SOLUTIONS

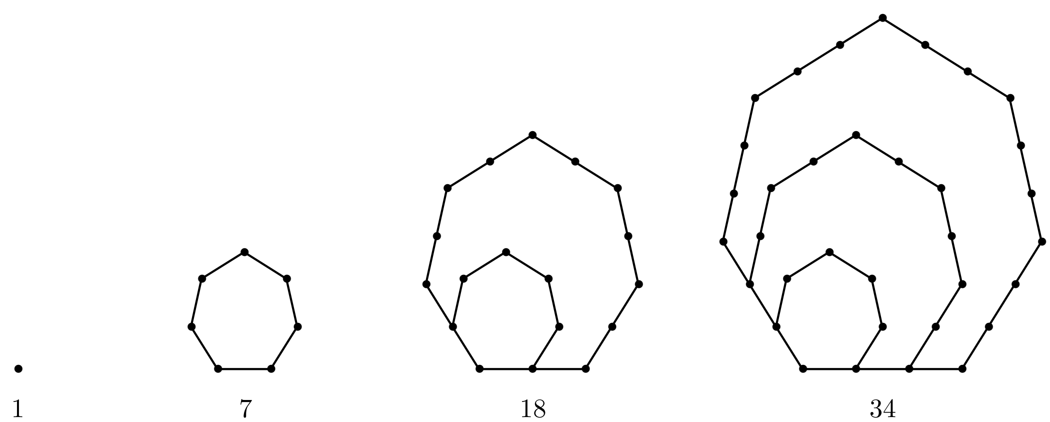
No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the problem editor.

67. [1994, 47; 1995, 43] *Proposed by Mohammad K. Azarian, University of Evansville, Evansville, Indiana.*

Show that one more than four times the product of two consecutive even or odd numbered triangular numbers is a square.

Comment by Joseph B. Dence, University of Missouri-St. Louis, St. Louis, Missouri.

Let $\{P(2n+1,k)\}_{k=1}^{\infty}$ be the polygonal numbers of order $2n+1$. For example, $\{P(3,k)\}_{k=1}^{\infty}$ are the triangular numbers: $P(3,1)=T_1=1$, $P(3,2)=T_2=3$, $P(3,3)=T_3=6, \ldots$. Similarly, the first few heptagonal numbers are $P(7,1)=1$, $P(7,2)=7$, $P(7,3)=18$, $P(7,4)=34, \ldots$.



By induction, $P(7,k)=k(5k-3)/2$, and in general any polygonal number of odd order is given explicitly by

$$P(2n+1,k)=\frac{k}{2}[(2n-1)k+(3-2n)], \quad \text{for} \quad n \geq 1.$$

After some algebra, we find

$$P(2n+1,k)P(2n+1,k+2)$$

$$=\frac{1}{4}\left([k[(2n-1)(k+2)+(3-2n)]]^2+2(3-2n)k[(2n-1)(k+2)+(3-2n)]\right),$$

and upon completing the square,

$$4P(2n+1,k)P(2n+1,k+2)+(3-2n)^2 = (k[(2n-1)(k+2)+(3-2n)]+(3-2n))^2 .$$

This is a generalization of Problem 67. For example, with the nonagonal numbers it yields

$$4P(9,k)P(9,k+2)+25 = (7k^2+9k-5)^2.$$

The constant term on the left-hand side is to be replaced by 1 in the analogous equations only in the cases of $n = 1$ (triangular numbers) and $n = 2$ (pentagonal numbers).

For polygonal numbers of even order (tetragonal numbers, hexagonal numbers, octagonal numbers, etc.), analogous reasoning to the above yields the explicit representation

$$P(2n,k) = k[(n-1)k-(n-2)],$$

and for the analogous product formula

$$P(2n,k)P(2n,k+2)+(n-2)^2 = (k[(n-1)(k+2)-(n-2)]-(n-2))^2 .$$

For example, with the octagonal numbers it yields

$$P(8,k)P(8,k+2)+4 = (3k^2+4k-2)^2.$$

Thus, Problem 67 is also extendable to polygonal numbers of even order.

Comment by Bob Prielipp, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin.

The following theorem is related to Problem 67.

