DIGITAL SOUP

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1. Introduction. Word patterns often suggest similar patterns in the case for numbers. Among the most interesting of words are those which read the same from left to right as from right to left (e.g., eye, noon, level, refer, deified, etc.). These are generally identified as palindromes, a word stemming from the Greek "palindromos" which means "running back again." More precisely, they are instances of "reciprocal palindromes." In a digital setting, reciprocal palindromes give rise to such appealing forms as the reversible prime (e.g., 101 or 15551), the reversible square (e.g., 121 or 40804), and the reversible cube (e.g., 343 or 1331). But another variation on palindromes presents itself. Suppose a different but valid word is formed by reversing letters (such as "flow" and "wolf," "diaper" and "repaid," "reward" and "drawer," "reviled" and "deliver," etc.). These are ordinarily labeled "recurrent palindromes." They also suggest a variety of mathematical questions.

Accordingly, are there numbers in a given category which produce different numbers in the same category by the reversing of digits? Examples are rather quickly found and include the following:

1. counting numbers,
2. multiples of 3 or of 9, and
3. multiples of 11.

Note that a counting number is still a counting number when its digits are reversed, a multiple of 3 (or 9) remains a multiple of 3 (or 9) by the reversing (even the permuting) of digits, and that a multiple of 11 becomes a number of the same kind by reversibility. Many encounters with recurrent palindromic numbers are vastly more subtle. Some actually extend into the area of presently unsolved problems. A look at several familiar classes of numbers reinforces the concept.

2. Exact Squares. It is apparent that such pairs as 144 and 441 or 169 and 961 are squares having the desired reversibility feature. For convenience, denote the recurrent palindromic relationship (in a given set context) by the connective "r." Hence, 144 r 441 and 169 r 961. A natural question concerns the cardinality of the set of such pairs. That is, how many exist? Building on the fact that 12^2 r 21^2, one can, by inserting appropriate in-between zeros, generate as many recurrent palindromic pairs as desired. More precisely,

$$[1(10)^n + 2]^2 \text{ r } [2(10)^n + 1]^2.$$

For example, if $n = 2$, expansion yields 10404 r 40401. If $n = 3$, it is noted that 1004004 r 4004001. Similarly,

$$[1(10)^n + 3]^2 \text{ r } [3(10)^n + 1]^2.$$

The reader may wish to pursue the problem concerning the cardinality of recurrent palindromic squares which deviate from the above mode of construction. Moreover, extended questions arise as in the case for higher powers (cubes, fourth powers, etc.).

3. Abundant Numbers. A number is abundant if it is exceeded by the sum of its proper divisors. Note that 12 is abundant as $1 + 2 + 3 + 4 + 6 > 12$. It is also the least abundant number. Again, it is not hard to find recurrent palindromic abundant numbers. Consider, for example, the small pair 24 and 42 and the pair 48 and 84. A simple number theory fact reveals that the set of such pairs is infinite. In particular, any multiple of a perfect number, other than itself, is abundant. Recall that a number is perfect if and only if it is equal to the sum of its proper divisors. As 6 is clearly perfect, then higher multiples will thus be abundant. As a consequence, one may quickly note that 24 r 42, 204 r 402, 2004 r 4002, etc. Generally,

$$[2(10)^n + 4] \text{ r } [4(10)^n + 2]$$

as both numbers are even and clearly multiples of 3 (the latter by the digital root criterion). The reader may also wish to consider the case for odd abundant numbers (945 is the least such number) along with the overall counterpart question for deficient numbers (a number whose proper divisor sum is less than the number itself). Several recurrent palindromic deficient number examples involve relatively small numbers (e.g., 17 r 71, 37 r 73, and 57 r 75).

4. Perfect Numbers. In examining a list of known even perfect numbers, various conjectures concerning patterns emerge. The first twelve such numbers are shown below.

6

28

496

8128

33550336

8589869056

137438691328

2305843008139952128

2658455991569831744654692615953842176

191561942608236107294793378084303638130997321548169216

131640364585696483372397534604458722910223472318386943117783728128

14474011154664524427946373126085988481573677491474835889066354349131199152128

The Perfect Number Tree

At one time it was thought that even perfect numbers alternately ended in 6 or 8. This was proved false as consecutive numbers in the list were gradually found. Each, beyond 6, appeared to yield a remainder of 1 when divided by 9. Today this is known to be true. Another conjecture relates to the task at hand. It seems that, excluding 6, no even perfect number gives rise to another such number by the reversing of digits. That is, one might conjecture that no even perfect numbers are recurrent palindromes. Indeed, the listing above strongly suggests just that.

