ACCELERATED MSOR METHOD

Saadat Moussavi

University of Wisconsin - Oshkosh

1. Abstract. Since the development of the "SOR" method by David Young [3], there has been a strong interest to use more than one parameter for SOR to improve the convergence [13], [14], [15] and [16].

D. Young himself considered a two parametric method called "MSOR". The two parameters weight the diagonal of positive-definite and consistently ordered 2-cyclic matrix [6], removing Young's hypothesis that the eigenvalues of *Jacobi* iteration matrix must all be real. We prove for certain cases that when "SOR" diverges, the two parametric method converges.

2. Introduction. To find the solution vector x to the linear system

$$Ax = b \ , \tag{2.1}$$

where A is a sparse $n \times n$ matrix and b is a given n-vector of complex n-space. Stationary iterative methods, including SOR, solve the $n \times n$ linear system (2.1) by first splitting A into two terms,

$$A = A_0 - A_1 \ , \tag{2.2}$$

where A_0^{-1} is easy to compute. Relation (2.2) can be written as:

$$A = A_0(I - A_0^{-1}A_1) = A_0(I - B) \ , \tag{2.3}$$

where $B = A_0^{-1}A_1$ is called the *iteration matrix.* Therefore, the linear system (2.1) can be written as

$$x = Bx + A_0^{-1}b \ . \tag{2.4}$$

Then, by choosing any arbitrary starting vector x_0, the equation (2.4) is used to generate the vector sequence $\{x_k\}$, constructed as

$$x_{k+1} = Bx_k + A_0^{-1}b \quad k = 0, 1, 2, \dots \ . \tag{2.5}$$

By relation (2.3), it is clear that if $\{x_k\}$ converges at all, it must converge to $x_{\text{sol}} = A^{-1}b$ (vector solution), where $Ax_{\text{sol}} = b$. Relation (2.3) shows that $\{x_k\}$ produced by (2.5) converges to $x_{\text{sol}} = A^{-1}b$ for any x_0 if and only if $\rho(B) < 1$, where $\rho(B)$ is the spectral radius of B [1]. The smaller $\rho(B)$, the faster the sequence $\{x_k\}$ converges to $x_{\text{sol}} = A^{-1}b$ (asymptotically).

The above splitting is called *stationary* since there is no altering of parameter from iteration to iteration. It is called *one part splitting* since each x_{k+1} depends only on one previous vector x_k.

Examples of one-part stationary splitting are represented in the following important iteration methods.

(i) *Successive Overrelaxation (SOR)* method was developed independently by Frankel [2] and Young [3], [4] in 1950.

S.O.R: Choose

$$A_0 = \frac{1}{\omega}D - L \quad , \quad A_1 = \left(\frac{1}{\omega} - 1\right) D + U$$

where D is the diagonal part of A and $-L$, $-U$, are strictly lower and upper triangular parts of A respectively. Then, iteration matrix B_ω is given by

$$B_\omega = (D - \omega L)^{-1}\big((1-\omega)D + \omega U\big)$$

(ii) *Modified Successive Overrelaxation (MSOR)* method first considered by Devogelaere [5] in 1958. Here is how it works. Consider the matrix A in the following form

$$A = \begin{pmatrix} D_1 & M \\ N & D_2 \end{pmatrix} ,$$

where D_1 and D_2 are square non-singular matrices. Use ω for the "red" equations corresponding to D_1 and ω' for the "black" equations corresponding to D_2 then

M.S.O.R: Choose

$$A_0 = \begin{pmatrix} \frac{1}{\omega}D_1 & 0 \\ N & \frac{1}{\omega'}D_2 \end{pmatrix} .$$

Therefore, iteration matrix $B_{(\omega,\omega')}$ is defined by

$$B_{(\omega,\omega')} = A_0^{-1}A_1 = \begin{pmatrix} (1-\omega)I_1 & \omega F \\ \omega'(1-\omega)G & \omega\omega' GF + (1-\omega')I_2 \end{pmatrix} ,$$

where $F = -D_1^{-1}M$ and $G = -D_2^{-1}N$. Young [6] has proved that if A is positive-definite, then

$$\rho(B_{\omega_b}) < \overline{\rho}(B_{(\omega,\omega')}) ,$$

where $\overline{\rho}(B_{(\omega,\omega')})$ is virtual spectral radius of $B_{(\omega,\omega')}$. Young also showed that B_1 (Gauss-Seidel iteration matrix) converges faster than MSOR if A is positive definite, $0 < \omega \leq 1$ and $0 < \omega' \leq 1$. Moussavi generalized Young's Theorem by considering $0 < \omega \leq 1$ or $0 < \omega' \leq 1$ [17]. Mcdowell [7] and Taylor [8] analyzed the convergence of the MSOR method and obtained slightly better convergence by considering $\rho(B_{(\omega,\omega')})$ instead of $\overline{\rho}(B_{(\omega,\omega')})$.

