SET VALUED MAPPINGS ON THE REALS

Kandasamy Muthuvel

University of Wisconsin-Oshkosh

1. Introduction. Let c be a fixed positive real number and let k be a cardinal smaller than $2^{\aleph_0}$. Let f, g, and h be functions from the set of all real numbers, $\mathbb{R}$, into the family of all subsets of $\mathbb{R}$, $P(\mathbb{R})$, such that, for every r in $\mathbb{R}$, $f(r)$ is a closed set of Lebesgue measure less than c, $g(r)$ is a nowhere dense subset of $\mathbb{R}$, and the cardinality of $h(r)$ is smaller than k. Two real numbers x and y are said to be free (independent) for f, if $x \notin f(y)$ and $y \notin f(x)$. A set of real numbers is said to be free for f, if every pair of real numbers in this set is free. It is known that each of these functions f, g, and h admits an infinite free set (see [5], [1], and [3]). The purpose of this paper is to investigate free sets (for mappings) when the above mentioned conditions are mixed. For example, using some set-theoretic axioms, we show that the function $W: \mathbb{R} \to P(\mathbb{R})$, defined by $W(r) = g(r) \cup h(r)$, admits an infinite free set. But the function $W: \mathbb{R} \to P(\mathbb{R})$, defined by $W(r) = f(r) \cup g(r)$, need not admit an infinite free set.

Throughout this paper, the set of all real numbers, the set of all positive integers, and the family of all subsets of the reals are denoted by $\mathbb{R}$, $\mathbb{N}$, and $P(\mathbb{R})$ respectively. The cardinality of any set A is denoted by $|A|$. k^+ is the cardinal successor of k.

Definition 1. Two real numbers x and y are said to be free for a function $f: \mathbb{R} \to P(\mathbb{R})$ if $x \notin f(y)$ and $y \notin f(x)$. A set of real numbers is said to be free for f if every pair of real numbers in this set is free.

Philosophical work concerning free sets can be found in [2].

Definition 2. Any countable union of nowhere dense subsets of $\mathbb{R}$ is called a set of first category (or meager). Any subset of $\mathbb{R}$ that is not of first category is called a set of second category (non meager). The complement of a first category subset of $\mathbb{R}$ is called a residual set. A set M is everywhere of second category if $M \cap I$ is of second category for every non-empty open interval I.

Definition 3. Martin's Axiom states that no compact Hausdorff space with the ccc (recall that a topological space has the ccc means that it has no uncountable collection of pairwise disjoint open sets) can be the union of less than $2^{\aleph_0}$ nowhere dense subsets.

The following example shows that if f and g are functions from $\mathbb{R}$ into $P(\mathbb{R})$ admitting infinite free sets, then the function W, defined by $W(r) = f(r) \cup g(r)$, need not admit an infinite free set.

Example 1. Let G be an infinite subset of $\mathbb{R}$ such that the complement of G is infinite. Define functions f and g from $\mathbb{R}$ into $P(\mathbb{R})$ by

$$f(r)=\begin{cases}\mathbb{R}, & \text{if } r\in G;\\ \emptyset, & \text{if } r\notin G\end{cases}$$

and

$$g(r)=\begin{cases}\emptyset, & \text{if } r\in G;\\ \mathbb{R}, & \text{if } r\notin G.\end{cases}$$

Then f admits an infinite free set $\mathbb{R}\backslash G$, and g admits an infinite free set G. But the function $W\colon\mathbb{R}\to P(\mathbb{R})$, defined by $W(r)=f(r)\cup g(r)$, does not admit an infinite free set, and indeed W does not admit a free pair.

