LINKS OF CERTAIN SINGULARITIES

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Abstract. A brief introduction to the topology of complex singularities is given. Then, it is shown how an elementary topological construction and some basic covering space theory can be used to analyze the connectivity of the link of certain 1-dimensional singularities in complex 3-space.

1. Introduction. It is the intent of this manuscript to provide a brief introduction to the topology of complex algebraic singularities and to show how what is usually a relatively complicated problem in this theory can be solved in certain cases by an easy topological construction. To see why singular points of complex algebraic varieties are topologically interesting we begin with two examples in knot theory and a third higher dimensional example.

Consider the polynomial $f(x,y) = x^2 + y^3$ in the two complex variables $x = x_1 + ix_2$ and $y = y_1 + iy_2$. Note that the origin, $(0,0) \in \mathbb{C}^2$, is a "singular point for f." That is, $(0,0)$ is a point on the variety $V(f) = \{(x,y) \in \mathbb{C}^2 \mid f(x,y) = 0\}$ at which *all* partial derivatives of f are simultaneously zero. For a point $(x,y) \in \mathbb{C}^2$, denote its modulus by

$$\|(x,y)\| = \sqrt{x_1^2 + x_2^2 + y_1^2 + y_2^2}.$$

Then the 3-sphere of radius $\epsilon > 0$ and center at $(0,0) \in \mathbb{C}^2$ is given by $S^3_\epsilon = \{(x,y) \mid \|(x,y)\| = \epsilon\}$. If one now looks at $K(f) = S^3_\epsilon \cap V(f)$ one obtains the trefoil knot. Now let $g(x,y) = xy$. Again note that $(0,0) \in \mathbb{C}^2$ is a "singular point for g." We have that $K(g) = S^3_\epsilon \cap V(g)$ is the Hopf Link. The objects $K(f)$ and $K(g)$ are called "links of the singularity at $(0,0)$."

Note that if one chooses a nonsingular point p on $V(f)$ (or on $V(g)$) and looks at the intersection of a 3-sphere of sufficiently small radius centered at p with $V(f)$ (or with $V(g)$), standard differential topology implies that all one obtains is an unknotted circle in the 3-sphere.

Finally, consider the polynomial $h(x) = x^2 + y^3 + z^5$ in three complex variables. It has an isolated singularity at the origin. Its "link" is $K(h) = V(h) \cap S^5$ where $V(h) = \{(x, y, z) \in \mathbb{C}^3 \mid h(x, y, z) = 0\}$ and S^5 is a 5-sphere of sufficiently small radius. It turns out that $K(h)$ is a famous 3-manifold. It is the Poincaré homology sphere. (See Milnor [4, §8].) These three examples illustrate what is generally true: topologically interesting objects are obtained as "links of singular points" on algebraic varieties.

Section 2 contains definitions and a description of the fundamental results governing the topology of singular points of complex algebraic varieties and provides the motivation for the situation where questions about the connectivity of the link are of interest. Section 3 gives the definition of 1-isolated singularity and a purely geometric/topological proof that the link of a 1-isolated singularity with transverse A_1 singularities away from the origin is (essentially) never simply connected. Some additional corollaries of the basic construction are also given. For more extensive bibliographies than are given here, the reader is encouraged to consult the references at the end of the manuscript, especially Milnor [4], Randell [13], Pellikaan [11], [12], and Siersma [18].