In this paper, a comparison of the spectral radii of iteration matrices $B_{(1,\omega')}$ and $B_{(\omega,1)}$ with B_1 is done for the case when the eigenvalues of B_j (Jacobi) are not all real (Theorem 3.1). It can also be shown that if A is positive-definite, then iteration matrices $B_{(1,\omega')}$ and $B_{(\omega,1)}$ induce faster convergence than B_1 (Gauss-Seidel) for $1 < \omega < 2$ and $1 < \omega' < 2$ (Corollary 3.3). If A is an irreducible, L-matrix with $\rho(B_j) < 1$, then a relationship can be found between $\rho(B_{(\omega,\omega')})$ and $\rho(B_1)$. A sufficient condition for $\rho(B_{(1,\omega')}) < 1$ can also be found (Theorem 3.7). Finally it is shown that if the SOR method is not convergent and the eigenvalues of SOR are in a certain region in the plane, then iteration matrix $B_{(1,2)}$ induces rapid convergence of $\{x_k\}$ (Theorem 3.9).

3. Accelerated MSOR Method.

Theorem 3.1. Suppose

$$A = \begin{pmatrix} D_1 & M \\ N & D_2 \end{pmatrix} ,$$

where D_1 and D_2 are non-singular matrices and let $\rho(B_j) < 1$. Then
(a) If the eigenvalues of B_1 with modulus $\rho(B_1)$ have the real part less than $\rho^2(B_1)$, then

$$\rho(B_{(\omega,1)}) > \rho(B_1) \quad \text{and} \quad \rho(B_{(1,\omega')}) > \rho(B_1)$$

for all $1 < \omega < 2$ and $1 < \omega' < 2$.

(b) If the eigenvalues of B_1 with modulus $\rho(B_1)$ have the real part greater than $\rho^2(B_1)$, then

$$\rho(B_{(\omega,1)}) > \rho(B_1) \quad \text{and} \quad \rho(B_{(1,\omega')}) > \rho(B_1)$$

for all $0 < \omega < 1$ and $0 < \omega' < 1$.

(c) If (a) and (b) hold together, then $\rho(B_1)$ is the smallest.

Proof. According to Young [6],

$$(\lambda + \omega - 1)(\lambda + \omega' - 1) = \lambda\omega\omega'\mu^2 \ ,$$

where

$$\lambda \in \sigma(B_{(\omega,\omega')}) \quad \text{and} \quad \mu \in \sigma(B_j) \ .$$

It is clear that $B_{(1,\omega')}$ and $B_{(\omega,1)}$ are Jacobi shifting of B_1, with parameters ω and ω', respectively [12]. Hence if

$$\xi \in \sigma(B_{(\omega,1)}) \quad \text{and} \quad \psi \in \sigma(B_{(1,\omega')}) \ ,$$

then

$$\xi = \omega\mu^2 + (1-\omega)\cdot 1 \tag{3.1.1}$$

$$\psi = \omega'\mu^2 + (1-\omega')\cdot 1 \tag{3.1.2}$$

(a) All the eigenvalues of B_1 with modulus $\rho(B_1)$ are on the arc TBT' (Figure 1), where ST and ST' are tangent lines to the circle C with center at the origin and radius $\rho(B_1)$. It is easy to show that x-coordinates of T and T' is $\rho^2(B_1)$. Hence if $\omega > 1$ or $\omega' > 1$, then μ^2 shifts to ξ or ψ outside the circle C on the line which passes through two points μ^2 and $S : (1, 0)$, respectively. This means that (3.1.1) or (3.1.2) gives a slower convergence. Of course in this case one could find $0 < \omega < 1$ or $0 < \omega' < 1$ such that

$$\rho(B_{(\omega,1)}) < \rho(B_1) \quad \text{and} \quad \rho(B_{(1,\omega')}) < \rho(B_1) \ .$$

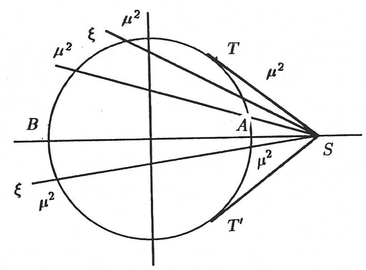

Figure 1.

