A-SETS AND ABCOHESIVE SPACES

David John

Missouri Western State College

Definitions. A space M is *abcohesive at a point* p *with respect to a point* q if there exists an open connected set U such that p is a point in U and U is a subset of $M - \{q\}$. The space M is *abcohesive at a point* p if it is abcohesive at p with respect to q for each q in $M - \{p\}$. The space M is *abcohesive* if it is abcohesive at p for each p in M.

Remarks. If p is a non-cut point of M, and M is T_1 then M is abcohesive at each point q in $M - \{p\}$ with respect to p. Hence, if each point of M is a non-cut point of M, then M is abcohesive. Also, if M is a locally connected T_1 space, then M is abcohesive. Sierpinski space is locally connected but not abcohesive. However, Sierpinski space is not T_1. For the remainder of this paper, we will assume the space M is Hausdorff. If M is a continuum, then there exist two points p and q in M such that M is abcohesive at each x in $M - \{p\}$ with respect to p and at each x in $M - \{q\}$ with respect to q.

Theorem 1. The space M is abcohesive at each point q in $M - \{p\}$ with respect to p if and only if each component of $M - \{p\}$ is open.

Proof. Suppose M is abcohesive at each point q in $M - \{p\}$ with respect to p. Let C be a component of $M - \{p\}$, and let x be a point in C. Since M is abcohesive at x with respect to p, there exists an open connected set K such that $x \in K$ and $K \subset M - \{p\}$. But $K \subset C$, and hence C is open.

If the components of $M - \{p\}$ are open, then for each q in $M - \{p\}$, there exists a component C such that $q \in C$ and $C \subset M - \{p\}$. Therefore, M is abcohesive at q with respect to p.

Theorem 2. If M is an abcohesive connected space and C is a component of $M - \{p\}$, then p is a limit point of C.

Proof. Let C be a component of $M - \{p\}$. If p is not a limit point of C, then C is both open and closed in M. This involves a contradiction. Hence p is a limit point of C.

Theorem 3. If M is an abcohesive space, then the components of M are open.

Proof. If M is connected, then M is the only component of the space. If M is not connected, then let C be a component of M and let p be a point in $M - C$. By Theorem

1, the components of $M - \{p\}$ are open. Since C is a subset of $M - \{p\}$, C is a component of $M - \{p\}$, and hence C is open.

Definitions. A space M is *aposyndetic at a point p with respect to a point q* if there exists a closed connected set H such that p is in the interior of H and H is a subset of $M - \{q\}$. The space M is *aposyndetic at a point p* if it is aposyndetic at p with respect to q for each q in $M - \{p\}$. The space M is *aposyndetic* if it is aposyndetic at p for each p in M. A space M is *semi-locally connected at a point p* of M if each open set containing p contains an open set V containing p such that $M - V$ has at most a finite number of components. The space M is *semi-locally connected* if it is semi-locally connected at each point p in M.

Theorem 4. If M is aposyndetic at each point q in $M - \{p\}$ with respect to p, then M is abcohesive at each point q in $M - \{p\}$ with respect to p.

Proof. Let q be a point in $M - \{p\}$ and let C be the component of $M - \{p\}$ containing q. There exists a closed connected set H such that q is in the interior of H and $H \subset M - \{p\}$. Now $H \subset C$, and hence C is open. Thus M is abcohesive at each point q in $M - \{p\}$ with respect to p.

Theorem 5. If M is an aposyndetic space, then M is an abcohesive space.

Jones [1] established Theorem 6.

Theorem 6. If the space M is semi-locally connected at p, then M is aposyndetic at each point q of $M - \{p\}$ with respect to p.

Theorem 7. If M is a semi-locally connected space, then M is an aposyndetic space.

Theorem 8 follows from Theorems 4 and 6.

Theorem 8. If the space M is semi-locally connected at p, then M is abcohesive at each point q in $M - \{p\}$ with respect to p.

Theorem 9. If M is a semi-locally connected space, then M is an abcohesive space.