2. Basic Definitions and Theorems. Let $\mathbb{C}[z_0, \dots, z_n]$ denote the ring of polynomials in $n+1$ complex variables. Think of n as being fixed in what follows. Note that n is actually one less than the number of complex variables in the polynomial. Let $f \in \mathbb{C}[z_0, \dots, z_n]$. The variety of f is the set $V(f) = \{(z_0, \dots, z_n) \in \mathbb{C}^{n+1} \mid f(z_0, \dots, z_n) = 0\}$. We *always* assume $f(0) = 0$. The *singular set* of f is defined to be

$$\Sigma(f) = \left\{ (z_0, \dots, z_n) \in f^{-1}(0) \mid \frac{df}{dz_k}(z_0, \dots, z_n) = 0, k = 0, \dots, n \right\}.$$

We *always* assume $0 \in \Sigma(f)$. Let $S^{2n+1} = \{(z_0, \dots, z_n) \in \mathbb{C}^{n+1} \mid \|(z_0, \dots, z_n)\| = \epsilon\}$ denote the $(2n+1)$-sphere of radius $\epsilon > 0$ centered at $0 \in \mathbb{C}^{n+1}$. The Cone Theorem states that the piece of $V(f)$ that is contained inside of S^{2n+1}, that is the intersection of $V(f)$ with the closed $(2n+2)$-ball with center at the origin, is homeomorphic to the cone on the link L with cone point at the origin. It is a consequence of the Cone Theorem and the Milnor Fibration Theorem, (see Randell [13] or Milnor [4]), that the local topology of the singularity at 0 is described by the objects

$$(*) \qquad K = S^{2n+1} \cap f^{-1}(0), \quad \frac{f}{\|f\|}: S^{2n-1} \backslash K \to S^1,$$

$$\text{and } F = \left(\frac{f}{\|f\|}\right)^{-1}(1) \subseteq S^{2n+1}\backslash K,$$

where S^{2n+1} is a sphere of sufficiently small radius, centered at $0 \in \mathbb{C}^{n+1}$. The main corollary of the Cone Theorem and the Fibration Theorem is that the objects in $(*)$ are all independent of the radius, ϵ, of S^{2n+1}, provided ϵ is sufficiently small. Furthermore, the Milnor Fibration Theorem and its corollaries state that the function

$$\frac{f}{\|f\|}\colon S^{2n+1}\backslash K \to S^1$$

is the projection of a locally trivial smooth fibre bundle with fibre

$$F = \left(\frac{f}{\|f\|}\right)^{-1}(1)$$

and that K is a compact $(2n-1)$-dimensional simplicial complex. Two other important results in Milnor [4] concern the connectivity of K and F. Milnor [4,5.2] shows that K is $(n-2)$-connected, that is $\pi_k(K) = 1$ for $1 \leq k \leq n-2$. He shows in [4,6.5] that when $\Sigma(f) = \{0\}$, the fibre F has the homotopy type of a bouquet of μ n-spheres. For the exact statements of these results see [4] or [13].

Note that in the three examples in Section 1, the singularity at the origin is "isolated." That is, $\Sigma(f) = \{0\}$. This is the simplest case and, as expected, the most detailed results available are for this case. By [4,6.1], K is a $(2n-1)$-dimensional compact manifold when the origin is an isolated singularity. The interested reader can consult Milnor [4, §6 – §10] to see additional results. In terms of the exposition in Section 3, it is of particular interest to note that by [4,8.2], for $n \neq 2$, the link K of an isolated singularity is a topological sphere if and only if the reduced integral homology group $\tilde{H}_{n-1}(K)$ is trivial. (When $n = 2$, we run into the Poincaré Conjecture.)

It is natural to wonder what happens to the topology of the link K and the Milnor fibre F when $\Sigma(f)$ is larger than a single point. Randell's survey paper [13] contains some results on the topology of non-isolated singularities. However, detailed results on the topology of non-isolated singularities for the class of non-isolated singularities in general are hard to come by so one begins by placing restrictions on the type of singular sets allowed. One such class of singularities, called k-isolated singularities with transverse A_1 singularities on $\Sigma(f)\backslash\{0\}$, have been studied in some detail. The structure of the complement $S^{2n+1}\backslash K$ and the Milnor fibre has been studied by Siersma [15], [16], [17], [18] especially, but also by Pellikaan [8], [9], [10], [11], [12], Schrauwen [14], and Van Straten [21]. The work by De Jong [1] contains an extension of the Siersma's techniques to other classes of non-isolated singularities in $\mathbb{C}^{n+1}$. The structure of the link K has received less study, but certain results about the homotopy groups, $\pi_*(K)$, and the integral homology groups $H_*(K)$ follow from Randell [13] and the Milnor Fibration Theorem. The work by Mumford [6] can be used to obtain information about $\pi_1(K)$ for singularities in complex projective space. For the case $n = 2$, Timm [20] provides a fairly detailed look at $\pi_1(K)$ and $H_*(K)$.