However, the following theorem shows that if the category condition and the cardinality condition are mixed, more precisely, if f and g are functions from $\mathbb{R}$ into $P(\mathbb{R})$ such that, for each r in $\mathbb{R}$, $f(r)$ is a nowhere dense subset of $\mathbb{R}$ and $|g(r)|<k$ for a fixed cardinal $k<|\mathbb{R}|$, then the function $W\colon\mathbb{R}\to P(\mathbb{R})$, defined by $W(r)=f(r)\cup g(r)$, admits an infinite free set, provided Martin's Axiom holds. It is shown in [1] that f admits an infinite free set and it is shown in [3] and [4] that g admits an infinite free set.

Theorem 1. Let k be a cardinal smaller than $|\mathbb{R}|$. Let f and g be functions from $\mathbb{R}$ into $P(\mathbb{R})$ such that, for every r in $\mathbb{R}$, $f(r)$ is a nowhere dense subset of $\mathbb{R}$ and $|g(r)|<k$. Then, under the assumption of Martin's Axiom, the function $W\colon\mathbb{R}\to P(\mathbb{R})$, defined by $W(r)=f(r)\cup g(r)$, admits an infinite free set and indeed an everywhere dense free set.

Lemma 1. Martin's Axiom implies that given any non-empty open interval I there is a second category subset S of I such that every subset of S of cardinality the continuum is of second category.

Proof. Let Ω be the initial ordinal number whose cardinality is $|\mathbb{R}|$. In addition, let $\{N_\alpha\mid\alpha<\Omega\}$ be the set of all closed nowhere dense subsets of $\mathbb{R}$. Inductively choose x_α in $I\backslash\cup_{\beta<\alpha}N_\beta$. To prove that $\{x_\alpha\mid\alpha<\Omega\}$ is the required set, let H be any subset of $\{x_\alpha\mid\alpha<\Omega\}$ such that $|H|=|\mathbb{R}|$. Suppose to the contrary that H is first category. Then H is a countable union of nowhere dense subsets of $\mathbb{R}$ and consequently, by using the fact that the immediate successor of an increasing sequence of type ω of ordinal numbers less than Ω is less than Ω, $H\subseteq\cup_{\beta<\sigma}N_\beta$ for some $\sigma<\Omega$. Since H is a subset of $\{x_\alpha\mid\alpha<\Omega\}$, $|H|=|\mathbb{R}|$ and $\sigma<\Omega$, there exists an ordinal number γ such that $\sigma<\gamma<\Omega$ and $x_\gamma\in H$, which contradicts the choice of $x_\gamma\in I\backslash\cup_{\beta<\sigma}N_\beta$. Thus, $\{x_\alpha\mid\alpha<\Omega\}$ is the required set.

The following lemma can be found in [4, Th. 2.5].

Lemma 2. Let S be an infinite set and let $\mathcal{F}$ be a family of subsets of S such that $|\mathcal{F}|<|S|$ and $|X|=|S|$ for every $X\in\mathcal{F}$. Let k be a cardinal smaller than $|S|$. Then every function $g: S \to \mathcal{P}(S)$ with the property $|g(r)| < k$ for every $r \in S$, admits a free set $A \subseteq S$ such that for every $X \in \mathcal{F}$, $|A \cap X| = |S|$ for every $X \in \mathcal{F}$.

As an application of Lemmas 1 and 2, we get the following.

Lemma 3. Let k be a cardinal smaller than $|\mathbb{R}|$ and let $g: \mathbb{R} \to P(\mathbb{R})$ be a function such that $|g(r)| < k$ for every r in $\mathbb{R}$. Then, under the assumption of Martin's Axiom, g admits a free set A that is everywhere of second category.

Proof. Let $\{I_1, I_2, \dots, I_n, \dots\}$ be the set of all nonempty open intervals having rational end points. According to Lemma 1, there is a second category subset S_n of I_n such that every subset of S_n of cardinality the continuum is of second category. Let $S = \cup_{n \in \mathbb{N}} S_n$ and $\mathcal{F} = \{S_n \mid n \in \mathbb{N}\}$. Then by Lemma 2, g admits a free set $A \subseteq S$ such that

$$|A \cap S_n| = |S| = |\mathbb{R}|.$$

Since, for every $n \in \mathbb{N}$, $A \cap S_n \subseteq A \cap I_n$, and $A \cap S_n$ is of second category, $A \cap I_n$ is of second category and consequently A is everywhere second category. This completes the proof of the lemma.