(b) All the eigenvalues of B_1 with modulus $\rho(B_1)$, are on the arc TAT' (Figure 1). Then $\omega < 1$ or $\omega' < 1$ shifts μ^2 toward point $S:(1,0)$ on the line which passes through two points μ^2 and $S:(1,0)$. Hence (3.1.1) and (3.1.2) will give a slower convergence. Of course in this case one could find $1 < \omega < 2$ or $1 < \omega' < 2$ such that

$$\rho(B_{(\omega,1)}) < \rho(B_1) \quad \text{or} \quad \rho(B_{(1,\omega')}) < \rho(B_1)\ .$$

(c) Clear by part (a) and part (b).

Lemma 3.2. $B_{(\omega,1)}$ is a Jacobi shifting of $B_{(1,\omega')}$ or vice versa.

Proof. By (3.1.1) and (3.1.2)

$$\mu^2 = \frac{\psi + \omega' - 1}{\omega'} = \frac{\xi + \omega - 1}{\omega}$$

or equivalently

$$\psi = \frac{\omega'}{\omega}\xi + \big(1 - \frac{\omega'}{\omega}\big) \cdot 1.$$

Corollary 3.3. Let

$$A = \begin{pmatrix} D_1 & M \\ N & D_2 \end{pmatrix}\ ,$$

where D_1 and D_2 are non-singular matrices. If $\mu_1 = \rho(B_j) < 1$ and all the eigenvalues of B_j are real, then

$$\rho(B_{(\omega,1)}) < \rho(B_1) \quad \text{or} \quad \rho(B_{(1,\omega')}) < \rho(B_1)$$

for $1 < \omega < 2$ or $1 < \omega' < 2$.

Proof. Since all the eigenvalues of B_1 are on the line segment $[0, \mu_1^2]$, it is clear by part (a) of Theorem 3.1 that $\rho(B_1)$ is greater than $\rho(B_{(\omega,1)})$ or $\rho(B_{(1,\omega')})$ for $1 < \omega < 2$ or $1 < \omega' < 2$.

Theorem 3.4. Let

$$A = \begin{pmatrix} D_1 & M \\ N & D_2 \end{pmatrix} ,$$

where D_1 and D_2 are non-singular matrices. If A is an irreducible L-matrix and $\rho(B_j) < 1$, then

$$\rho(B_{(1,\omega')}) < \rho(B_{(\omega,\omega')})$$

for $0 < \omega < 1$ and $0 < \omega' < 1$.

Proof. To get $B_{(1,\omega')}$ and $B_{(\omega,\omega')}$, split matrix A in the following ways. $A = A_0 - A_1$, where

$$A_0 = \begin{pmatrix} D_1 & 0 \\ N & \frac{1}{\omega'} D_2 \end{pmatrix}$$

and

$$A_1 = \begin{pmatrix} 0 & -M \\ 0 & (\frac{1}{\omega'} - 1) D_2 \end{pmatrix}$$

and $A = A_0' - A_1'$, where

$$A_0' = \begin{pmatrix} \frac{1}{\omega} D_1 & 0 \\ N & \frac{1}{\omega'} D_2 \end{pmatrix}$$

and

$$A_1' = \begin{pmatrix} (\frac{1}{\omega} - 1) D_1 & -M \\ 0 & (\frac{1}{\omega'} - 1) D_2 \end{pmatrix} .$$

Since A is an L-matrix, D_1 and D_2 are positive, and N and M are non-positive matrices, thus $-M$ is non-negative. Since $0 < \omega < 1$ and $0 < \omega' < 1$,

$$\left(\frac{1}{\omega} - 1\right) D_1 \text{ and } \left(\frac{1}{\omega} - 1\right) D_2$$

are positive, hence $A'_1 \geq A_1 > 0$. Since A is an L-matrix and $\rho(B_j) < 1$, A is an M-matrix [6]. That is, $A^{-1} \geq 0$. But because A is also irreducible, $A^{-1} > 0$ [9]. By Varga's Theorem 3.15,

$$\rho(B_{(1,\omega')}) < \rho(B_{(\omega,\omega')})$$

for $0 < \omega < 1$ and $0 < \omega' < 1$ [9].

