# ZERO-DIMENSIONAL NEARNESS SPACES AND EXTENSIONS OF TOPOLOGICAL SPACES

Rhonda L. McKee

Central Missouri State University

**Abstract.** Zero-dimensional nearness spaces are defined. It is shown that the subtopological zero-dimensional nearness spaces are precisely those nearness spaces which are induced by a strict zero-dimensional extension of the underlying topology.

**1. Introduction.** Nearness structures were first introduced by Herrlich in [4] and have proven to be quite useful, especially in the study of extensions of topological spaces (see [1], [2], and [3]).

<u>Definition 1</u>. Let $X$ be a set and $\mu$ a collection of covers of $X$, called uniform covers, which satisfy the following conditions:

(N1) $\mathcal{A} \in \mu$ and $\mathcal{A}$ refines $\mathcal{B}$ implies $\mathcal{B} \in \mu$.

(N2) $\{X\} \in \mu$.

(N3) If $\mathcal{A} \in \mu$ and $\mathcal{B} \in \mu$, then $\mathcal{A} \wedge \mathcal{B} \in \mu$, where $\mathcal{A} \wedge \mathcal{B} = \{A \cap B \mid A \in \mathcal{A} \text{ and } B \in \mathcal{B}\}$.

(N4) If $\mathcal{A} \in \mu$, then $\text{int } \mathcal{A} \in \mu$, where $\text{int } \mathcal{A} = \{\text{int } A \mid A \in \mathcal{A}\}$ and $\text{int } A = \{x \in X \mid \{A, X \setminus x\} \in \mu\}$.

Then, $(X, \mu)$ is called a nearness space.

<u>Definition 2</u>. Given a set $X$, a map, $A \to \text{int } A$, of $P(X)$ into $P(X)$ is called an interior operator in $X$ if the following conditions hold:

(1) $\text{int } A \subset A$.

(2) $\text{int } (\text{int } A) = \text{int } A$.

(3) $\text{int } (A \cap B) = \text{int } A \cap \text{int } B$.

(4) $\text{int } X = X$.

Each interior operator in a set $X$ defines a topology for $X$, in which a set $A$ is open if and only if $\text{int } A = A$. It is easily verified that if $(X, \mu)$ is a nearness space, then the operator int in Definition 1 (N4) is an interior operator in $X$ and thus defines a topology on $X$. This topology is denoted by $t_\mu$. It is a symmetric topology. That is, if $x \in \text{cl}\{y\}$ then $y \in \text{cl}\{x\}$. Conversely, if $(X, t)$ is any symmetric topological space, then there exists at least one compatible nearness structure on $X$.

Definition 3. A nearness space $(X, \mu)$ is said to be topological if every $\mathcal{A} \subset P(X)$ with $\bigcup \mathrm{int}\, \mathcal{A} = X$ is in $\mu$.

Definition 4. A dense embedding $e\colon (X, t) \to (Y, s)$ of $(X, t)$ into a topological space $(Y, s)$ is called an extension of a topological space $(X, t)$ into a topological space $(Y, s)$.

Definition 5. An extension $e$ from $(X, t)$ into $(Y, s)$ is a strict extension if $\{\mathrm{cl}\ e(A) \mid A \subset X\}$ is a base for the closed sets in $Y$.

Several different types of structures are induced on $X$ by an extension, the most useful of which is the nearness structure induced on $X$ by $e$. Bentley and Herrlich [1] have shown that a nearness structure induced by a strict extension contains so much information that the extension $(Y, s)$ can be recovered from it by simply taking the completion.

Definition 6. Let $e$ be an extension of a topological space $(X, t)$ into a topological space $(Y, s)$. The nearness structure induced on $X$ by $e$ is determined by: $\mathcal{A}$ is a uniform cover of $X$ if and only if $\mathcal{A} \subset P(X)$ and there exists an open cover $\mathcal{B}$ of $Y$ such that $\{B \cap e(X) \mid B \in \mathcal{B}\}$ refines $e(\mathcal{A})$.