As a corollary to the homology and fundamental group calculations of Timm [20] one obtains that except for certain "trivial" cases, the link of a 1-isolated singularity with transverse A_1 singularities on $\Sigma(f)\backslash\{0\}$ in $\mathbb{C}[z_0, z_1, z_2]$ is never simply connected. Note that in this case $n = 2$. So, all that the Milnor Fibration Theorem implies is that K is connected. In the light of this fact, it is natural to wonder if links of such non-isolated singularities can ever be simply connected. In Section 3, the focus is on 1-isolated singularities in three complex variables with transverse A_1 singularities on $\Sigma(f)\backslash\{0\}$. The basic definitions are given and an alternate, elementary, topological proof of the non-simply connectivity of the links of such singularities is given. The proof given here requires neither the differential techniques of Milnor's general result nor the algebraic techniques of Timm [20]. Instead, an elementary construction and basic covering space theory are used.

3. 1-Isolated Singularities in Three Complex Variables. We being with two examples. Consider first the polynomial $f(x, y, z) = xy^2 + z^2$ in three complex variables. The variety defined by f, $V(f) = \{(x, y, z) \in \mathbb{C}^3 \mid f(x, y, z) = 0\}$ has complex dimension $\dim_{\mathbb{C}} V(f) = 2$. We have that $\Sigma(f) = \{(x, 0, 0) \mid x \in \mathbb{C}\}$, that is $\Sigma(f)$ is the x-axis in $\mathbb{C}^3$. Observe that $\dim_{\mathbb{C}} \Sigma(f) = 1$. The fact that $\Sigma(f)$ has complex dimension 1 is what is meant by a "1-isolated singularity." Recall that a function is biholomorphic if and only if it is a homeomorphism and both it and its inverse are analytic. Observe that, for each $x \neq 0$, there is an open ball U_x about $(x, 0, 0) \in \mathbb{C}^3$, an open subset $V \subset \mathbb{C}^3$, and a biholomorphic function $\phi_x\colon U_x \to V_x$ such that for $(x, y, z) \in U$, $(f \circ \phi_x)(x, y, z) = y^2 + z^2$. (For example, let $\sqrt{}$ denote a fixed branch of the complex valued square root function that contains x in its domain. Let U_x be a ball of fixed radius less than $\|x\|$ that is contained entirely in the domain of $\sqrt{\ }$. Define $\phi_x\colon U_x \to \mathbb{C}^3$ by $\phi_x(x,y,z) = (x, y/\sqrt{x}, z)$.) The function ϕ_x is called a "local analytic, or biholomorphic, coordinate system for $\mathbb{C}^3$ about $(x,0,0)$." It is the existence of such local coordinate systems, ϕ_x, for each $x \in \Sigma(f)\backslash\{0\}$ that says that f has "transverse A_1 singularities on $\Sigma(f)\backslash\{0\}$." Let $K(f) = S^5 \cap V(f)$ be the link of the singularity at $0 \in \mathbb{C}^3$.

