MIXED INSURANCE RISK MODELS

Liansheng Chen and Jinhua Tao

1. Introduction. Traditionally, the distribution of aggregate claims of a portfolio has been a central topic in risk theory. The discussion has focused on two problems: the choice of the distribution and its numerical evaluation by means of an approximation. Consider the collective risk model where the aggregate claim random variable S for a portfolio of insurance policies over a fixed period can be expressed as

$$S = X_1 + X_2 + \cdots + X_N.$$

In this model, $X_1, X_2, \cdots$ are claim size random variables and N is the claim frequency random variable. The case of when N is a simple counting random variable with fixed parameters has been extensively studied. In this paper we generalize the above model to a mixed risk model where the random variable N depends on another random risk parameter. The parameter itself is assumed to be distributed over the population of risks under consideration in accordance with some distribution. There are several situations in which this might be a useful way to consider the distribution of N. For example, consider a population of insureds where various classes of insureds within the population generate numbers of claims according to different distributions.

2. Probability Distribution of S. Let $S = X_1 + \cdots + X_N$, where N is a counting random variable (r.v.) whose distribution depends on some random parameter Λ. In order to make the model mathematically attractive, we assume the following conditions.

1) $X_1, X_2, \cdots$ are identically distributed with common cumulative distribution function (c.d.f.) $F_X(x)$.

2) The random variables $N, X_1, X_2, \cdots$ are mutually independent (this implies that $\Lambda, X_1, X_2, \cdots$ are mutually independent).

We further denote the conditional probability density function (p.d.f.) of non-negative r.v. N given that $\Lambda = \lambda$ by $f_{N|\lambda}(n)$, for $n = 0, 1, 2, \cdots$. Also, we denote the c.d.f. of Λ by $F_\Lambda(\lambda)$. Using the law of total probability we can get the following propositions about the distribution, expectations, and moment generating function of S.

Proposition 2.1. The c.d.f of S can be written as

$$F_S(x) = P(S \leq x) = \int \sum_{n=0}^{\infty} F_X^{*n}(x) f_{N|\lambda}(n) dF_\Lambda(\lambda).$$

Proof.

$$P(S \leq x) = P(X_1 + \cdots + X_N \leq x)$$

$$= \int \sum_{n=0}^{\infty} P(X_1 + \cdots + X_N \leq x | N = n \,\&\, \Lambda = \lambda) f_{N|\lambda}(n) dF_\Lambda(\lambda)$$

where the integral is the Riemann-Stieltjes integral. Note that for each fixed λ and n, $P(X_1 + \cdots + X_N \leq x) = F_X^{*n}(x)$, the nth convolution of the c.d.f. of X_1. Therefore the result holds.

Proposition 2.2.

$$E(S) = E_\Lambda[E(N|\Lambda)] \cdot E(X_1).$$

Proof. Since

$$\begin{aligned} E(S) &= E(X_1 + \cdots + X_N) \\ &= E_\Lambda\{E_N[E(X_1 + \cdots + X_N|N)|\Lambda]\} \\ &= E_\Lambda\{E_N[N \cdot E(X_1)|\Lambda]\} \\ &= E_\Lambda[E(N|\Lambda)] \cdot E(X_1). \end{aligned}$$

Note that $E_\Lambda[E(N|\Lambda)]$ represents the mean claim number over the entire population and $E(N|\lambda)$ is the mean claim number over the population class of risk component λ.