Also in [1], Bentley and Herrlich study various properties of topological spaces with the intention of determining which nearness spaces give rise, by taking the completion, to extensions of the underlying topology with the restriction that the extension have the given topological property. In this paper, we characterize those subtopological nearness spaces which give rise to strict zero-dimensional extensions of the underlying topological space.

## 2. Zero-Dimensional Extensions.

Definition 7. A nearness space $(X, \mu)$ is zero-dimensional if for every $\mathcal{U} \in \mu$ there exists $\mathcal{V} \in \mu$ such that $\mathcal{U}$ refines $\mathcal{V}$ and for every $V \in \mathcal{V}$, $\{V, X \setminus V\} \in \mu$.

Theorem 1. If $(X, \mu)$ is a 0-dimensional nearness space, then $(X, t_\mu)$ is a 0-dimensional topological space.

Proof. Let $x \in X$ and let $U$ be an open neighborhood (in $t_\mu$) of $x$. Define, $\mathcal{U} = \{U, X \setminus x\}$. Then $\mathcal{U} \in \mu$. Since $(X, \mu)$ is 0-dimensional, there exists $\mathcal{V} \in \mu$ such that $\mathcal{V}$ refines $\mathcal{U}$ and for each $V \in \mathcal{V}$, $\{V, X \setminus V\} \in \mu$. Choose $V_0 \in \mathcal{V}$ such that $x \in V_0$. Then, $x \in V_0 \subset U$ (since either $V_0 \subset U$ or $V_0 \subset X \setminus x$), and $\{V_0, X \setminus x\} \in \mu$. Now, if $y \in V_0$, $\{V_0, X \setminus V_0\}$ refines $\{V_0, X \setminus y\}$, so $\{V_0, X \setminus y\} \in \mu$. Thus, $y \in \mathrm{int}\ V_0$, and $V_0$ is open.

But, if $y \in X \setminus V_0$, then $\{V_0, X \setminus V_0\}$ refines $\{X \setminus y, X \setminus V_0\}$. So, $\{X \setminus y, X \setminus V_0\} \in \mu$ and $y \in \mathrm{int}\ (X \setminus V_0)$. Thus, $X \setminus V_0$ is also open.

Theorem 2. Suppose $(X, \mu)$ is a nearness space such that $\mu$ contains all finite covers which are open in $t_\mu$. Then $(X, \mu)$ is 0-dimensional if and only if $(X, t_\mu)$ is 0-dimensional.

Proof. If $(X, \mu)$ is 0-dimensional, then $(X, t_\mu)$ is 0-dimensional by Theorem 1.

Suppose $(X, t_\mu)$ is 0-dimensional and let $\mathcal{U} \in \mu$. Then int $\mathcal{U} \in \mu$. If $x \in X$, there exists $U_x \in \mathcal{U}$ such that $x \in \text{int } U_x$. Since $(X, t_\mu)$ is 0-dimensional, there exists $V_x$ which is both open and closed such that $x \in V_x \subset \text{int } U_x$. Let $\mathcal{V} = \{V_x \mid x \in X\}$. Then $\mathcal{V}$ refines $\mathcal{U}$ and for each $V_x \in \mathcal{V}$, $\{V_x, X \setminus V_x\}$ is a finite open cover of $X$, and hence, a uniform cover.

Notice that any topological nearness space would contain all finite open covers, since it would contain all open covers.

Definition 8. A nearness space $(X, \mu)$ is regular if $\mathcal{U} \in \mu$ implies that $\{B \subset X \mid \{U, X \setminus B\} \in \mu \text{ for some } U \in \mathcal{U}\} \in \mu$.

The following proposition was proven by Bentley and Herrlich [1].

Proposition 1. Let $(X, \mu)$ be a dense nearness subspace of a regular nearness space $(Y, \nu)$. For each $A \subset X$, let op $A = \text{int}_\nu[A \cup (Y \setminus X)] = Y \setminus \text{cl}_\nu(X \setminus A)$, the largest open subset $B$ of $Y$ with $B \cap X = \text{int}_\nu A$. Then for any $\mathcal{A} \subset P(Y)$, $\mathcal{A} \in \nu$ if and only if $\{B \subset X \mid \text{op } B \subset A \text{ for some } A \in \mathcal{A}\} \in \mu$.

