ON THE REFLECTION PROPERTIES OF THE CONIC SECTIONS

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Each of the three types of non-degenerate conic sections: the ellipse, the hyperbola, and the parabola, has its own reflection property. In this paper we present an elementary proof that no other curve shares any of these three properties. This has been proved directly in [2] §17.2.2.4., but the proof involves the first variation formula from differential geometry. Our proof, rather, is based on the uniqueness of the solution of an initial-value problem for an ordinary differential equation, and so makes an interesting and non-trivial application of this uniqueness theorem. Our presentation, however, assumes that the students already are familiar with the reflection properties of the conic curves. If not, this would be an opportunity for the students to learn about these important properties and their applications (e.g., the use of parabolic mirrors in telescopes, the parabolic reflectors of a car's headlights). An elegant proof of the reflection property of the ellipse is given in [3].

1. The Ellipse. Light emanating from one focus of an elliptic mirror will pass through the other focus. Since the angle of incidence equals the angle of reflection, this is equivalent to saying that the tangent line at a point P is the external bisector of the angle $\angle F_1PF_2$, where F_1 and F_2 are the foci. Let us say that a curve $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ satisfies the reflection property (of the ellipse) if there are two points F_1 and F_2 such that the tangent line at any point P on the curve is the external bisector of the angle $\angle F_1PF_2$. The external bisector is unique, so this tangent line must coincide with the tangent line of the ellipse passing through P with foci F_1 and F_2. The idea of the proof is that $\mathbf{r}$ is an integral curve of the direction field whose direction at any point (except for the foci) is given by this external bisector. If this field is sufficiently smooth, then there is only one integral curve which passes through each point, and so this curve must coincide with an arc of the ellipse which passes through this point and has these foci. What needs to be done, then, is to show that the direction field satisfies the conditions of the uniqueness theorem for an initial value problem.

Take the coordinates of the foci to be $(\pm c, 0)$, $c > 0$, and compute the direction (i.e., the slope of the external bisector) at the point $P(x_0, y_0)$ which does not coincide with one of the foci. To compute this direction consider the ellipse $x^2/a^2 + y^2/b^2 = 1$ with these foci and passing through the point (x_0, y_0). Then $A = a^2$ and $B = b^2$ satisfy the equations $x_0^2/A + y_0^2/B = 1$ and $A - B = c^2$. Solving this for B we obtain the quadratic equation

$$B^2 + (c^2 - x_0^2 - y_0^2)B - y_0^2 c^2 = 0,$$

which has exactly one positive solution if $y_0 \neq 0$. The quadratic formula shows that A and B are smooth functions of x_0 and y_0 away from the foci. The slope of the external bisector (which is computed by implicitly differentiating the equation of the ellipse) at (x_0, y_0) is given by $dy/dx = -Bx_0/Ay_0$, that is, the direction field is given by $y' = f(x, y)$ where

$$f(x, y) = -\frac{B(x, y)x}{A(x, y)y}.$$

Since $f(x, y)$ and $\partial f/\partial y$ are continuous for $y \neq 0$, it follows by the basic theorem for initial-value problems ([1] Theorem 2.2) that in the complement of the x-axis there is a unique integral curve passing through any particular point. An integral curve of the direction field in the complement of the segment $\overline{F_1F_2}$ must be an ellipse, since an ellipse with these foci intersects the x-axis in a point $(x, 0)$ with $|x| > c$, and since there is only one ellipse containing this point. The segment $\overline{F_1F_2}$ can be considered a degenerate ellipse which satisfies the reflection property with angle of incidence and angle of reflection equal to zero.

2. The Hyperbola. A tangent line of a hyperbola at P is the internal bisector of $\angle F_1PF_2$; F_i again being the foci. Essentially the same argument as for the ellipse shows that no curve other than the hyperbola enjoys this property. The differential equation in this case is $dy/dx = b^2x/a^2y$, where a^2 and b^2 are solutions of the equations $x_0^2/a^2 - y_0^2/b^2 = 1$ and $a^2 + b^2 = c^2$.

3. The Parabola. We show that the parabola is the only curve with the following reflection property: there is a ray R such that the reflection of any ray which is parallel to R and has the same direction as R will pass through a certain point F (the focus). There is no loss of generality in assuming that F is at the origin and that R is parallel to the y-axis and is pointed downwards (i.e., in the negative direction).

Any parabola with focus at the origin and directrix given by $y = -a$, $a > 0$ enjoys this reflection property. As in the case of the ellipse and hyperbola, the direction at any point (x_0, y_0) is determined by the reflection property and must coincide with the direction of the tangent line of the parabola whose focus is at the origin, whose directrix is given by $y = -a$, and which passes through the point (x_0, y_0). A parabola with this focus and directrix passes through the points $(\pm a, 0)$ and $(0, -a/2)$, so it has the equation

$$y + \frac{a}{2} = \frac{1}{2a}x^2. \tag{1}$$

The slope of the tangent line at (x_0, y_0) is given by

$$dy/dx = \frac{x_0}{a}, \tag{2}$$

where a depends smoothly on x_0 and y_0 in the complement of the positive y-axis, and does not vanish there. Indeed, solving the equation (1) gives $a = -y + \sqrt{x^2 + y^2}$.

The function $f(x, y) = x/a(x, y)$ and its partial derivative $\partial f/\partial y$ are continuous in the complement of the positive y-axis, so the basic existence and uniqueness theorem for ordinary differential equations shows that the only curve satisfying the differential equation (2) is a parabola. (The positive y-axis can be considered a degenerate parabola for which the angle of incidence is 0.) We note that the result here can be used to show that the circular paraboloid is the only surface for which all rays parallel to a given ray pass through the same point after reflection by the surface.

References

1. W. Boyce and R. DiPrima, *Elementary Differential Equations and Boundary Value Problems,* 4th ed., John Wiley and Sons, New York, 1986.
2. M. Berger, *Geometry,* vols 1 and 2, Springer-Verlag, Berlin, 1987.
3. J. Kitchen, *Calculus of One Variable,* Addison-Wesley, 1968.