SOLUTIONS

No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the problem editor.

69. *Proposed by Mohammad K. Azarian, University of Evansville, Evansville, Indiana.*

Let G be a group such that whenever $g_1, g_2, g_3 \in G$ and $g_1g_2 = g_3g_1$, then $g_2 = g_3$. Show that:

(a) If G has two elements of order 2, then G must contain the Klein four group.

(b) The set $H = \{g \mid g^k = 1, \text{where } k \text{ is some integer}\}$ is a subgroup of G.

Composite solution by Kandasamy Muthuvel, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin; N. J. Kuenzi, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin; Jayanthi Ganapathy, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin; and the proposer.

First we shall prove that G is abelian. Let a and b be elements of G. Since $b(ab) = (ba)b$, it follows from the given condition that $ab = ba$ for all $a, b \in G$. Thus, G is abelian.

(a) Suppose a and b are distinct elements of G of order 2. If $ab = 1$, then $a = a \cdot 1 = aab = 1 \cdot b = b$. Hence, $ab \neq 1$ and since

$$(ab)^2 = a^2b^2 = 1,$$

the order of ab is 2. Finally, ab is distinct from a and b because if $ab = a$ or $ab = b$, then $b = 1$ or $a = 1$. Now it can be easily seen that the set consisting of the elements $1, a, b$, and ab is isomorphic to the Klein four group.

Comment by the editor. The editor is responsible for some poor wording in part (b) of this problem. As a result, two different solutions to the (b) part of this problem were given. The editor should have worded part (b) in one of the following ways.

(b1) Let k be a fixed integer. The set

$$H = \{g \mid g^k = 1\}$$

is a subgroup of G.

Solution by Kandasamy Muthuvel, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin and the proposer.

Clearly the identity element is in H. Now, suppose x and y are in H. It is enough to show that

$$(xy^{-1})^k = 1.$$

But,

$$(xy^{-1})^k = x^k(y^{-1})^k = (y^k)^{-1} = 1.$$

This completes the proof.

(b2) The set

$$H = \{g \mid g \text{ has finite order }\}$$

is a subgroup of G.

Solution by Jayanthi Ganapathy, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin and N. J. Kuenzi, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin.

Clearly, H is nonempty, since $1 \in H$. Let $g_1, g_2 \in H$. Then for some integers k_1 and k_2,

$$g_1^{k_1} = 1 \quad \text{and} \quad g_2^{k_2} = 1.$$

Thus,

$$(g_2^{-1})^{k_2} = 1.$$

Therefore,

$$(g_1g_2^{-1})^{k_1k_2} = g_1^{k_1k_2}(g_2^{-1})^{k_1k_2} = (g_1^{k_1})^{k_2}((g_2^{-1})^{k_2})^{k_1} = 1^{k_1}1^{k_2} = 1.$$

Hence,

$$g_1g_2^{-1} \in H.$$

This proves that H is a subgroup of G. H is called the torsion subgroup of G.

70. *Proposed by Herta T. Freitag, Roanoke, Virginia.*

The perfect number 28 can be expressed as $1^3 + 3^3$. Another perfect number, $496 = 1^3 + 3^3 + 5^3 + 7^3$. Generalize to give the conditions, if any, under which perfect numbers can be expressed as sums of consecutive odd cubes, and give this representation.

Solution by Bob Prielipp, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin; Joseph B. Dence, University of Missouri-St. Louis, St. Louis, Missouri; Joe Howard, New Mexico Highlands University, Las Vegas, New Mexico; Joseph Weiner and Miguel Parades (jointly), University of Texas-Pan American, Edinburg, Texas; and the proposer.

We shall show that if $n > 6$ is an even perfect number, then n can be expressed as a sum of consecutive odd cubes. (It is NOT known if there are any odd perfect numbers.)