Theorem 10. If M is an abcohesive connected space and M has two cut points, then there exist disjoint closed sets H and K such that $M - H$ and $M - K$ are connected and H and K have non-empty interiors.

Proof. Let p and q be cut points of M. Let E be the component of $M - \{p\}$ containing q and let F be the component of $M - \{q\}$ containing p. Let $\mathfrak{A}$ be the collection of all components of $M - \{p\}$ different from E and let $\mathfrak{B}$ be the collection of all components of $M - \{q\}$ different from F. Since M is abcohesive, A and B are open for each A in $\mathfrak{A}$ and B in $\mathfrak{B}$. For each A in $\mathfrak{A}$, $A \cup \{p\}$ is a connected subset of $M - \{q\}$. Thus $A \cup \{p\} \subset F$. Also for each B in $\mathfrak{B}$, $B \cup \{q\}$ is a connected subset of $M - \{p\}$. Thus $B \cup \{q\} \subset E$. Hence

$A \cup \{p\}$ and $B \cup \{q\}$ are disjoint for each A in $\mathfrak{A}$ and B in $\mathfrak{B}$. Let $H = \cup\mathfrak{A} \cup \{p\}$ and let $K = \cup\mathfrak{B} \cup \{q\}$. Now, H and K are disjoint, $M - H = E$, and $M - K = F$, and both H and K have non-empty interiors.

It is well-known that every non-degenerate continuum has at least two non-cut points. Theorem 11 is a generalization of this well-known theorem.

Theorem 11. If M is a non-degenerate abcohesive connected space, then there exist two disjoint closed connected sets H and K such that H and K have degenerate boundaries and $M - H$ and $M - K$ are connected.

Proof. If M contains two non-cut points p and q, then let $\{p\} = H$ and $\{q\} = K$. If M does not contain two non-cut points, then M has two cut points and Theorem 11 follows from Theorem 10.

Definition. Let M be a connected space. An *A-set* of M is a closed subset of M such that $M - A$ is the union of a collection of open sets each bounded by a single point of A.

Theorem 12. If M is an abcohesive connected space and A is a closed set in M, then A is an A-set if and only if for each component C of $M - A$, C is open and there is a point p of A such that $\partial C = \{p\}$.

Proof. Let A be an A-set of M. let $\mathfrak{U}$ be a collection of open sets such that $M - A = \cup\mathfrak{U}$ and ∂U is a degenerate subset of A for each U in $\mathfrak{U}$. Suppose C is a component of $M - A$. There exists an element U in $\mathfrak{U}$ such that $C \subset U$. Let $\partial U = \{p\}$, and let K be the component of $M - \{p\}$ containing C. Since M is abcohesive, K is open. K is connected, and U is separated from $(M - \{p\}) - U$, and so $K \subset U$. Now K is a connected subset of $M - A$, which implies the component C of $M - A$ must contain K. Hence $K = C$, and each component of $M - A$ is open and has a degenerate boundary in A.

The converse is easy.

The proof of Theorem 13 follows from the proof of Theorem 12.

Theorem 13. If M is an abcohesive connected space, A is an A-set of M, and C is a component of $M - A$, then for some point p of A, C is a component of $M - \{p\}$.

Theorem 14. If M is an abcohesive connected space, $p \in M$, and C is a component of $M - \{p\}$, then $C \cup \{p\}$ is an A-set of M.

Proof. The components of $M - \{p\}$ are open and each has boundary $\{p\}$. Then $M - (C \cup \{p\})$ is the union of all components of $M - \{p\}$ different from C, and hence $C \cup \{p\}$ is an A-set of M.

Theorem 15. If M is an abcohesive continuum, A is an A-set of M, and Z is a subcontinuum of M, then $Z \cap A$ is a continuum.

