longrightarrow x-f_1+f_2-y = x-\frac{1}{2}(y_2-y_1+x_1+x_2)$$

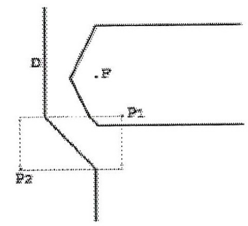
$$\Longrightarrow y = -f_1+f_2+\frac{1}{2}(y_2-y_1+x_1+x_2).$$

For any other $(x,y) \in R^2$, $d_T((x,y),F) = d_T((x,y),D)$ will lead us to an empty set.

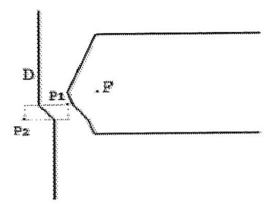
Therefore, $M = \bigcup\limits_{i=1}^{4} A_i$.

For each of the remaining subcases, we are going to show only the graph of M with its hypotheses.

(B) If $y_2 < \frac{1}{2}(f_2-f_1+\frac{1}{2}(y_2+y_1+x_1+x_2)) < y_1$, then the graph of M is



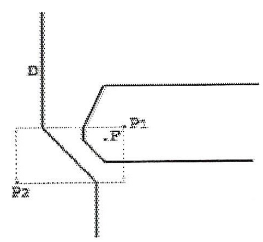
(C) If $\frac{1}{2}(f_2 - f_1 + \frac{1}{2}(y_2 + y_1 + x_1 + x_2)) \leq y_2 < y_1$, then the graph of M is



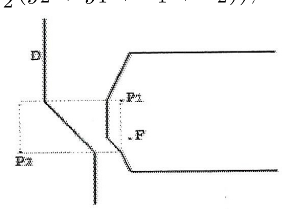
Similar to the subcases of Case 2.1, only the graph of M and its hypotheses are presented in each of the following cases and their subcases.

Case 2.2. Let $y_2 < f_2 < y_1$.

(A) If $y_2 \leq \frac{1}{2}(f_2 - f_1 + \frac{1}{2}(y_2 + y_1 + x_1 + x_2))$, then the graph of M is

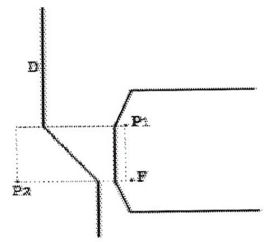


(B) If $y_2 > \frac{1}{2}(f_2 - f_1 + \frac{1}{2}(y_2 + y_1 + x_1 + x_2))$, then the graph of M is



Case 2.3. Let $y_2 = f_2$.

(A) If $y_1 < f_2 + f_1 - \frac{1}{2}(y_2 - y_1 + x_1 + x_2)$, then the graph of M is



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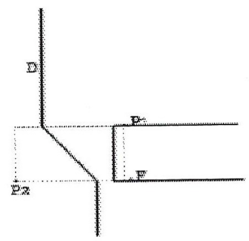
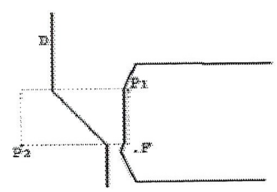
(B) If $y_1 = f_2 + f_1 - \frac{1}{2}(y_2 - y_1 + x_1 + x_2)$, then the graph of M is



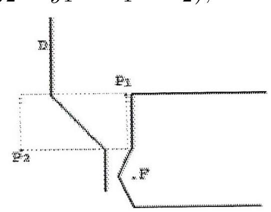
In both Case 2.2 and Case 2.3, if $y_1 > f_2 + f_1 - \frac{1}{2}(y_2 - y_1 + x_1 + x_2)$, then F has to be on the left side of D which is contradictory to our assumption.

Case 2.4. Let $f_2 < y_2$.