Such a conjecture proves to be true as no two even perfect numbers can have the same number of digits. Beginning with 496, any even perfect number is over ten times its predecessor. Thus, it must have at least one digit more. To justify this, the Euclid-Euler characterization suffices. That is, any number of this type is of the form $2^{n-1}(2^n-1)$ where 2^n-1 is prime – and conversely. In order for 2^n-1 to be prime, it is necessary that n also be prime. [Primes of the form 2^n-1 are called Mersenne primes.]

Let us now compare an even perfect number (greater than 28), say $2^{y-1}(2^y-1)$, with its predecessor $2^{x-1}(2^x-1)$, noting that as x and y are prime, y is greater than or equal to $x+2$. More precisely, consider the ratio r where

$$r = \frac{2^{y-1}(2^y-1)}{2^{x-1}(2^x-1)}.$$

The ratio

$$\frac{2^{y-1}}{2^{x-1}}$$

must equal or exceed 4. It is seen, by long division, that the ratio of the Mersenne primes exceeds 4 as

$$\frac{(2^y-1)}{(2^x-1)}$$

equals

$$2^{y-x} + \frac{2^{y-x}-1}{2^x-1}.$$

Recall that $y-x$ is greater than or equal to 2. Overall, the ratio of any such even perfect number to its predecessor is greater than 16 (i.e., $r > 16$), thus necessitating at least one additional digit in the conventional Hindu-Arabic representation.

Accordingly, no even perfect number can produce another even perfect number by the reversing of digits. In still other words, there are no recurrent palindromes in the set of even perfect numbers. This argument, in and of itself, does not preclude the possibility of reciprocally palindromic even perfect numbers. (However, the reciprocal case has an easy disposition if the number of digits is even as divisibility by 11 would be implied.) Clearly, no even perfect number can have a prime (Mersenne) divisor of 11 (i.e., 2^n-1 is never equal to 11 for integral n). The case for an odd number of digits proves more difficult.

As is well known, many unsolved problems surround the tantalizing class of numbers given the name "perfect." For example, the cardinality of the set of even perfect numbers is today unknown. Thirty-two have now been found, the largest known being

$$(2^{756838})(2^{756839}-1).$$

It is based on a Mersenne prime of $227,832$ digits and boggles the mind when compared to the four such numbers known in ancient times. Today no odd perfect numbers are known. Should any exist, they would prove in excess of 10^{300} (the cube of a googol).

A more challenging question involving recurrent palindromes concerns the set of perfect numbers in general (whether even or odd). Is it possible for the reversal of the digits of an even perfect number to yield an odd perfect number? For example, the gargantuan perfect number

$$2^{21700}(2^{21701} - 1)$$

has a last digit of 6 but a leading digit of 1. Is an odd perfect number formed by digital reversal? Though unanswered, this question (especially when generalized) provides an interesting variation within the broader domain of discussion (of perfect numbers regardless of parity).

5. A Digital Mix. Mathematics is often found in the strangest of places. As mentioned earlier, word patterns easily suggest number patterns. Consider now the digital counterpart to alphabet soup. Rather than reverse the order of letters, suppose letters can be permuted in any manner whatever. Such close-at-hand anagram examples as "range" and "anger," "least" and "steal," "state" and "taste," or "night" and "thing" come quickly to mind. The mathematical problem thus raised concerns numbers in a given category and their yielding numbers in the same category by some digital rearrangement. Primes provide an excellent illustration (such as 241 and 421), or the absolute prime (numbers which are prime regardless of the digital arrangement such as 199 and 331 and 733, etc.). The latter is much akin to the absolute anagram (such as "no" and "oh") which culminates in valid words by all permutations of letters.

Since no two even perfect numbers can have the same number of digits, then no arrangement of digits in one even perfect number can give rise to another even perfect number. The fact that 496, for example, is perfect, leads immediately to the conclusion that 469, 649, 694, 946, and 964 are not. It is a case of near absoluteness but in a negative manner. [By way of comparison, multiples of 9 afford an example of absoluteness but in a positive setting.] Various other number classes allow for a similar kind of analysis.

6. Summary. Linguistic units called "words" have letters as their building blocks. Correspondingly, arithmetic units called "numerals" have digits as their fundamental elements of construction. The fascination which frequently attaches to "words" is easily extended to the world of numbers. Students of language may note that the word "instantaneous" contains all of the vowels and thus try to find the first such word, in an alphabetical ordering, which has this property. Students of mathematics may likewise note that the prime

$$86,759,222,313,428,390,812,218,077,095,850,708,048,977$$

contains all ten of the digits at least once and accordingly try to find the smallest full-digit prime. [The set of full-digit primes is known to be infinite.] Or, in noting that the tenth perfect number, shown earlier, contains all of the digits, resultingly look for the next full-digit perfect number. Taking a closer look at these and similar digital arrangements opens the door to a vast assortment of questions concerning patterns. Some are easy to answer; some are not. So often, the open door leads to the unification of concepts, mathematical discovery, and a deeper insight.

References

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