ON THE EVALUATION OF MULTIPLE INTEGRALS

K. L. D. Gunawardena

University of Wisconsin - Oshkosh

1. Introduction. The students in my Mathematical Statistics course came across the following double integrals, and were unable to evaluate them using techniques that they had learned in calculus courses. The two integrals were

$$\int_0^1 \int_0^1 \frac{\theta^2 (x_1 x_2)^{\theta-1}}{\ln(x_1 x_2)} dx_1\, dx_2 \quad \text{and} \quad \int_0^\infty \int_0^\infty \frac{\theta^2 [(1+x_1)(1+x_2)]^{-(\theta+1)}}{\ln[(1+x_1)(1+x_2)]} dx_1\, dx_2$$

where $\theta > 0$. The computer algebra systems Derive and Mathematica were unable to evaluate the integrals. However, using results from Probability Theory the two integrals can be evaluated. We will generalize the two double integrals and instead consider the two multiple integrals

$$\int_0^1 \cdots \int_0^1 \frac{\theta^n (x_1 \cdots x_n)^{\theta-1}}{\ln(x_1 \cdots x_n)} dx_1 \cdots dx_n$$

and

$$\int_0^\infty \cdots \int_0^\infty \frac{\theta^n [(1+x_1)\cdots(1+x_n)]^{-(\theta+1)}}{\ln[(1+x_1)\cdots(1+x_n)]} dx_1 \cdots dx_n$$

where $\theta > 0$ and n is an integer greater than or equal to 2.

The key probability result needed to evaluate the integrals is the following.

Let $X_1, \ldots, X_n$ be n random variables with joint probability density function (pdf) $f(x_1, \ldots, x_n)$ and $Y = g(X_1, \ldots, X_n)$. The expected value $E(Y)$ can be obtained in one of two ways. We could directly calculate

$$Eg(X_1, \ldots, X_n) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(x_1, \ldots, x_n) f(x_1, \ldots, x_n) dx_1 \cdots dx_n$$

or we could also find the pdf $f_Y(y)$ of $Y = g(X_1, \dots, X_n)$ and

$$Eg(X_1, \dots, X_n) = E(Y) = \int_{-\infty}^{\infty} y f_Y(y) dy.$$

2. Evaluation of $\int_0^1 \cdots \int_0^1 \frac{\theta^n (x_1 \cdots x_n)^{\theta-1}}{\ln(x_1 \cdots x_n)} dx_1 \cdots dx_n$.

Let $X_1, \dots, X_n$ be a random sample of size n from a distribution with pdf $f(x) = \theta x^{\theta-1}$ if $0 < x < 1$, and zero otherwise; $\theta > 0$.

Define the random variable $Y = \left(\sum_{i=1}^{n} \ln X_i \right)^{-1}$.

Since $X_1, \dots, X_n$ is a random sample, the joint pdf of $X_1, \dots, X_n$ is

$$f(x_1, \dots, x_n) = \prod_{i=1}^{n} f(x_i) = \theta^n (x_1 \cdots x_n)^{\theta-1}, \quad 0 < x_i < 1$$

and

$$E\left(\sum_{i=1}^{n} \ln X_i \right)^{-1} = \int_0^1 \cdots \int_0^1 \left(\sum_{i=1}^{n} \ln x_i \right)^{-1} \theta^n (x_1 \cdots x_n)^{\theta-1} dx_1 \cdots dx_n$$

$$= \int_0^1 \cdots \int_0^1 \frac{\theta^n (x_1 \cdots x_n)^{\theta-1}}{\ln(x_1 \cdots x_n)} dx_1 \cdots dx_n.$$

If the random variable X has pdf $f(x) = \theta x^{\theta-1}$ when $0 < x < 1$, and zero otherwise, where $\theta > 0$, then the random variable $W = -\ln X$ has an exponential distribution with mean $1/\theta$, since the pdf of W, $f_W(w)$ is given by

$$f_W(w) = f(e^{-w}) \left| \frac{d}{dw} e^{-w} \right| = \theta (e^{-w})^{\theta-1} | - e^{-w} | = \theta e^{-\theta w}$$

for $w > 0$.