Corollary 3.5. Under the assumption of Theorem 3.4, if all the eigenvalues of B_1 with modulus $\rho(B_1)$ have the real part greater than $\rho^2(B_1)$, then

$$\rho(B_{(\omega,\omega')}) > \rho(B_1)$$

for $0 < \omega < 1$ and $0 < \omega' < 1$.

Proof. Clear by Theorem 3.4 and part (b) of Theorem 3.1.

Lemma 3.6. Suppose that

$$A = \begin{pmatrix} I_1 & M \\ N & I_2 \end{pmatrix} .$$

Then

$$(I - B_{(\omega,\omega')})^{-1}$$

exists if and only if $(I - NM)^{-1}$ exists.

Proof. Since

$$B_{(\omega,\omega')} = \begin{pmatrix} (1-\omega)I_1 & \omega M \\ \omega'(1-\omega)N & \omega\omega' NM + (1-\omega')I_2 \end{pmatrix} ,$$

$$I - B_(\omega,\omega') = \begin{pmatrix} \omega I_1 & -\omega M \\ -\omega'(1-\omega)N & -\omega\omega' NM + \omega' I_2 \end{pmatrix} .$$

Suppose that the matrix

$$\begin{pmatrix} X & U \\ Y & V \end{pmatrix}$$

is the inverse of

$$(I - B_{(\omega,\omega')}) \ .$$

Then

$$\begin{aligned} \omega X - \omega M Y &= I_1 \\ -\omega'(1-\omega) N X - \omega\omega' N M Y + \omega' Y &= 0 \end{aligned} \tag{3.6.3}$$

$$\begin{aligned} \omega U - \omega M V &= 0 \\ -\omega'(1-\omega) N U - \omega\omega' N M V + \omega' V &= I_2 \end{aligned} \tag{3.6.4}$$

By (3.6.3) and (3.6.4) one gets

$$\begin{aligned} (I - B_{(\omega,\omega')})^{-1} &= \begin{pmatrix} X & U \\ Y & V \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{\omega} I_1 + \frac{1-\omega}{\omega} M (I - NM)^{-1} N & \frac{1}{\omega'} M (I - NM)^{-1} \\ \frac{1-\omega}{\omega} (I - NM)^{-1} & \frac{1}{\omega'} (I - NM)^{-1} \end{pmatrix} \ . \end{aligned}$$

Note that

$$(I - B_{(\omega,\omega')})^{-1} \ \text{ and } \ (I + B_{(\omega,\omega')})$$

are commutative.

Theorem 3.7. Let

$$A = \begin{pmatrix} I_1 & M \\ N & I_2 \end{pmatrix}$$

and γ be the eigenvalue of $(I - NM)^{-1}$ with the smallest real part, i.e., $0 < \mathrm{Re}\gamma \le \mathrm{Re}\lambda$ for all $\lambda \in \sigma((I - NM)^{-1})$. Let $\rho(B_j) < 1$. Then $\rho(B_{(1,\omega')}) < 1$ if and only if $0 < \omega' < 2\mathrm{Re}\gamma$.

Proof. If μ is an eigenvalue of B_j, then μ^2 is an eigenvalue of NM. Hence $\rho(NM) < 1$, which implies that $(I - NM)^{-1}$ exists [9]. First we show that $(I - NM)^{-1}$ is N-stable, which means all the eigenvalues of $(I - NM)^{-1}$ have positive real parts. Suppose that γ is an eigenvalue of $(I - NM)^{-1}$, then one can write it in the form

$$\gamma = \frac{1}{1 - \mu^2} \ ,$$

where $\mu \in \sigma(B_j)$. Let $\mu = x + yi$. Then

$$\text{Re}\gamma = \frac{1 - x^2 + y^2}{(1 - x^2 + y^2) + 4x^2y^2} \ , \tag{3.7.5}$$

since $x^2 + y^2 < 1$, (3.7.5) is positive. Let

$$H = (I - B_{(\omega,\omega')})^{-1}(I + B_{(\omega,\omega')}) = 2(I - B_{(\omega,\omega')})^{-1} - I \ .$$