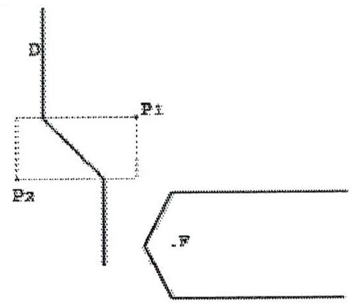
(A) If $y_1 < f_2 + f_1 - \frac{1}{2}(y_2 - y_1 + x_1 + x_2)$, then the graph of M is



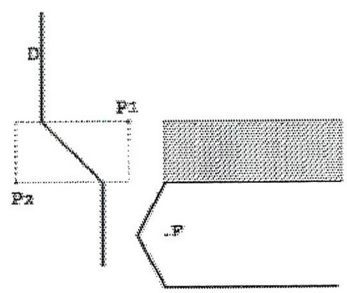
(B) If $y_1 = f_2 + f_1 - \frac{1}{2}(y_2 - y_1 + x_1 + x_2)$, then the graph of M is



(C) If $y_2 > f_2 + f_1 - \frac{1}{2}(y_2 - y_1 + x_1 + x_2)$, then the graph of M is

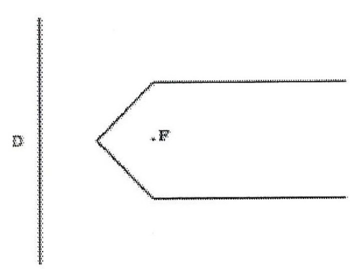


(D) If $y_2 = f_2 + f_1 - \frac{1}{2}(y_1 - y_2 + x_1 + x_2)$, then the graph of M is

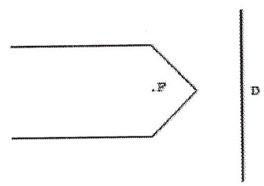


3. One-line Directrix. The second type of taxicab bisector is either a horizontal line or a vertical line. In the first two cases, D is the taxicab bisector of $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ where $y_1 = y_2$ while the last two cases, $x_1 = x_2$. Again, we will only present the assumptions and the graphs.

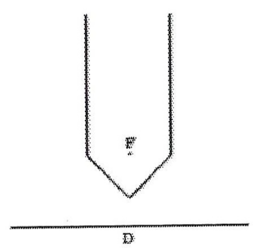
Case 3.1. If $D = \{(x, y)|x = \frac{1}{2}(x_1 + x_2)\}$ and $f_1 > \frac{1}{2}(x_1 + x_2)$, then the graph of M is



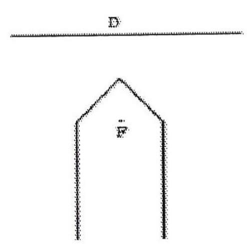
Case 3.2. If $D = \{(x, y)|x = \frac{1}{2}(x_1 + x_2)\}$ and $f_1 < \frac{1}{2}(x_1 + x_2)$, then the graph of M is



Case 3.3. If $D = \{(x, y)|y = \frac{1}{2}(y_1 + y_2)\}$ and $f_2 > \frac{1}{2}(y_1 + y_2)$, then the graph of M is



Case 3.4. If $D = \{(x, y)|y = \frac{1}{2}(y_1 + y_2)\}$ and $f_2 < \frac{1}{2}(y_1 + y_2)$, then the graph of M is



The same mechanics can be used for the discussion of the case that $|x_1 - x_2| > |y_1 - y_2|$, $x_2 < x_1$, and $y_2 > y_1$, and the case $|x_1 - x_2| < |y_1 - y_2|$. When

$|x_1 - x_2| = |y_1 - y_2|$, the directrix D is more complicated than the previous cases, thus it will be saved for the sequel of this paper.

References

1. B. Reynolds, "Taxicab Geometry" *Pi Mu Epsilon Journal*, 7 (1980), 77–88.
2. J. Moser and J. Kramer, "Lines and Parabolas in Taxicab Geometry," *Pi Mu Epsilon Journal*, 7 (1982), 441–444.
3. D. Iny, "Geometry: Another Look at Conic Sections," *Pi Mu Epsilon Journal*, 7 (1984), 645–647.
4. G. Chen, *Lines and Circles in Taxicab Geometry*, CMSU Thesis (1992).
5. Y. Liu, *Conic Sections in Taxicab Geometry*, CMSU Preprint.