Now consider the function $g(x,y,z) = yz$. Let $V(g)$ denote the variety defined by g. We have that $\dim_{\mathbb{C}} V(g) = 2$. Observe that $\Sigma(g) = \{(x,0,0) \mid x \in \mathbb{C}\}$ is again the x-axis in $\mathbb{C}^3$. Finally, define the global biholomorphic (in fact, complex linear) coordinate change $\phi\colon \mathbb{C}^3 \to \mathbb{C}^3$ by $\phi(x,y,z) = (x, y+iz, y-iz)$ and note that $(g \circ \phi)(x,y,z) = y^2 + z^2$. Thus, g is also a 1-isolated singularity with transverse A_1 singularities on $\Sigma(g)\backslash\{0\}$. We have $K(g)$ is the link of the singularity at the origin.

What are the topological consequences of the above observations? Begin with the function g. Note that $V(g)$ is the union of the xz-plane and xy-plane in complex 3-space along the x-axis. So topologically, $V(g)$ is the union of two copies of $\mathbb{R}^4 = \{(a_1,a_2,a_3,a_4) \mid a_j \in \mathbb{R}\}$ along the real two dimensional vector subspace $\mathbb{R}^2 = \{(a_1,a_2,0,0) \mid a_j \in \mathbb{R}\}$. Note that this implies that at points in the singular set $\Sigma(g)$, that is along the complex x-axis, each point has a neighborhood in $V(g)$ that is homeomorphic to two copies of the four ball

$$B^4 = \left\{(a_1,a_2,a_3,a_4) \mid a_j \in \mathbb{R} \text{ and } \sqrt{\sum_{j=1}^{4} a_j} \leq 1\right\}$$

glued together along the two 2-disk

$$D^2 = \left\{(a_1,a_2,0,0) \mid a_j \in \mathbb{R} \text{ and } \sqrt{\sum_{j=1}^{2} a_j} \leq 1\right\}$$

via the obvious identification. Finally, note that $\Sigma(g) \cap K(g)$ is the circle $S^1 = \{(x,0,0) \in \mathbb{C}^3 \mid \|x\| = 1\}$. The entire link $K(g) = V(g) \cap S^5$ is the union of two 3-spheres, S^3, along an unknotted circle. Note that along the singular circle $\Sigma = \Sigma(g) \cap K(g)$, each point in Σ has neighborhoods homeomorphic to two three balls

$$B^3 = \left\{ (b_1, b_2, b_3) \in \mathbb{R}^3 \mid \sqrt{\sum_{j=1}^{2} b_j} \leq 1 \right\}$$

glued together along the interval $I = \{(b_1, 0, 0) \in \mathbb{R}^3 \mid |b_1| \leq 1\}$ via the obvious identification. In particular, by the Milnor Fibration Theorem and the above observation it follows that the link, $K(g)$, is a 3-dimensional complex that is not a manifold. It is a manifold at all points except those in Σ.

Now look at the singularity $f(x, y, z) = xy^2 + z^2$. Again $V(f)$ has real dimension 4 and the singular set is a real 2-dimensional plane. Furthermore, since f has transverse A_1 singularities on $\Sigma(f) \backslash \{0\}$, all points in the singular set, except the origin, have neighborhoods in $V(f)$ that look like two four balls glued together along a two disk, just like the case for g above. Again, note that $\Sigma = \Sigma(g) \cap K(g)$ is a circle and that each point in Σ has a neighborhood in $V(g)$ that is homeomorphic to two three balls glued together along an interval. Again the Milnor Fibration Theorem and this observation imply that $K(g)$ is a manifold except along the singular circle Σ.

Observe that if p is a nonsingular point in $V(f)$ (or $V(g)$) and we look at the intersection of a sufficiently small 5-sphere, centered at p, with $V(f)$ (or $V(g)$) standard differential topology implies that what is obtained is an unknotted 3-sphere in the 5-sphere.

We give the formal definitions of the ideas introduced above and make a few additional general comments about the topology of 1-isolated singularities with transverse A_1 singularities away from the origin. Since we are only concerned about 1-isolated singularities that satisfy the indicated transverse condition we will abuse the notation as indicated in Definition 3.1.