By hypothesis, $n > 6$ is an even perfect number so

$$n = 2^{k-1}(2^k - 1),$$

where $2^k - 1$ is a prime number and k is an odd prime number. Because

$$\sum_{j=1}^{t} (2j-1)^3 = t^2(2t^2-1),$$

$$\sum_{j=1}^{2^{(k-1)/2}} (2j-1)^3 = 2^{k-1}(2 \cdot 2^{k-1} - 1) = 2^{k-1}(2^k - 1) = n.$$

Comment by the editor. What can be said if n is an odd perfect number?

71. *Proposed by Ronnie Gupton, Larry Hoehn, and Jim Ridenhour, Austin Peay State University, Clarksville, Tennessee.*

Provide a non-calculus solution to the following problem on page 530 of James Stewart's *Calculus* (2nd ed.), Brooks/Cole Publishing Company, 1991.

"A cow is tied to a silo with radius r by a rope just long enough to reach the opposite side of the silo. Find the area available for grazing by the cow."

Solution by the proposers.

The area grazed by the cow is the shaded region in Figure 1.

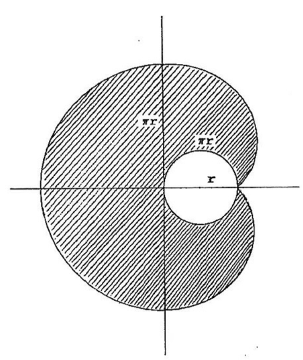

Figure 1.

We inscribe a regular polygon with $2n$ sides in the circular silo and consider the related problem of finding the area available to the cow around this polygonal silo in the first quadrant. This area is the sum of the n circular sectors shown in Figure 2 for $n = 5$.

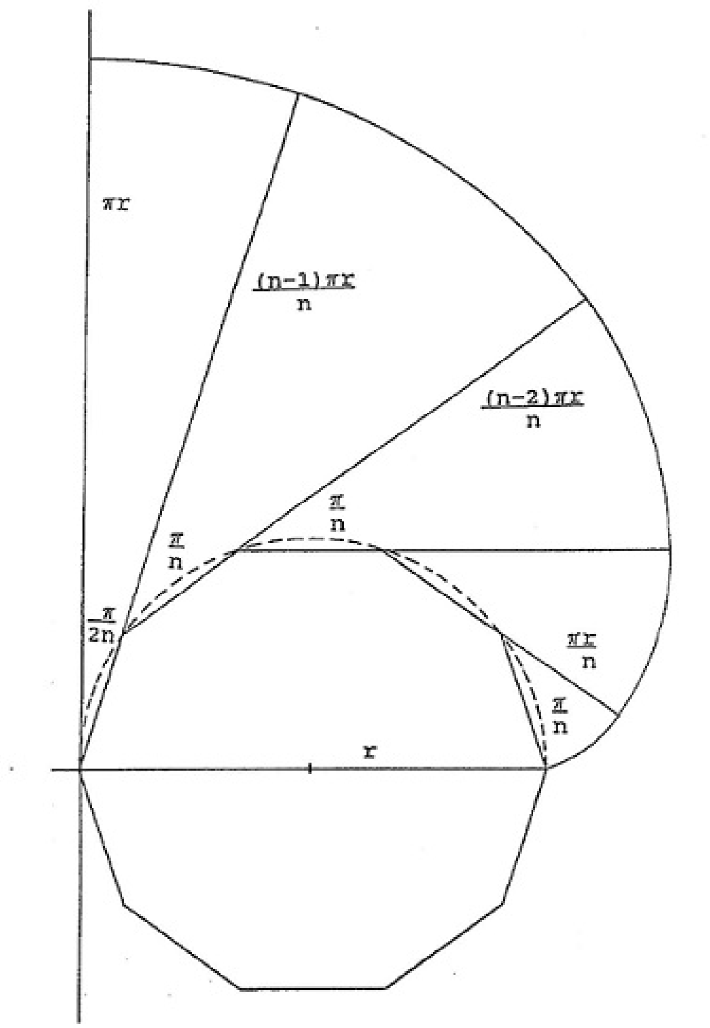

Figure 2.