Proof of the Theorem. By Lemma 3, g admits a free set A that is everywhere of second category. That is, for any two real numbers x, y in A, $x \notin g(y)$ and $y \notin g(x)$. It follows from the proof of Theorem 1 in [1] (see Remark 1 in [1]) that f admits an everywhere dense free subset B of A. That is, $x \notin f(y)$ and $y \notin f(x)$ for any two real numbers x, y in B. Thus, for any two real numbers x, y in B, $x \notin f(y) \cup g(y) = W(y)$ and $y \notin f(x) \cup g(x) = W(x)$. Therefore, B is an infinite (indeed everywhere dense) free set for W. Thus, the proof is complete.

We prove the following theorem without the assumption of Martin's Axiom.

Theorem 2. Let k be a cardinal smaller than $|\mathbb{R}|$. If $f: \mathbb{R} \to P(\mathbb{R})$ is a function such that, for every r in $\mathbb{R}$, $f(r)$ is a nowhere dense subset of $\mathbb{R}$ or $|f(r)| < k$, then f admits an infinite free set.

Proof. Let $M = \{r \mid f(r) \text{ is nowhere dense. }\}$. If M is everywhere second category, then by [1, Remark 1], M contains an infinite free subset. If M is not everywhere second category, there is a nonempty open interval I such that $M \cap I$ is first category. Hence, $|I \backslash M| = |I| = |\mathbb{R}|$, and since $|f(r)| < k$ for every $r \in I \backslash M$, by [4, Th. 2.5], there is an infinite free subset of $I \backslash M$.

The following theorem shows that the conclusion of Theorem 2 is not true if the category condition and the measure-theoretic condition are mixed.

Theorem 3. Let c be a positive real number. Then there is a function $f: \mathbb{R} \to P(\mathbb{R})$ such that, for every $r \in \mathbb{R}$, $f(r)$ is a closed set of (Lebesgue) measure less than c or $f(r)$ is a nowhere dense set, and the cardinality of any free set is at most 2.

Proof. It is known that there is a measure zero set M such that $\mathbb{R} \backslash M$ is first category [6, Th. 1.6]. It follows from [7, p. 56] that there exists a sequence (I_n) of closed intervals such that $M \subseteq \cup_{n \in \mathbb{N}} I_n$ and measure of $\cup_{n \in \mathbb{N}} I_n$ is less than c. If $r \in \cup_{n \in \mathbb{N}} I_n$, define $f(r) = \cup_{n=1}^{m} I_n$, where m is the smallest positive integer for which $r \in I_m$. Since $\mathbb{R} \backslash M$ is first category, $\mathbb{R} \backslash \cup_{n \in \mathbb{N}} I_n$ is first category and $\mathbb{R} \backslash \cup_{n \in \mathbb{N}} I_n = \cup_{n \in \mathbb{N}} N_n$, where N_n is nowhere dense. If $r \in \mathbb{R} \setminus \cup_{n \in \mathbb{N}} I_n$, define $f(r) = \cup_{n=1}^{m} N_n$, where m is the smallest positive integer for which $r \in N_m$. It can be easily seen that, for all $r \in \mathbb{R}$, $f(r)$ is nowhere dense or $f(r)$ is a closed set of measure less than c. It is clear that if $x, y \in \cup_{n \in \mathbb{N}} I_n$ or $x, y \in \cup_{n \in \mathbb{N}} N_n$, then $y \in f(x)$ or $x \in f(y)$. But if $x \in \cup_{n \in \mathbb{N}} I_n$ and $y \in \cup_{n \in \mathbb{N}} N_n$, then $x \notin f(y)$ and $y \notin f(x)$. Thus, f admits no free set of size more than 2.