Definition 3.1. Let $f(x, y, z) \in \mathbb{C}[x, y, z]$. Say that f is a 1-*isolated singularity* if and only if $\dim_{\mathbb{C}} \Sigma(f) = 1$ and for each point $p \in \Sigma(f) \backslash \{0\}$ there is a local biholomorphic coordinate system $\phi_p : U_p \to V_p$ such that for all $(x, y, z) \in U_p$, $(f \circ \phi_p)(x, y, z) = y^2 + z^2$.

It then follows that for 1-isolated singularities, $\Sigma(f)$ is the union of a finite collection of real 2-dimensional surfaces that only intersect at the origin of $\mathbb{C}^3$. This implies that $\Sigma = \Sigma(f) \cap K(f)$ is a finite disjoint union of circles. (That is, Σ is a link of circles in $K(f)$.)

Furthermore, $K(f)$ is a 3-complex that is a manifold except at points along the singular link Σ. Each point in Σ has a neighborhood in $K(f)$ that is homeomorphic to the union of two 3-balls along an interval. Finally, as promised in the introduction, we avoid certain "trivial" situations by assuming that the polynomial f is *analytically irreducible*, that is, if there are two analytic functions $g(x,y,z)$ and $h(x,y,z)$ such that $f(x,y,z) = g(x,y,z)h(x,y,z)$ then either h is a unit or g is a unit in the ring of analytic functions.

The topological consequences of the definition of 1-isolated singularities mentioned above, together with the assumption that f is irreducible, allow the use of an object introduced by Orlik and Wagreich [7], called the covering manifold of a singufold, to construct connected, nontrivial, covering spaces of the 1-isolated singularity, $K(f)$. We have the following definitions. This idea is a further generalization of the notion of branched covering given by Fox [2].

Definition 3.2. Let $(\theta, \overline{K}, \overline{\Sigma}, K, \Sigma)$ be a quintuple with $(\overline{K}, \overline{\Sigma})$ a connected, closed, topological 3-manifold pair and $\overline{\Sigma}$ a link of circles in $\overline{K}$. Let $\theta: \overline{K} \to K$ be a closed topological immersion (that is, each point $p \in \overline{K}$ has an open ball, B_p, about it such that the restriction $\theta: B_p \to \theta(B_p)$ is a one-to-one map of B_p onto its image) of $\overline{K}$ onto K with $\theta(\overline{\Sigma}) = \Sigma$ and $\theta: \overline{K} \backslash \overline{\Sigma} \to K \backslash \Sigma$ a homeomorphism. Assume that $\theta: \overline{\Sigma} \to \Sigma$ is a double cover. The triple $(\theta, \overline{K}, \overline{\Sigma})$, or just $\overline{K}$, is called the *covering manifold* of (K, Σ) and Σ is called the *singular set* of K.

Theorem 3.3. K is a "singufold" and $(\overline{K}, \theta)$ its covering manifold in the sense of Orlik and Wagreich [7]. In particular, $(\overline{K}, \theta)$ is the unique manifold such that if $\mu: L \to K$ is any closed topological immersion of a manifold L onto K such that $\mu^{-1}(K \backslash \Sigma)$ is dense in L then there is a unique map $\overline{\mu}: L \to \overline{K}$ such that $\mu = \theta \circ \overline{\mu}$.

Proof. Observe that $K \backslash \Sigma$ is a 3-manifold and that each $p \in \Sigma$ has a neighborhood that looks like two 3-balls glued together along a diameter. Note that p is contained in this common diameter. Thus, K is, by definition, a "singufold" and $\overline{K}$ its covering manifold.