Since each central angle of a regular polygon is congruent to each exterior angle, the central angles of the sectors are each π/n radians with the exception that the left-most one is half the others. Since the length of the rope is πr and each side of the polygon is $\pi r/n$, the successive sides of the sectors are πr, $(n-1)\pi r/n$, $(n-2)\pi r/n$, $\ldots$, $\pi r/n$. Hence, the area, A, of these sectors is

$$\begin{aligned}
A &= \frac{\pi(\pi r)^2}{4n} + \frac{\pi}{2n}\left(\frac{(n-1)\pi r}{n}\right)^2 + \frac{\pi}{2n}\left(\frac{(n-2)\pi r}{n}\right)^2 + \cdots + \frac{\pi}{2n}\left(\frac{1\cdot\pi r}{n}\right)^2 \\
&= \frac{\pi^3 r^2}{2n^3}\left(\frac{1}{2}n^2 + (n-1)^2 + (n-2)^2 + \cdots + 2^2 + 1^2\right) \\
&= \frac{\pi^3 r^2}{2n^3}\left(\frac{n^2}{2} + \frac{(n-1)n(2n-1)}{6}\right) \\
&= \frac{\pi^3 r^2}{12}\left(2 + \frac{1}{n^2}\right).
\end{aligned}$$

Therefore, the area grazed about the circular silo in the first quadrant is

$$\lim_{n\to\infty} A = \frac{\pi^3 r^2}{6},$$

so the total area grazed by the cow is $5\pi^3 r^2/6$.

Also solved by Alan H. Rapoport, M.D., Ashford Medical Center, Santurce, Puerto Rico. Three incorrect solutions were also received.

72. *J. Sriskandarajah, University of Wisconsin Center-Richland, Richland Center, Wisconsin.*

Show that one more than four times the product of two consecutive odd numbered pentagonal numbers is a square.

Solution by Bob Prielipp, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin; Russell Euler, Northwest Missouri State University, Maryville, Missouri; Kandasamy Muthuvel and Kevin McDougal, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin; Herta T. Freitag, Roanoke, Virginia; and Joseph B. Dence, University of Missouri-St. Louis, St. Louis, Missouri.

We shall show that one more than four times the product of two consecutive odd-numbered or two consecutive even-numbered pentagonal numbers is a perfect square.

Let P_n denote the nth pentagonal number. Then

$$P_n = \frac{3n^2 - n}{2}.$$

Thus,

$$\begin{aligned}
4P_nP_{n+2} + 1 &= 4 \cdot \frac{3n^2 - n}{2} \cdot \frac{3(n+2)^2 - (n+2)}{2} + 1 \\
&= n(3n-1) \cdot (n+2)(3n+5) + 1 = [n(3n+5)][(n+2)(3n-1)] + 1 \\
&= (3n^2 + 5n)[(3n^2 + 5n) - 2] + 1 = (3n^2 + 5n)^2 - 2(3n^2 + 5n) + 1 \\
&= [(3n^2 + 5n) - 1]^2 = (3n^2 + 5n - 1)^2.
\end{aligned}$$

Also solved by Donald P. Skow, University of Texas-Pan American, Edinburg, Texas; N. J. Kuenzi, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin; Bella Wiener, University of Texas-Pan American, Edinburg, Texas; Joe Howard, New Mexico Highlands University, Las Vegas, New Mexico; and the proposer.

Comment by Donald P. Skow, University of Texas-Pan American, Edinburg, Texas. If h_n denotes the nth hexagonal number, then $h_n = n(2n-1)$ and the sequence $1, 6, 15, 28, 45, 66, \ldots$ has a similar property, i.e.,

$$4h_nh_{n+1} + 1 = (4n^2 + 2n - 1)^2$$

where $n \geq 1$.