Proof. Assume there exist non-empty separated sets H_1 and H_2 such that

$$Z \cap A = H_1 \cup H_2 \ , \ Z \cap A \cap H_1 \neq \emptyset \ , \text{ and } Z \cap A \cap H_2 \neq \emptyset.$$

Now each component of $Z - A$ has a boundary point in $Z \cap A$. Let $\mathfrak{C}_1$ be the collection of all components of $Z - A$ with at least one boundary point in H_1, and let $\mathfrak{C}_2$ be the collection of all components of $Z - A$ with at least one boundary point in H_2. Let $\mathfrak{K}_1$ be the collection of all components of $M - A$ with at least one boundary point in H_1, and let $\mathfrak{K}_2$ be the collection of all components of $M - A$ with at least one boundary point in H_2. By Theorem 13, for each component C of $M - A$, there exists a point x in A such that C is a component of $M - \{x\}$. Since M is abcohesive, C is open. Now $\cup\mathfrak{K}_1$ and $\cup\mathfrak{K}_2$ are separated sets. Since each member of $\mathfrak{C}_1$ is in some member of $\mathfrak{K}_1$ and each member of $\mathfrak{C}_2$ is in some member of $\mathfrak{K}_2$, $\cup\mathfrak{C}_1$ is separated from $\cup\mathfrak{C}_2$. Let $Z_1 = \cup\mathfrak{C}_1 \cup H_1$, and let $Z_2 = \cup\mathfrak{C}_2 \cup H_2$. Then $Z = Z_1 \cup Z_2$.

Suppose $\cup\mathfrak{C}_2$ is not separated from H_1. Since H_1 is closed, $\cup\mathfrak{C}_2 \cap \overline{H_1} = \emptyset$. Now there must exist a net N in $\mathfrak{C}_2$ and a point p in H_1 such that $p \in \limsup N$. Let $\overline{N}$ be the net of closures of the elements of N. $\overline{N}$ is a net of continua and each element of $\overline{N}$ has a point in H_2. Since $p \in \limsup \overline{N}$, some subnet of $\overline{N}$ converges to a continuum K containing p. $\overline{N}$ is a net in the compact space Z, and so $K \subset Z$. Since $K \cap H_1 \neq \emptyset$ and $K \cap H_2 \neq \emptyset$, it follows that $K \subset A$. Now $K \subset Z \cap A = H_1 \cup H_2$, and this is a contradiction. Hence $\cup\mathfrak{C}_2$ is separated from H_1, and similarly $\cup\mathfrak{C}_1$ is separated from H_2. Therefore Z_1 is separated from Z_2, which contradicts the fact that Z is connected. Hence $Z \cap A$ is connected.

The proof of the following theorem is similar to Whyburn's proof of this theorem in [2], and is omitted here.

Theorem 16. If M is a semi-locally connected continuum and A is an A-set of M, then A is a semi-locally connected continuum.

The next theorem follows from Theorem 15.

Theorem 17. If M is an abcohesive continuum, then each A-set of M is a continuum.

Theorem 18. If M is an abcohesive continuum, A is an A-set of M, a and b are points in A, and L is an irreducible continuum from a to b, then $L \subset A$.

Proof. By Theorem 17, A is a continuum. By Theorem 15, $L \cap A$ is a continuum, and hence $L \cap A$ is a subcontinuum of A containing a and b. Therefore, $L \cap A = L$ and $L \subset A$.

Theorem 19. If M is an abcohesive connected space and A is a closed set in M, then A is an A-set of M if and only if each component of $M - A$ has a degenerate boundary.

Proof. If A is an A-set, then by Theorem 12, each component of $M - A$ has exactly one boundary point. Let C be a component of $M - A$, let $\partial C = \{p\}$, and let K be the component of $M - \{p\}$ containing C. Since M is abcohesive, K is open. Now

$$M - \{p\} = C \cup [(M - \{p\}) - C],$$

C is separated from $(M - \{p\}) - C$, and K is a connected subset of $M - \{p\}$, and so $K \subset C$. Hence $K = C$ and $M - A$ is the union of open sets, each with a degenerate boundary in A. Therefore, A is an A-set.

References

1. F. B. Jones, "Aposyndetic Continua and Certain Boundary Problems," *American Journal of Mathematics,* 63 (1941), 545–553.

2. G. T. Whyburn, *Analytic Topology,* 1st ed., American Mathematical Society Colloquium Publications 28, American Mathematical Society, Providence, RI, 1942.