Proposition 2.3.

$$Var(S) = E_\Lambda[E(N|\Lambda)] \cdot Var(X_1) + \{E_\Lambda[Var(N|\Lambda)] + Var_\Lambda[E(N|\Lambda)]\} \cdot [E(X_1)]^2$$

Proof. Using the law of total probability, the independence condition of X_i's, and the definition of variance: $Var(S) = E(S^2) - [E(S)]^2$, it is easy to prove that

1)

$$\begin{aligned} Var(N) &= E(N^2) - [E(N)]^2 \\ &= E_\Lambda[E(N^2|\Lambda)] - [E_\Lambda(E(N|\Lambda))]^2 \\ &= E_\Lambda[E(N^2|\Lambda)] - E_\Lambda[E(N|\Lambda)]^2 + E_\Lambda[E(N|\Lambda)]^2 - [E_\Lambda(E(N|\Lambda))]^2 \\ &= E_\Lambda[Var(N|\Lambda)] + Var_\Lambda[E(N|\Lambda)]. \end{aligned}$$

2) Similarly,

$$\begin{aligned} Var(S) &= E_N[Var(X_1 + \cdots + X_N|N)] + Var_N[E(X_1 + \cdots + X_N|N)] \\ &= E_N[N \cdot Var(X_1)] + Var_N[E(N \cdot X_1)] \\ &= E(N) \cdot Var(X_1) + Var(N) \cdot [E(X_1)]^2. \end{aligned}$$

The result of Proposition 2.3 follows from 1) and 2).

Proposition 2.4.

$$M_S(t) = E_\Lambda[M_{N|\Lambda}(\log M_{X_1}(t))]$$

where $M_S(t)$ and $M_{X_1}(t)$ are the moment generating functions (MGF) for random variables S and X_1 respectively, and $M_{N|\Lambda}(t)$ is the MGF for the conditional random variable N given Λ.

Proof. By the definition of MGF and the independence condition of X_i's,

$$\begin{aligned} M_S(t) &= E(e^{tS}) \\ &= E_\Lambda\{E_N[E(e^{t(X_1+\cdots+X_N)}|N)|\Lambda]\} \\ &= E_\Lambda\{E_N[(M_{X_1}(t))^N|\Lambda]\} \\ &= E_\Lambda\{E_N[exp(N \log M_{X_1}(t))|\Lambda]\} \\ &= E_\Lambda[M_{N|\Lambda}(\log M_{X_1}(t))]. \end{aligned}$$

3. Mixed Compound Poisson Model. In practice, people usually choose the Binomial, Poisson or Negative Binomial random variables to model the counting random variable N. Because of the special structure and the mathematical simplicities of the expectations of Poisson distribution, not only do Propositions 2.1–2.4 have simpler forms but also the distribution of S has a special recursive property. In particular, we will study the mixed compound Poisson model in this section, i.e., N is a Poisson random variable with a random mean Λ whose p.d.f. is denoted by $f_\Lambda(\lambda)$.

a) Distribution function:

$$F_S(x) = \int \sum_{n=0}^{\infty} F_X^{*n}(x) \cdot \frac{e^{-\lambda}\lambda^n}{n!} f_\Lambda(\lambda) d\lambda$$

b) Expectation: $E(S) = E(\Lambda) \cdot E(X_1)$

c) Variance: $Var(S) = E(\Lambda) \cdot E(X_1^2) + Var(\Lambda) \cdot [E(X_1)]^2$

d) Moment generating function:

$$M_S(t) = E_\Lambda[e^{\Lambda(M_{X_1}(t)-1)}] = M_\Lambda(M_{X_1}(t) - 1)$$

e) Recursion formula for p.d.f. of S: Usually, the calculation of convolutions involved in 1) is tedious. However, Harry Panjer [6] has derived a recursion formula for the distribution of the compound Poisson model (λ is a fixed parameter) when the claim size (X_1) distribution is on the positive integers. This formula can reduce computational time dramatically when a large number of claims are expected to occur. For completeness, we present both the theorem and its proof here.

Theorem 1. The compound Poisson distribution with the claim size distribution defined on the positive integers with probability function $f_X(x)$, $x = 1, 2, 3, \cdots$ satisfies

$$f_S(x) = \frac{\lambda}{x} \sum_{k=1}^{x} k f_X(k) f_S(x-k), \ \ x = 1, 2, 3, \cdots$$

with $f_S(0) = e^{-\lambda}$.