Theorem. The product of four consecutive terms of an arithmetic progression of integers plus the fourth power of its common difference is always a perfect square.

Proof.

$$\begin{aligned} n(n+d)(n+2d)(n+3d)+d^4 &= [n(n+3d)][(n+d)(n+2d)]+d^4 \\ &= (n^2+3nd)[(n^2+3nd)+2d^2]+d^4 \\ &= (n^2+3nd)^2+2(n^2+3nd)d^2+(d^2)^2 \\ &= [(n^2+3nd)+d^2]^2 \\ &= (n^2+3nd+d^2)^2. \end{aligned}$$

73. [1994, 160; 1995, 39] *Proposed by Herta T. Freitag, Roanoke, Virginia.*

Let F_n and L_n denote the nth Fibonacci and Lucas numbers, respectively. Consider a right triangle such that the diameter of its circumcircle equals F_n and for its leg a, $a^2 = (L_{2n-1} + (-1)^n)/5$. Let p and q denote the segments formed on the hypotenuse by the footpoint of the height and let K denote the area of the triangle.

(a) Prove that the measures of p, q, and c (the hypotenuse) are Fibonacci numbers.

(b) Prove that the squares of a, b, and $2K$ are products of Fibonacci numbers.

Solution by Bob Prielipp, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin; Russell Euler, Northwest Missouri State University, Maryville, Missouri; Lawrence Somer, The Catholic University of America, Washington, D.C.; Joseph B. Dence, University of Missouri-St. Louis, St. Louis, Missouri; and the proposer.

We begin by establishing the following preliminary result.

Lemma. For each positive integer n,

$$L_{2n-1} + (-1)^n = 5F_nF_{n-1}.$$

Proof. Let $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$. Then,

$$F_k = \frac{\alpha^k - \beta^k}{\sqrt{5}}, \quad L_k = \alpha^k + \beta^k, \quad \alpha\beta = -1, \text{ and } \alpha + \beta = 1$$

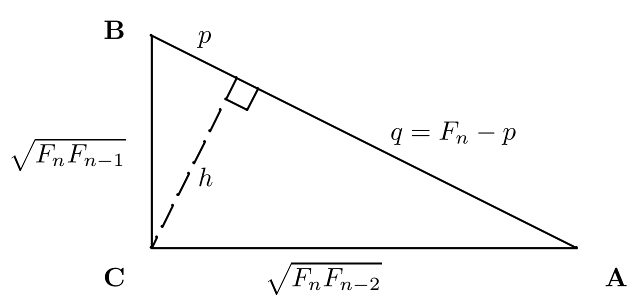
(where k is an arbitrary non-negative integer). Thus,

$$5F_nF_{n-1} = (\alpha^n - \beta^n)(\alpha^{n-1} - \beta^{n-1}) = (\alpha^{2n-1} + \beta^{2n-1}) - (\alpha\beta)^{n-1}(\alpha+\beta) = L_{2n-1} + (-1)^n.$$

From our Lemma and the hypothesis that $a^2 = (L_{2n-1} + (-1)^n)/5$, $a^2 = F_nF_{n-1}$ so $a = \sqrt{F_nF_{n-1}}$. Since the midpoint of the hypotenuse of a right triangle is the circumcenter of the triangle and the diameter of the circumcircle of the triangle if F_n, the measure of the hypotenuse c is F_n. From the Pythagorean Theorem

$$b^2 = c^2 - a^2 = F_n^2 - F_nF_{n-1} = F_n(F_n - F_{n-1}) = F_nF_{n-2},$$

making $b = \sqrt{F_nF_{n-2}}$. We may now construct the following diagram.



By the Pythagorean Theorem $F_nF_{n-1} - p^2 = h^2 = F_nF_{n-2} - (F_n - p)^2$. Thus

$$\begin{aligned}
F_nF_{n-1} - F_nF_{n-2} &= 2F_np - F_n^2\\
F_{n-1} - F_{n-2} &= 2p - F_n\\
(F_n - F_{n-2}) + F_{n-1} &= 2p\\
2F_{n-1} &= 2p\\
p &= F_{n-1}.
\end{aligned}$$

Hence, $q = F_n - p = F_n - F_{n-1} = F_{n-2}$ and $(2K)^2 = 4K^2 = 4(\frac{1}{2}ab)^2 = a^2b^2 = (F_nF_{n-1})(F_nF_{n-2}) = F_n^2F_{n-1}F_{n-2}$.