It is interesting to compare the following corollary with Theorem 1.

Corollary 1. Let c be a positive real number. Let f and g be functions from $\mathbb{R}$ into $P(\mathbb{R})$ such that, for every $r \in \mathbb{R}$, $f(r)$ is a closed set of measure less than c and $g(r)$ is a nowhere dense subset of $\mathbb{R}$. Then the function $W: \mathbb{R} \to P(\mathbb{R})$, defined by $W(r) = f(r) \cup g(r)$, need not admit an infinite free set (indeed it need not admit a free set of size 3).

Proof. Proof of the corollary follows from the proof of Theorem 3 with appropriate minor modifications. As in the proof of Theorem 3, if $r \in \cup_{n \in \mathbb{N}} I_n$, define $f(r) = \cup_{n=1}^{m} I_n$, where m is the smallest positive integer such that $r \in I_m$. Otherwise, define $f(r) = \emptyset$. Similarly if $r \in \cup_{n \in \mathbb{N}} N_n$, define $g(r) = \cup_{n=1}^{m} N_n$, where m is the smallest positive integer such that $r \in N_m$. Otherwise, define $g(r) = \emptyset$. Then it is not hard to see that W admits no free set of size 3.

Even with the assumption of Martin's Axiom, we are unable to solve the following problem.

Problem 1. Let k be a cardinal smaller than $|\mathbb{R}|$ and let c be a positive real number. Let f and g be functions from $\mathbb{R}$ into $P(\mathbb{R})$ such that, for every $r \in \mathbb{R}$, $f(r)$ is a closed set of measure less than c and $|g(r)| < k$. Does there exist an infinite free set for the function $W: \mathbb{R} \to P(\mathbb{R})$, defined by $W(r) = f(r) \cup g(r)$?

However, under the assumption of Martin's Axiom, we prove the following theorem which is the measure-theoretic analogue of Theorem 2.

Theorem 4. Let k be a cardinal smaller than $|\mathbb{R}|$ and let c be a positive real number. If $f: \mathbb{R} \to P(\mathbb{R})$ is a function such that, for every $r \in \mathbb{R}$, $f(r)$ is a closed set of measure less than c or $|f(r)| < k$. Then, under the assumption of Martin's Axiom, f admits an infinite free set.

Proof. Let $M = \{r \mid |f(r)| < k\}$. If $|M| \geq k^+$, then by [4, Th. 2.5], f admits an infinite free set. Suppose $|M| < k^+$. Theorem 3 in [8] together with the statement in [8, line 29 on p. 611] implies that any set of reals of cardinality less than $|\mathbb{R}|$ is of measure zero. Since $|M| < k^+ \leq |\mathbb{R}|$, the measure of M is zero. By [7, p. 56], there is a sequence (I_n) of closed intervals such that $M \subset \cup_{n \in \mathbb{N}} I_n$ and the measure of $\cup_{n \in \mathbb{N}} I_n$ is less than c. Define $F(r) = f(r)$ if $r \in \mathbb{R} \setminus \cup_{n \in \mathbb{N}} I_n$, and $F(r) = \cup_{n=1}^{m} I_n$ if m is the smallest positive integer for which $r \in I_m$. Then, for each $r \in \mathbb{R}$, $F(r)$ is a closed set of measure less than c. Since, for each $r \in \mathbb{R}$, $F(r)/c$ is a closed set of measure less than 1, $F(cr)/c$ is a closed set of measure less than 1 for each $r \in \mathbb{R}$. Now by Corollary 1 in [5], the function $F(cr)/c$ admits an infinite free set, say M. That is, for any two real numbers a, b in M, $a \notin F(cb)/c$ and $b \notin F(ca)/c$, which is equivalent to $ca \notin F(cb)$ and $cb \notin F(ca)$. Hence, cM is an infinite free set for the function F. Since, $x \in F(y)$ or $y \in F(x)$ for $x, y \in \cup_{n \in \mathbb{N}} I_n$ (in other words, any two elements in $\cup_{n \in \mathbb{N}} I_n$ are not free), and F admits an infinite free set, F admits an infinite free subset of $\mathbb{R}\setminus \cup_{n\in\mathbb{N}} I_n$. But F and f agree on $\mathbb{R}\setminus \cup_{n\in\mathbb{N}} I_n$. Thus, f admits an infinite free set.

