A NOWHERE ANALYTIC C^∞ FUNCTION

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1. Introduction. Early on in a typical calculus course, just after we show that a differentiable function must necessarily be continuous, we note that the converse is not true: a continuous function need not be differentiable. Of course, we provide an example of this phenomenon, usually $f(x) = |x|$. Our students should be forgiven if they are led by this example, in fact, by their whole experience, to believe that even though a continuous function may fail to be differentiable at a few "bad" points, it should still be differentiable at "most" points. After all, it would have been easy for most mathematicians in the middle of the 19th century to hold to that belief, until they were confronted with Weierstrass's example of a *nowhere* differentiable continuous function. At a later time, in a later course, we begin the discussion of power series. We show that an analytic function (by which I mean a function given locally by a power series) must necessarily be infinitely differentiable, but we note that the converse is not true: an infinitely differentiable function need not be analytic. Once again, we back up this assertion with an example, usually

$$f(x) = e^{-1/x^2} \ .$$

Once again, our students might believe that infinitely differentiable functions must nonetheless be analytic "in most places", and, once again, this belief, however reasonable, would be false. The example of

$$f(x) = e^{-1/x^2}$$

serves exactly the same purpose in the discussion of analyticity as does the example of $f(x) = |x|$ in the discussion of differentiablity. To complete the discussion of differentiability, we need Weierstrass's nowhere differentiable continuous function, and to complete the discussion of analyticity, we need a *nowhere analytic infinitely differentiable function.* There are several ways of producing such an example, in fact, we could take the example we already have and use an enumeration of the rationals to pile up singularities near every point on the line. (This is accomplished in [4, p. 127]). We could also give an argument based on the Baire Category Theorem to the effect that "most" infinitely differentiable functions are nowhere analytic (see [1, p. 301]). In contrast to these approaches, I provide a concrete example motivated by Weierstrass's nowhere differentiable continuous function.

Weierstrass's example is a *lacunary trigonometric series:*

$$f(x) = \sum_{k=0}^{\infty} a^k \cos\bigl(b^k x\bigr)$$

where $a < 1$, $b > 1$ is an integer, and $ab > 1 + \frac{3\pi}{2}$ (see [3, p. 258]). This last condition is an artifact of a particular proof; Hardy showed in [2] that it can be weakened to $ab > 1$. What Hardy saw was that the lacunarity of the series (the fact that "most of the terms are missing") forces a great uniformity on its behavior. Any property that such a function has at one point is a property that it is likely to have everywhere. My example is also a lacunary trigonometric series, and I will take advantage of Hardy's insight.

2. Results. I define a complex-valued function $f(x)$ of a real variable to be *analytic* at a point x_0 if there is a number $\epsilon > 0$ and a power series

$$P(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

such that the power series has radius of convergence at least ϵ and

$$P(x) = f(x) \quad \text{for} \quad |x - x_0| < \epsilon \ .$$

(Many authors would call such a function *real analytic,* but that sounds a little strange as long as I'm allowing complex values.) To lessen potential confusion, I use the word *holomorphic* for functions with a complex derivative.

Theorem. There exists a function $f(x)$ which is infinitely differentiable for every real number x, but which is not analytic at any point x_0. To be specific, take the following function:

$$f(x) = \sum_{k=0}^{\infty} 2^{-2^{k/2}} e^{i2^k x} \ .$$

That this function is infinitely differentiable may be seen from the uniform convergence (by the Weierstrass M-test) of the series for the nth derivative.

To show that it cannot be analytic at any point, I employ a series of lemmas, mostly variations of standard arguments from a first course in complex variables.

Lemma 1. If $f(x)$ is a function of a real variable that is analytic at a point x_0 with radius of convergence ϵ, then it may be uniquely extended to be holomorphic on the set

$$\{z : |z - x_0| < \epsilon\}$$

in the complex plane.

Proof. Just use the same power series and the uniqueness of power series.

Lemma 2. Let

$$f(x) = \sum_{n=0}^{\infty} a_n e^{inx} \quad \text{for} \quad \sum_{n=0}^{\infty} |a_n| < \infty \ ,$$

and suppose that f is analytic in a neighborhood of x_0. Then the function

$$g(z) = \sum_{n=0}^{\infty} a_n z^n$$

is holomorphic on the set $\{z : |z| < 1\}$, continuous on the set $\{z : |z| \leq 1\}$, and may be extended to be holomorphic on a neighborhood of the point

$$z_0 = e^{ix_0} \ .$$

Proof. That $g(z)$ is holomorphic on the open ball is basic; that it is continuous on the closed ball is Abel's theorem. The holomorphic extension comes from the extension promised by Lemma 1, together with the fact that the composition of holomorphic functions (in this case, composition of the extension of f with $\phi(z) = e^{iz}$) is again holomorphic.

Lemma 3. If $g(z)$ is a function holomorphic on the set $\{z : |z| < \delta\}$, then g is given on that set by the power series

$$g(z) = \sum_{n=0}^{\infty} a_n z^n, \quad \text{for which} \quad \limsup_{n \to \infty} |a_n|^{\frac{1}{n}} \leq \frac{1}{\delta} \ .$$

A standard argument based on the Cauchy integral formula gives the power series representation, and the estimate for the coefficients is the usual method of computing the radius of convergence of a power series.

Lemma 4. Let

$$f(x) = \sum_{k=0}^{\infty} a_k e^{i2^k x} \ .$$

If $f(x)$ is analytic at any point, then $f(x)$ is analytic at every point.