Then

$$H_{(1,\omega')} = \begin{pmatrix} I_1 & \frac{2}{\omega'}M(I - NM)^{-1} \\ 0 & \frac{2}{\omega'}(I - NM)^{-1} - I_2 \end{pmatrix} \ . \tag{3.7.6}$$

Therefore, the eigenvalues of $H_{(1,\omega')}$ are the same as the eigenvalues of its diagonal submatrices. Hence

$$\sigma(H_{(1,\omega')}) = 1 \cup \left\{ \frac{2}{\omega'}\nu - 1 \middle| \nu \in \sigma((I - NM)^{-1}) \right\} \ .$$

$H_{(1,\omega')}$ is N-stable if and only if all the real parts of its eigenvalues are positive, that is,

$$\frac{2}{\omega}\text{Re}\gamma - 1 > 0$$

or equivalently $0 < \omega' < 2\text{Re}\gamma$. Then it is clear that $\rho(B_{(1,\omega')}) < 1$ (see Theorem 1.5 in [6]).

Lemma 3.8. Let

$$A = \begin{pmatrix} D_1 & M \\ N & D_2 \end{pmatrix} \ ,$$

where D_1 and D_2 are non-singular matrices. Let ξ be an eigenvalue of $B_{(1,\omega')}$ and λ be an eigenvalue of $B_{\omega'}$, then eigenvalues ξ of $B_{(1,\omega')}$ and λ of $B_{\omega'}$ are related by the following relation

$$(3.8.7) \qquad \xi = \frac{1}{\omega'}\lambda + \frac{(\omega'-1)^2}{\omega'}\frac{1}{\lambda} - \frac{(\omega'-1)(\omega'-2)}{\omega'} \ .$$

Moreover, $\xi = (l_1(g(l_2(\lambda))))$, where

$$l_1(\lambda) = \pm\left(\frac{1}{\omega'-1}\right)\lambda \ , \quad g(\lambda) = \lambda + \frac{1}{\lambda}$$

and

$$l_2(\lambda) = \pm\frac{(\omega'-1)}{\omega'}\lambda - \frac{(\omega'-1)(2-\omega')}{\omega'} \ .$$

<u>Proof</u>. Suppose that ψ is an eigenvalue of $B_{(\omega,\omega')}$ and λ is an eigenvalue of $B_{\omega'}$. According to Young [9],

$$(\psi+\omega-1)(\psi+\omega'-1) = \psi\omega\omega'\mu^2 \ \text{ and } \ (\lambda+\omega'-1)^2 = \lambda\omega'\mu^2 \ ,$$

which implies

$$(3.8.8) \qquad \psi\lambda(\omega'\psi-\omega\lambda) + \lambda\omega(\omega'-\omega)(\omega'-2) + (\omega'-1)(\omega'\lambda(\omega-1) - \omega\psi(\omega'-1)) = 0 \ .$$

Then if ξ is an eigenvalue of $B_{(1,\omega')}$,

$$\lambda\xi(\omega'\xi-\lambda) + (\omega'-1)(\omega'-2)\lambda\xi + (\omega'-1)(-\xi(\omega'-1)) = 0$$

by (3.8.8). If $\xi \neq 0$, then (3.8.7) holds. Moreover, suppose that $\xi = (l_1(g(l_2)))(\lambda)$, where l_1 and l_2 are linear functions, say $l_1(\lambda) = k\lambda + l$, $l_2(\lambda) = b\lambda + c$ and $g(\lambda) = \lambda + \frac{1}{\lambda}$. Then

$$(3.8.9) \qquad \xi = (l_1(g(l_2(\lambda)))) = (bk)\lambda + \frac{b}{k\lambda+l} + (bl+c) \ .$$

Comparing (3.8.9) with (3.8.7),

$$(3.8.10) \qquad bk = \frac{1}{\omega'} \ , \ \frac{b}{k\lambda+l} = \frac{(\omega'-1)^2}{\omega'}\frac{1}{\lambda} \ , \ bl+c = \frac{(\omega'-1)(2-\omega')}{\omega'} \ .$$

By choosing $l = 0$ in (3.8.10),

$$k = \pm\left(\frac{1}{\omega' - 1}\right) \quad \text{and} \quad b = \pm\frac{(\omega' - 1)}{\omega'}$$

that implies

$$l_1(\lambda) = \pm\left(\frac{1}{\omega' - 1}\right)\lambda\ , \quad l_2(\lambda) = \pm\frac{(\omega' - 1)^2}{\omega'}\lambda - \frac{(\omega' - 1)(2 - \omega')}{\omega'}$$

and $g(\lambda) = \lambda + \frac{1}{\lambda}$.