Theorem 5. There is a function $f\colon \mathbb{R} \to P(\mathbb{R})$ such that, for every $r \in \mathbb{R}$, $f(r)$ is a proper subgroup of the additive group $\mathbb{R}$, and the cardinality of any free set is at most 1.

Proof. Well order $\mathbb{R} = (r_\xi)_{\xi<\Omega}$. For each $\xi < \Omega$, define $f(r_\xi)$ to be the set of all finite linear combinations of elements from $\{r_\nu \mid \nu < \xi\}$ with integer coefficients. Since $|f(r_\xi)| < |\mathbb{R}|$, $f(r_\xi)$ is a proper subgroup of $\mathbb{R}$. Since $r_\nu \in f(r_\mu)$ for $\nu < \mu$, f does not admit a free pair and thus, the proof is complete.

Recall that a subset H of $\mathbb{R}$ is called a Hamel basis for $\mathbb{R}$ if every element of $\mathbb{R}$ is a finite linear combination of elements from H with rational coefficients and the set H is linearly independent over the set of rationals. As our final result we have the following theorem.

Theorem 6. Every function $f\colon \mathbb{R} \to P(\mathbb{R})$ with the property that, for every $r \in \mathbb{R}$, $f(r)$ is a Hamel basis, admits an infinite free set.

Proof. Suppose, to the contrary, that f does not admit an infinite free set. Let M_1 be a maximal free subset of $\mathbb{Q}$ and let M_2 be a maximal free subset of $\mathbb{Q}\setminus(M_1 \cup \cup_{m\in M_1} f(m))$. Note that since $f(r)$ is a Hamel basis for $\mathbb{R}$, $|f(r)\cap\mathbb{Q}| \leq 1$. So, if we let $M = M_1 \cup M_2$, then $\mathbb{Q}\setminus(M \cup \cup_{m\in M} f(m)) \neq \emptyset$. Let $x \in \mathbb{Q}\setminus(M \cup \cup_{m\in M} f(m))$. By the maximality of M_1, there exists an element, say $a \in M_1$, such that $a \in f(x)$, and by the maximality of M_2, there exists an element, say $b \in M_2$, such that $b \in f(x)$. Hence, $f(x)$ contains at least two rational numbers, namely a and b. This contradicts the fact that $f(x)$ is linearly independent over the set of rationals. Thus, the proof of the theorem is complete.

References

1. F. Bagemihl, "The Existence of an Everywhere Dense Independent Set," *Michigan Mathematical Journal*, 20 (1973), 1–2.
2. C. Freiling, "Axioms of Symmetry: Throwing Darts at the Real Number Line," *The Journal of Symbolic Logic*, 51 (1986), 190–200.
3. A. Hajnal, "Proof of a Conjecture of S. Ruziewicz," *Fundamenta Mathematicae*, 50 (1961), 123–128.
4. S. Hechler, "Directed Graphs Over Topological Spaces: Some Set-Theoretical Aspects," *Israel Journal of Mathematics*, 11 (1972), 231–247.
5. L. Newelski, J. Pawlikowski, and N. Seredynski, "Infinite Free Set for Small Measure Set Mappings," *Proceedings of the American Mathematical Society*, 100 (1987), 335–339.
6. J. C. Oxtoby, *Measure and Category*, 2nd edition, Springer-Verlag, New York, 1980.
7. H. L. Royden, *Real Analysis*, 2nd edition, Macmillan Company, New York, 1968.
8. J. R. Shoenfield, "Martin's Axiom," *American Mathematical Monthly*, 82 (1975), 610–617.