74. [1994, 160] *Proposed by Curtis Cooper and Robert E. Kennedy, Central Missouri State University, Warrensburg, Missouri.*

Let $s(n)$ denote the digital sum (base 10) of the positive integer n. Prove that if n is divisible by 41, then $s(n) \geq 5$.

Solution by Joseph B. Dence, University of Missouri-St. Louis, St. Louis, Missouri.

First, we note that 1/41 has a period length of 5; the possible remainders when 41 is divided into powers of 10 are the elements of the modulo 41 multiplicative group $G = \{1, 10, 16, 18, 37\}$.

Let $n = 41k$; if $k \equiv 5, 6, 7, 8$, or $9 \pmod{10}$, the theorem is automatic. If $k \equiv 4 \pmod{10}$, then $41k$ must contain (besides 4) at least one other nonzero digit, so $s(41k) \geq 5$ again holds. Since $(10, 41) = 1$, it is not necessary to consider the case $k \equiv 0 \pmod{10}$.

Now suppose the theorem fails for some integer $41k$, where $k \equiv 1, 2$, or $3 \pmod{10}$. Then $41k = A \cdot 10^a + B \cdot 10^b + C \cdot 10^c + D$, where $D = 1, 2$, or 3 and $2 \leq A + B + C + D \leq 4$. Modulo 41, this equation yields the following sets of congruences, where each $x_i \in G$:

$$\underline{s(41k) = 2}$$

$$x_i + 1 \equiv 0 \pmod{41}$$

$$\underline{s(41k) = 3}$$

$$\begin{aligned}
2x_i + 1 &\equiv 0 \pmod{41}\\
x_i + x_j + 1 &\equiv 0 \pmod{41}\\
x_i + 2 &\equiv 0 \pmod{41}
\end{aligned}$$

$\underline{s(41k) = 4}$

$$
\begin{aligned}
3x_i+1 &\equiv 0 \pmod{41}\\
2x_i+x_j+1 &\equiv 0 \pmod{41}\\
x_i+x_j+x_k+1 &\equiv 0 \pmod{41}\\
2x_i+2 &\equiv 0 \pmod{41}\\
x_i+x_j+2 &\equiv 0 \pmod{41}\\
x_i+3 &\equiv 0 \pmod{41}.
\end{aligned}
$$

But with the values for the x_i's given earlier, it is an easy matter arithmetically to show that none of the above congruences hold. The contradiction implies that there is no integer $41k$, as described above, for which the theorem fails, and we are done.

Comment by Joseph B. Dence, University of Missouri-St. Louis, St. Louis, Missouri.

It is apparent that the theorem works, in part, because G is of low order, and thus there is only a small number of linear combinations of the x_i's over the set $\{0, 1, 2, 3\}$. Contrast the case of the prime 41 with that of the prime 31, whose group G of remainders is of order 15. Repetition of the procedure used to solve Problem 74 leads to the result that if $31|n$, then $s(n) \geq 3$ (ex: $31k = 10^7 + 10^1 + 1 = 31 \cdot 322581$). More spectacularly, in the case of the prime 23, the group G has order 22, and therefore $23|n$ implies $s(n) \geq 2$ automatically (ex: $23k = 10^{11} + 1 = 23 \cdot 4347826087$).

Also, since the order of 10 modulo 41 is 5 (a prime), and $10^5 - 1 = 3^2 \cdot 41 \cdot 271$, then the order of 10 modulo 271 is also 5. We might expect that if $271|n$, then the minimum value for $s(n)$ should be no less than that for the case of the prime 41 (and certainly no greater than 10). There are more cases to check now, but a few moments' work shows, at the least, that $s(271k) > 5$.

Also solved by Elizaveta Kuznetsova, University of Notre Dame and the proposers.