Construction 3.4. Let K be a singufold and $\overline{K}$ its covering manifold as in 3.2 and 3.3. We construct the double cover of K. Note that the assumption that the restriction that $\theta: \overline{\Sigma} \to \Sigma$ be a double cover of Σ, which is a link of circles, implies that $\overline{\Sigma}$ is also a link of circles and that the inverse image of each component of Σ is either two circles each of which is mapped homeomorphically onto the given component in Σ or this inverse image is a single circle that is a double cover of the given component of Σ. Define a map $\alpha: \overline{\Sigma} \to \overline{\Sigma}$ as follows: $\alpha(x) = y$ if and only if $\theta(x) = \theta(y)$ and $x \neq y$. Note that α just swaps the two points of $\theta^{-1}(\theta(x))$. The points x, y with $\theta(x) = \theta(y)$ and $x \neq y$ will be called *antipodal* points of $\overline{\Sigma}$. Since θ is continuous so is α. Also note that $\alpha \circ \alpha =$ id. Take two copies $\overline{K}_j$, $j = 1, 2$, of $\overline{K}$ and glue them together along $\overline{\Sigma}_1$ and $\overline{\Sigma}_2$ via $\alpha\colon \overline{\Sigma}_1 \to \overline{\Sigma}_2$. Call the resulting space $\tilde{K}$. The common image of the $\overline{\Sigma}_j$ will be denoted by $\tilde{\Sigma}$. A set $U \subset \tilde{K}$ is open if and only if $U \cap \overline{K_j}$ is open in $\overline{K}_j$ for $j = 1$ and $j = 2$. Define a map $p\colon \tilde{K} \to K$ in the obvious manner, namely $p(x) = \theta(x)$. Then p is clearly continuous and 2-to-1. Since θ is a local homeomorphism of $\overline{\Sigma}$, so is p. By construction, p is a local homeomorphism at points $x \in \tilde{\Sigma}$. Thus, $p\colon \tilde{K} \to K$ is a connected double cover of K. Note that the closure of each fundamental region of the projection is a copy of $\overline{K}$. The process just described will be referred to in what follows as "attaching $\overline{K}_1$ to $\overline{K}_2$ by identifying each point in $\overline{\Sigma}_1$ with its antipodal point in $\overline{\Sigma}_2$."

Some immediate corollaries of the construction follow. The first is on the simple connectivity of links of irreducible 1-isolated singularity in three complex variables.

<u>Corollary 3.5</u>. Let K be the link of an analytically irreducible 1-isolated singularity $f \in \mathbb{C}[x, y, z]$. Then $\pi_1(K)$ is not trivial, that is, K is not simply connected.

<u>Proof</u>. By the remarks on the topology of K preceding Definition 3.2, K is a singufold. Since f is analytically irreducible, it has, according to Orlik and Wagreich [7], a connected covering manifold $\overline{K}$. For the remainder of the proof we use the symbolism of Construction 3.4. We have $\tilde{K} = \overline{K}_1 \cup_\alpha \overline{K}_2$. Define a map $f\colon \tilde{K} \to \tilde{K}$ by $f(x) = y$ if and only if $p(x) = p(y)$ and $x \neq y$. It is an easy exercise to see that f is continuous, is a covering translation, and $f \circ f =$ id. Therefore, $(\tilde{K}, p)$ is a regular double cover. Thus, Aut $(\tilde{K}) \cong \mathbb{Z}/2\mathbb{Z} \cong \pi_1(K)/p{*}\pi_1(\tilde{K})$. So $\pi_1(K)$ always has a subgroup of index 2. So K is never simply connected.

The reader may feel that the assumption of irreducibility of the 1-isolated singularity f is a severe restriction. In practice it is not. If one inspects the lists of Mond [5] and Timm [19], one discovers that the only polynomial that is eliminated from their lists by the assumption of analytic irreducibility is the polynomial $g(x, y, z) = yz$ (or equivalently, $g(x, y, z) = y^2 + z^2$). This situation is easy to analyze. The link $K(g)$ is two copies of S^3 joined along an unknotted circle, the covering manifold is a disjoint union $S^3 \cup S^3$, and so, by Van Kampen's Theorem, $\pi_1(K(g)) = 1$.