Since X_i, $i = 1, \ldots, n$ is a random sample $W_i = -\ln X_i$ are independent. Also, since W_i, $i = 1, \ldots, n$ are independent exponential random variables with mean $1/\theta$, $\sum_{i=1}^{n} W_i$ has a gamma distribution with parameters $1/\theta$ and n. Thus, the random variable $U = -\sum_{i=1}^{n} \ln X_i$ has pdf

$$f_U(u) = \frac{\theta^n u^{n-1} e^{-\theta u}}{\Gamma(n)} \quad u \geq 0, \quad \text{and zero otherwise,}$$

and

$$E(Y) = E(-1/U) = \int_0^\infty -\frac{1}{u} \frac{\theta^n u^{n-1} e^{-\theta u}}{\Gamma(n)} du$$

$$= -\frac{1}{\Gamma(n)} \int_0^\infty \theta^n u^{n-2} e^{-\theta u} du = -\frac{\theta \Gamma(n-1)}{\Gamma(n)} = -\frac{\theta}{n-1}.$$

Thus,

$$\int_0^1 \cdots \int_0^1 \frac{\theta^n (x_1 \cdots x_n)^{\theta - 1}}{\ln(x_1 \cdots x_n)} dx_1 \cdots dx_n = -\frac{\theta}{n-1}.$$

3. Evaluation of $\int_0^\infty \cdots \int_0^\infty \frac{\theta^n [(1+x_1)\cdots(1+x_n)]^{-(\theta+1)}}{\ln[(1+x_1)\cdots(1+x_n)]} dx_1 \cdots dx_n$.

Let $X_1, \ldots, X_n$ be a random sample of size n from a distribution with pdf $f(x) = \theta(1+x)^{-(\theta+1)}$ if $x > 0$, and zero otherwise, where $\theta > 0$.

Define the random variable

$$Y = \left(\sum_{i=1}^{n} \ln(1 + X_i) \right)^{-1}.$$

Since $X_1, \ldots, X_n$ is a random sample, the joint pdf of $X_1, \ldots, X_n$ is

$$f(x_1, \ldots, x_n) = \prod_{i=1}^{n} f(x_i) = \theta^n [(1+x_1)\cdots(1+x_n)]^{-(\theta+1)}, \quad x_i > 0$$

and

$$E\left(\sum_{i=1}^{n} \ln(1+X_i)\right)^{-1}$$

$$= \int_0^\infty \cdots \int_0^\infty \left(\sum_{i=1}^{n} \ln(1+x_i)\right)^{-1} \theta^n [(1+x_1)\cdots(1+x_n)]^{-(\theta+1)} dx_1 \cdots dx_n$$

$$= \int_0^\infty \cdots \int_0^\infty \frac{\theta^n [(1+x_1)\cdots(1+x_n)]^{-(\theta+1)}}{\ln[(1+x_1)\cdots(1+x_n)]} dx_1 \cdots dx_n.$$

If the random variable X has pdf $f(x) = \theta(1+x)^{-(\theta+1)}$ when $x > 0$, and zero otherwise, where $\theta > 0$, then the random variable $W = \ln(1+X)$ has an exponential distribution with mean $1/\theta$, since the pdf of W, $f_W(w)$ is given by

$$f_W(w) = f(e^w - 1)\left|\frac{d}{dw}(e^w - 1)\right| = \theta e^{-(\theta+1)w}|e^w| = \theta e^{-\theta w} \quad \text{for} \quad w > 0.$$

As before

$$U = \sum_{i=1}^{n} \ln(1+X_i)$$

has a gamma distribution with parameters $1/\theta$ and n, and

$$E(Y) = E(1/U) = \int_0^\infty \frac{1}{u} \frac{\theta^n u^{n-1} e^{-\theta u}}{\Gamma(n)} du$$

$$= \frac{1}{\Gamma(n)} \int_0^\infty \theta^n u^{n-2} e^{-\theta u} du = \frac{\theta \Gamma(n-1)}{\Gamma(n)} = \frac{\theta}{n-1}.$$

Thus,

$$\int_0^\infty \cdots \int_0^\infty \frac{\theta^n [(1+x_1)\cdots(1+x_n)]^{-(\theta+1)}}{\ln[(1+x_1)\cdots(1+x_n)]} dx_1 \cdots dx_n = \frac{\theta}{n-1}.$$

These integrals were encountered in the derivation of unbiased estimates for the parameter θ of the two pfds considered. The probability results used can be found in any standard text on Mathematical Statistics [1]. It would be of interest to know whether these integrals can be evaluated without using results from Probability Theory.

Reference

1. L. J. Bain and M. Engelhardt, *Introduction to Probability and Mathematical Statistics*, 2nd ed., PWS-Kent, Boston, MA, 1992.