Theorem 3.9. Let

$$A = \begin{pmatrix} D_1 & M \\ N & D_2 \end{pmatrix}\ ,$$

where D_1 and D_2 are non-singular matrices. Suppose that eigenvalues of SOR lie inside the shaded area of Figure 2, where the circles C_1 and C_3 both have radius

$$\frac{1+R}{2}$$

with centers at

$$\left(\frac{1-R}{2}, 0\right)\ ,\quad \left(\frac{-1+R}{2}, 0\right)\ ,$$

respectively. Moreover, the circles C_2 and C_4 both have radius

$$\frac{1+\frac{1}{R}}{2}$$

with centers at

$$\left(\frac{1-\frac{1}{R}}{2}, 0\right)\ ,\quad \left(\frac{-1+\frac{1}{R}}{2}, 0\right)\ ,$$

respectively. Then the eigenvalues of $B_{(1,2)}$ are inside the shaded area of Figure 3.

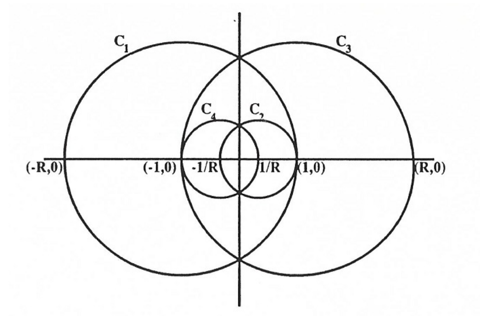

Figure 2.

Furthermore, if $1 < R < 3 + 2\sqrt{2}$, then $\rho(B_{(1,2)}) < 1$.

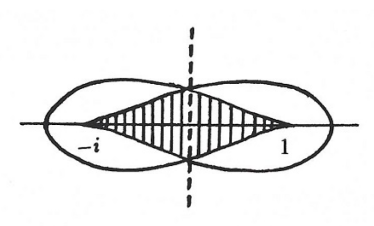

Figure 3.

Proof. Since $\omega' = 2$, by (3.8.7)

$$\xi = \frac{1}{2}\left(\lambda + \frac{1}{\lambda}\right) . \tag{3.9.11}$$

(3.9.11) can be written in the following form [10],

$$\frac{\xi - 1}{\xi + 1} = \left(\frac{\lambda - 1}{\lambda + 1}\right)^2$$

by the following auxiliary transformations

$$(i)\ Z_1 = \frac{\lambda - 1}{\lambda + 1} \quad (ii)\ Z_2 = Z_1^2 \quad (iii)\ \frac{\xi - 1}{\xi + 1} = Z_2\ .$$

The image of circle which passes through two points $(1,0)$ and $(-R,0)$, (i.e., circle C_1) under the transformation (i) is a circle say C, which goes through two points $(0,0)$ and

$$\left(\frac{-R-1}{-R+1}, 0 \right)\ .$$

The image of circle C under the transformation (ii) is a cardioid with the following equation

$$\rho = \frac{(R-1)^2}{(R+1)^2}(1 + \cos\varphi)\ .$$

Finally, the image of this cardioid under the transformation (iii) is a symmetric *Joukowski airfoil* with respect to the real axis, which passes through two points $(1,0)$ and $(-\frac{1}{2}(R + \frac{1}{R}), 0)$ [11] (Figure 4).

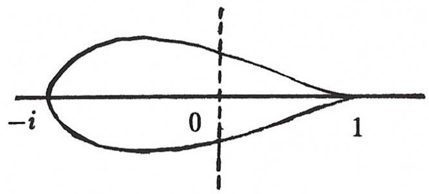

Figure 4.

Obviously the image of the circle C_2 under transformation (i) is the circle C. Then the image of the circle C_1 and C_2 under transformation (3.9.11) coincide. Also it is clear that all the points outside the circle C_2 and inside of the circle C_1 (i.e., all points belong to $C_1 - C_2$) map inside the Joukowski airfoil. The same argument holds for circles C_3 and C_4, i.e., all points belong to $C_3 - C_4$ map inside the symmetric Joukowski airfoil about the real axis which passes through the points $(-1, 0)$ and $(-\frac{1}{2}(R + \frac{1}{R}), 0)$.

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