<u>Corollary 3.6</u>. Let K be a link of an irreducible 1-isolated singularity with covering manifold $(\overline{K}, \theta)$ as in 2.4. Assume Σ is such that the lifting of each component of Σ to $\overline{K}$ is a single circle. Let $(\tilde{K}, p)$ be the double cover of 2.4 and let $i\colon \overline{K} \to \tilde{K}$ be the inclusion

of $\overline{K}$ onto $\overline{K}_1 \subset \tilde{K}$. Then $i_*: \pi_1(\overline{K}) \to \pi_1(\tilde{K})$ is injective. Hence, $\theta_*: \pi_1(\overline{K}) \to \pi_1(K)$ is injective.

Proof. Pick a base point $x_1 \in \tilde{\Sigma}$. Connect it to every other component of $\tilde{\Sigma}$, if any, via an arc in $\overline{K}_1 \backslash \tilde{\Sigma}$. Let $\tilde{\Sigma}_+$ denote the union of $\tilde{\Sigma}$ and the arcs. It has the homotopy type of a bouquet of circles. Let $\tilde{K}_{2+} = \tilde{K}_2 \cup \tilde{\Sigma}_+$. Write $\tilde{K} = \overline{K}_1 \cup \overline{K}_{2+}$. Then $\overline{K}_1 \cap \overline{K}_{2+} = \tilde{\Sigma}_+$. Apply Van Kampen's Theorem and standard results in combinatorial group theory, e.g. [4,4.3], to obtain a presentation of $\pi_1(\tilde{K})$ in terms of $\pi_1(\overline{K}_1)$ and $\pi_1(\overline{K}_{2+})$ and the injectivity of i_*. The injectivity of θ_* then follows since $\tilde{K}$ is a covering space. (A different proof of the injectivity of θ_* is given in [21,3.7].)

Corollary 3.7. Let K be a link of an irreducible 1-isolated singularity and $\theta: \overline{K} \to K$ its covering manifold. Assume that Σ is a single circle and its lifting $\overline{\Sigma}$ is also a single circle. Then the following are equivalent: (1) $\pi_1(K)$ is abelian. (2) $\pi_1(\overline{K})$ is cyclic and is generated by the homotopy class of $\overline{\Sigma}$. (3) $\pi_1(K)$ is cyclic and is generated by the homotopy class of Σ.

Corollary 3.8. Let $(\theta, \overline{K}, \overline{\Sigma}, K, \Sigma)$ be as in Definition 3.2. Write

$$\overline{\Sigma} = \bigcup_{j=1}^{n} \overline{\Sigma}_j,$$

a disjoint union of circles. If $n \geq 2$, then K has covering spaces of arbitrarily high finite order.

Proof. Let $\tilde{K}$ be the regular double cover of K constructed in Construction 3.4 above. We have

$$\tilde{\Sigma} = \tilde{\Sigma}_1 \cup \tilde{\Sigma}_2 \cup \cdots .$$

Observe that the $\tilde{\Sigma}_j$ are the components of $\tilde{\Sigma}$ and that $\tilde{\Sigma}_1$ has an open neighborhood N that is the union of two solid 3-tori glued together along their center circles via α. $\tilde{\Sigma}_1$ is closed in $\tilde{K}$ and separates N but does not separate $\tilde{K}$. That is $\tilde{\Sigma}_1$ is a closed subset that separates $\tilde{K}$ locally but not globally. So by [3,7.2], $\tilde{K}$ has regular k-fold covering spaces for all positive integers k. So K has $2k$-fold covering spaces for all positive integers k.

Remark 3.9. Actually more can be said in most instances. If some component Σ_j of Σ lifts to two circles in $\overline{K}$, then Σ_j is a subset of K that separates K locally but not globally.

So in this situation, by Jungck [3,7.2], K not just $\overline{K}$, has *regular* k-fold covering spaces for all positive integers k.

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