Lemma 1. Every 0-dimensional nearness space is regular.

Proof. Let $(X, \mu)$ be a 0-dimensional nearness space and let $\mathcal{U} \in \mu$. Then, there exists $\mathcal{V} \in \mu$ such that $\mathcal{V}$ refines $\mathcal{U}$ and for all $V \in \mathcal{V}$, $\{V, X \setminus V\} \in \mu$. Thus, if $V \in \mathcal{V}$, there exists $U \in \mathcal{U}$ such that $V \subset U$. Hence, $\{V, X \setminus V\}$ refines $\{U, X \setminus V\}$, so that $\{U, X \setminus V\} \in \mu$. This shows that $\mathcal{V} \subset \{B \subset X \mid \{U, X \setminus B\} \in \mu \text{ for some } U \in \mathcal{U}\}$, and hence, $\{B \subset X \mid \{U, X \setminus B\} \in \mu \text{ for some } U \in \mathcal{U}\} \in \mu$.

Definition 9. A nearness space $(X, \mu)$ is a subspace of a nearness space $(Y, \nu)$ if $X \subset Y$ and $\mu = \{\mathcal{V} \wedge \{X\} \mid \mathcal{V} \in \nu\}$.

Lemma 2. If a dense nearness subspace of a regular nearness space is 0-dimensional, then the nearness space itself is 0-dimensional.

Proof. Let $(X, \mu)$ and $(Y, \nu)$ be nearness spaces such that $\text{cl}_\nu X = Y$, $\mu$ is 0-dimensional and $\nu$ is regular. Let $\mathcal{U} \in \mu$. Then, by Proposition 1, $\mathcal{B} = \{B \subset X \mid \text{op } B \subset U \text{ for some } U \in \mathcal{U}\} \in \mu$. Since $\mu$ is 0-dimensional, there exists $\mathcal{V} \in \mu$ such that $\mathcal{V}$ refines $\mathcal{U}$ and, for every $V \in \mathcal{V}$, $\{V, X \setminus V\} \in \mu$. Let op $\mathcal{V} = \{\text{op } V \mid V \in \mathcal{V}\}$. Then, by Proposition 1, op $\mathcal{V} \in \nu$.

We now show that op $\mathcal{V}$ refines $\mathcal{U}$. Let op $V \in$ op $\mathcal{V}$. Since $\mathcal{V}$ refines $\mathcal{B}$, there exists $B \in \mathcal{B}$ such that $V \subset B$. Thus, op $V \subset$ op $B$, and op $B \subset U$ for some $U \in \mathcal{U}$. So, op $\mathcal{V}$ refines $\mathcal{U}$.

We must also show that $\{\text{op } V, Y \setminus \text{op } V\} \in \nu$ for all $V \in \mathcal{V}$. By Proposition 1, $\{V, X \setminus V\} \in \mu$ implies that $\{\text{op } V, \text{op } (X \setminus V)\} \in \nu$. We show that $\text{op } (X \setminus V) \subset Y \setminus \text{op } V$. Notice that $Y \setminus \text{op } V = \text{cl}_\nu(X \setminus V)$ and $\text{op } (X \setminus V) = Y \setminus \text{cl}_\nu V$. Let $y \in \text{op } (X \setminus V) = Y \setminus \text{cl}_\nu V$ and suppose $y \notin Y \setminus \text{op } V = \text{cl}_\nu(X \setminus V)$. Then $y \notin \text{cl}_\nu V \cup \text{cl}_\nu(X \setminus V) = \text{cl}_\nu(V \cup X \setminus V) = \text{cl}_\nu X = Y$, a contradiction. Thus, $\{\text{op } V, \text{op } (X \setminus V)\}$ refines $\{\text{op } V, Y \setminus \text{op } V\}$ which implies that $\{\text{op } V, Y \setminus \text{op } V\} \in \nu$. This shows that $\nu$ is 0-dimensional.

Corollary 1. The completion of a 0-dimensional space is 0-dimensional. (For a description of the completion of a nearness space, see [4].)