Proof. Let $f(x)$ be analytic at x_0. In turn, this means there is a number $\epsilon > 0$ such that $f(x)$ is analytic on the set

$$\{x : |x - x_0| < \epsilon\} \ .$$

Now choose n so large that $2^{-n}\pi < \epsilon$. Write

$$f(x) = \sum_{k=0}^{n} a_k e^{i2^k x} + \sum_{k=n+1}^{\infty} a_k e^{i2^k x} = f_1(x) + f_2(x) \ .$$

Then $f_1(x)$, being a trigonometric polynomial, is analytic everywhere. (In fact, it is *entire*; that is, it is given by a power series with an infinite radius of convergence.) On the other hand, $f_2(x)$ is periodic of period $2^{-n}\pi$. Hence, $f_2(x)$ is analytic on each of the sets

$$J_m = \{x : |x - x_m| < \epsilon\} ,$$

where $x_m = x_0 + 2^{-n}\pi m$, m ranging over the integers. But then $f_2(x)$ (and thus $f(x)$) is analytic on the set

$$\bigcup_{m=-\infty}^{\infty} J_m ,$$

which is the whole real line.

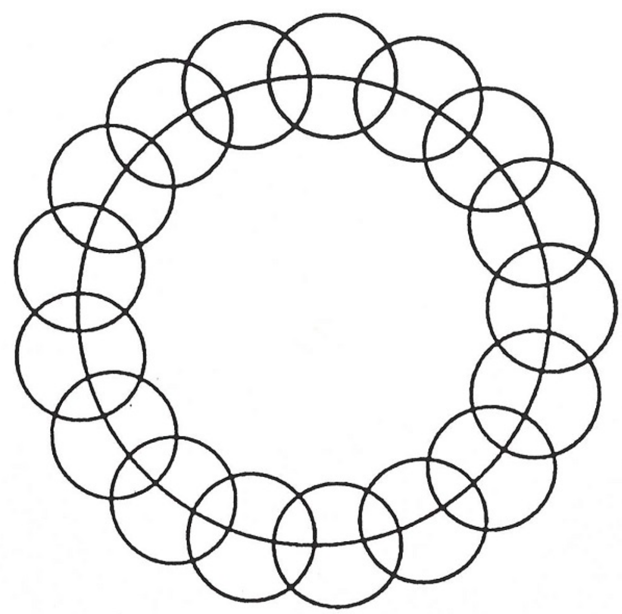

Figure 1.

Proof of the Theorem. Let $f(x)$ be as in the statement of the theorem. Assume that $f(x)$ is analytic at some one point. Then by Lemma 4, $f(x)$ is analytic on the whole real line. Let

$$g(z) = \sum_{k=0}^{\infty} 2^{-2^{k/2}} z^{2^k} \ .$$

By Lemma 2, $g(z)$ is holomorphic on the set $\{z : |z| < 1\}$, continuous on the set $\{z : |z| \leq 1\}$, and may be extended to be holomorphic on a neighborhood of *every* point in the set $\{z : |z| = 1\}$. Hence, by the uniqueness of holomorphic functions, $g(z)$ may be extended to be holomorphic on an open set which contains $\{z : |z| \leq 1\}$. In particular, this open set contains a ball $\{z : |z| < \delta\}$ for some $\delta > 1$ (see Figure 1). Then Lemma 3 implies that

$$\limsup_{n\to\infty} \left|a_n\right|^{\frac{1}{n}} = \lim_{k\to\infty} \left|2^{-2^{k/2}}\right|^{1/2^k} = 1 \leq \frac{1}{\delta} \ .$$

But this is false, and so my assumption that $f(x)$ was analytic at some point must also be false.

3. Epilogue. The reader is entitled to one complaint; wouldn't it be more æsthetically pleasing to have an example which is (like Weierstrass's example) real-valued? In particular, isn't the real part of my function, namely

$$h(x) = \sum_{k=0}^{\infty} 2^{-2^{k/2}} \cos\left(2^k x\right)$$

also a nowhere analytic infinitely differentiable function? As you might expect, it is, but its proof requires me to use slightly more advanced material. I would need to show that whenever

$$h(x) = \sum_{n=1}^{\infty} a_n \cos nx$$

is analytic at a point x_0, then

$$\tilde{h}(x) = \sum_{n=1}^{\infty} a_n \sin nx$$

is also analytic at the same point. To prove this, I would note that $\tilde{h}(x)$ is the *conjugate function,* or *Hilbert transform* of $h(x)$, and can be written as

$$\tilde{h}(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} h(x-y) \cot\left(\frac{y}{2}\right) dy$$

in which the integral is to be interpreted as a Cauchy principal value integral (see [5, p. 89–91]). The required analyticity can then be deduced from this formula.

References

1. J. Dugundji, *Topology,* Allyn and Bacon, 1966.
2. G. H. Hardy, *Trans. Amer. Math. Soc.,* 17 (1916), 301–325.
3. E. Hewitt and K. Stromberg, *Real and Abstract Analysis,* Springer-Verlag, 1965.
4. Z. A. Melzak, *Mathematical Ideas, Modeling & Applications* (Volume II of *Companion to Concrete Mathematics*), John Wiley & Sons, 1976.
5. Y. Katznelson, *An Introduction to Harmonic Analysis,* Dover, 1968.