75. [1994, 160] *Proposed by Leonard L. Palmer, Southeast Missouri State University, Cape Girardeau, Missouri.*

"Prove that $n(n+1)$ is never a square for $n > 0$" is a problem in *Elementary Number Theory* by Underwood Dudley. Generalize this problem by showing that $n(n+1) \neq t^k$ for t an integer and $k \geq 2$.

Solution by Bob Prielipp, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin; Joseph B. Dence, University of Missouri-St. Louis, St. Louis, Missouri; Bella Wiener, University of Texas-Pan American, Edinburg, Texas; Russell Euler, Northwest Missouri State University, Maryville, Missouri; Joe Howard, New Mexico Highlands University, Las Vegas, New Mexico; Elizaveta Kuznetsova, University of Notre Dame; Lawrence Somer, The Catholic University of America, Washington, D.C.; R. P. Sealy, Mount Allison University, Sackville, New Brunswick, Canada; and the proposer.

Suppose that $n(n+1) = t^k$ where n is a positive integer and t and k are integers with $k \geq 2$. We shall use the following known result.

If a and b are two relatively prime positive integers whose product is the m^{th} power of a positive integer (that is, $ab = c^m$ where c and m are positive integers) then a and b are both m^{th} powers of positive integers.

[For a proof of this result, see Theorem 8 and its proof on pp. 17 and 18 of Sierpinski, *Elementary Theory of Numbers*, Hafner Publishing Company, New York, 1964.]

Now this statement tells us that if $n(n+1) = t^k$ where t and k are positive integers with $k \geq 2$, then $n = a^k$ and $n+1 = b^k$ with a and b positive integers and $b > a$. It follows that

$$1 = (n+1) - n = b^k - a^k = (b-a)(b^{k-1} + b^{k-2}a + \cdots + ba^{k-2} + a^{k-1}).$$

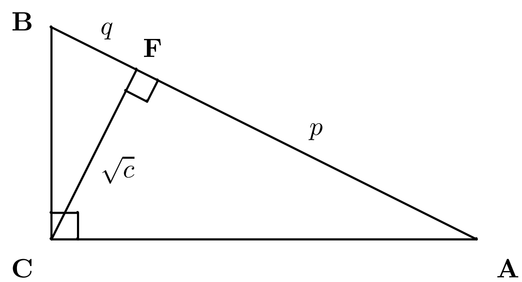
But since, a and b are positive integers, the second factor on the right must be greater than or equal to 3. This is a contradiction.

76. [1994, 160] *Proposed by Herta T. Freitag, Roanoke, Virginia.*

Let the hypotenuse c of a right triangle ABC equal G^3 (where $G = (\sqrt{5}+1)/2$, the golden ratio) and $\overline{CF}$ (where F is the footpoint of the height on the hypotenuse) equals $\sqrt{c}$. Obtain the length of the shorter leg of triangle ABC and prove or disprove that its length equals $\max(\overline{AF}, \overline{FB})$.

Solution by Bella Wiener, University of Texas-Pan American, Edinburg, Texas; Bob Prielipp, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin; Joe Howard, New Mexico Highlands University, Las Vegas, New Mexico; and the proposer.

Let $\overline{AF} = p$ and $\overline{FB} = q$. This information allows us to draw the following diagram.



Thus, $p + q = c$ and $pq = \overline{CF}^2 = c$. Hence, p and q are the roots of the quadratic equation $x^2 - cx + c = 0$. On the other hand, the golden ratio G is the positive root of the equation $x^2 - x - 1 = 0$. So, $G^2 = G + 1$ and $c = G^3 = G^2 + G = 2G + 1$. This enables one to write the equation for p and q in the form

$$x^2 - (2G+1)x + G(G+1) = 0 \quad \text{or} \quad (x - G)(x - G - 1) = 0.$$

Assuming $p > q$, we obtain

$$p = \max(\overline{AF}, \overline{FB}) = G + 1, \quad q = G.$$

The shorter leg of triangle ABC is

$$\begin{aligned} \min(\overline{AC}, \overline{CB}) &= (\overline{CF}^2 + q^2)^{1/2} \\ &= (c + q^2)^{1/2} = (2G + 1 + G^2)^{1/2} \\ &= G + 1 = (3 + \sqrt{5})/2. \end{aligned}$$