Lemma 3. Every subspace of a 0-dimensional nearness space is 0-dimensional.

Proof. Let $(X, \mu)$ be a subspace of a 0-dimensional nearness space $(Y, \nu)$. Let $\mathcal{U} \in \mu$. Then, there exists $\mathcal{V} \in \nu$ such that $\mathcal{U} = \{X\} \wedge \mathcal{V}$. Since $\nu$ is 0-dimensional, there exists $\mathcal{W} \in \nu$ such that $\mathcal{W}$ refines $\mathcal{V}$ and for all $W \in \mathcal{W}$, $\{W, Y \setminus W\} \in \nu$. But then, $\{X\} \wedge \mathcal{W} \in \mu$ and $\{X \cap W, X \cap (Y \setminus W)\} \in \mu$ for each $W \in \mathcal{W}$, which is equivalent to $\{X \cap W, X \cap (X \setminus W)\} \in \mu$. Also, $\{X\} \wedge \mathcal{W}$ refines $\{X\} \wedge \mathcal{V} = \mathcal{U}$ since $\mathcal{W}$ refines $\mathcal{V}$. Thus, $(X, \mu)$ is 0-dimensional.

Propositions 2 through 4 are all due to Bentley and Herrlich [1].

Proposition 2. For any nearness space $(X, \mu)$, the following are equivalent:

(i) $\mu$ is a nearness structure induced on $X$ by a strict extension.

(ii) The completion of $(X, \mu)$ is topological.

Definition 10. A nearness space is called concrete provided it satisfies the equivalent conditions of Proposition 2.

Definition 11. A nearness space $(X, \mu)$ is called subtopological if it is a nearness subspace of a topological nearness space.

Proposition 3. Every regular, subtopological nearness space is concrete.

Proposition 4. If $\mu$ is the nearness structure induced on $X$ by a strict extension $e\colon X \to Y$, then the completion of $(X, \mu)$ is topological and the canonical embedding, $e'$, of $X$ into the completion $(X^*, \mu^*)$ of $(X, \mu)$ is equivalent to $e$. That is, there exists a homeomorphism $h\colon Y \to X^*$ with $h \circ e = e'$.

The following theorem is the main result of this paper.

Theorem 3. Let $(X, \mu)$ be a nearness space. Then the following are equivalent.

(i) $\mu$ is the nearness structure induced by a strict 0-dimensional extension of $(X, t_\mu)$.

(ii) $(X, \mu)$ is subtopological and 0-dimensional.

(iii) The completion of $(X, \mu)$ is both topological and 0-dimensional.

Proof. (iii) implies (ii). This follows from Lemma 3.

(ii) implies (iii). By Corollary 1, the completion of a 0-dimensional space is 0-dimensional, and, since 0-dimensional spaces are regular (Lemma 1), Proposition 3 implies that $(X, \mu)$ is concrete.

(iii) implies (i). By Proposition 2, $\mu$ is induced by a strict extension. Then, by Proposition 4, the extension is homeomorphic to the completion, and hence is 0-dimensional (in the topological sense).

(i) implies (iii). Let $Y$ be the strict 0-dimensional extension of $X$. Then by Proposition 4, the completion $(X^*, \mu^*)$ is topological and homeomorphic to $Y$. Thus, $(X^*, t_{\mu^*})$ is 0-dimensional (in the topological sense), and therefore $(X^*, \mu^*)$ is 0-dimensional, since it is topological.

## *References*


1. H. L. Bentley and H. Herrlich, "Extensions of Topological Spaces," Topology, Proc. Ninth Annual Spring Conf., Memphis State Univ., Memphis, Tenn. 1975, *Lecture Notes in Pure and Appl. Math.*, Marcel Dekker, New York, 24 (1976), 129–184.
2. J. W. Carlson, "H-Closed and Countably Compact Extensions," *Pacific J. Math.*, 81 (1979), 317–326.
3. J. W. Carlson, "Baire Space Extensions," *Top. Appl.*, 15 (1983), 19–28.
4. H. Herrlich, "A Concept of Nearness," *General Topology and Appl.*, 5 (1975), 191–212.