Proof. For the Poisson distribution,

$$nP\{N=n\} = \lambda P\{N=n-1\},\ n=1,2,3,\cdots .$$

Multiplying each side by $[M_X(-z)]^{n-1}M'_X(-z)$, (note that $M_X(-z)$ is actually the Laplace transform of p.d.f. of X) and summing over n yields

$$\sum_{n=1}^{\infty} nP\{N=n\}[M_X(-z)]^{n-1}M'_X(-z)$$

$$= \lambda \sum_{n=1}^{\infty} P\{N=n-1\}[M_X(-z)]^{n-1}M'_X(-z).$$

Recognizing that $M_S(-z) = \sum_{n=0}^{\infty} P\{N=n\}[M_X(-z)]^n$, the above equation can be written as

$$M'_S(-z) = \lambda M'_X(-z)M_S(-z).$$

Taking the inverse Laplace transform and using the fact that X is defined only on positive integers yields

$$xf_S(x) = \lambda \sum_{k=1}^{x} kf_X(k)f_S(x-k),$$

and the proof is complete.

Applying this theorem to each risk class in the population and using the law of total probability, we have the following corollary.

Corollary. Let S be a finite mixture of the compound Poisson model (i.e., $f_\Lambda(\lambda)$ is defined only on m values $\lambda_1,\cdots,\lambda_m$). Then when the claim size distribution is defined on the positive integers, the distribution of S can be computed recursively as follows:

$$f_S(x) = \sum_{i=1}^{m} f_{S|\lambda_i}(x)f_\Lambda(\lambda_i),\ \ x=1,2,3,\cdots$$

and

$$f_{S|\lambda_i}(x) = \frac{\lambda_i}{x}\sum_{k=1}^{x} k f_X(k) f_{S|\lambda_i}(x-k),$$

with $f_{S|\lambda_i}(0) = e^{-\lambda_i}\ \ i = 1, \cdots, m.$
f) Relation between mixed compound Poisson model and compound negative binomial model.

The following theorem states that for some special mixing functions the mixed compound Poisson model can be viewed as a compound risk model. Therefore, all the results for the compound model apply to this particular mixed model.

Theorem 2. For the mixture of a compound Poisson model, if Λ is a random variable with Gamma distribution, then the aggregate claim random variable S has a compound negative binomial distribution.

Proof. Recall that a Gamma random variable with parameters α and β has a p.d.f.

$$f(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)}\lambda^{\alpha-1}e^{-\beta\lambda},\ \ \lambda > 0$$

where $\Gamma(\cdot)$ is the Gamma function, and a MGF

$$M_\Lambda(t) = \left(\frac{\beta}{\beta - t}\right)^\alpha,\ \ t < \beta.$$

Also recall that a negative binomial random variable N, with parameters p and r, has a MGF

$$M_N(t) = \left(\frac{p}{1 - qe^t}\right)^r$$

where $q = 1 - p$.

Since the MGF uniquely determines the distribution, we only need to show that the MGF of the mixed Poisson with Gamma mixing is equal to the MGF of a compound negative binomial.

From part d) of this section and the MGF of Gamma distribution we have

$$M_S(t) = M_\Lambda(M_{X_1}(t) - 1)$$

$$= \left(\frac{\beta}{\beta - M_{X_1}(t) + 1}\right)^\alpha$$

$$= \left(\frac{p}{1 - qM_{X_1}(t)}\right)^r,$$

with $p = \frac{\beta}{\beta+1}$, $q = \frac{1}{\beta+1}$ and $r = \alpha$.

It is clear to see that the last expression is the MGF for the compound negative binomial model with parameters p, q and r.

References

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Jinhua Tao
Department of Mathematics and Computer Science
Central Missouri State University
Warrensburg, MO 64093-5045
email: tao@cmsuvmb.cmsu.edu

Liansheng Chen
Department of Mathematics and Computer Science
Central Missouri State University
Warrensburg, MO 64093-5045