# SOME PROPERTIES RELATED TO DENDRITIC SPACES

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**1. Introduction.** L. E. Ward [1] investigated a large number of dendritic properties. He proceeded to establish sufficient conditions for these properties to be equivalent. Ward was apparently unable to show equivalence for all the properties he investigated in any space weaker than Peano continua. He also presented some other dendritic properties he did not investigate. We answer some of the questions he raised. Unless explicitly stated to the contrary, $M$ will always denote a Hausdorff space.

**2. Definitions.** A *continuum* is a compact connected Hausdorff space. A connected space $M$ is said to be *dendritic* if and only if each pair of distinct points of $M$ is separated in $M$ by a third point of $M$. A compact dendritic space is a *dendrite.* An element $p$ of $M$ is a *cutpoint* of $M$ if $M - \{p\}$ is not connected. If $M - \{p\}$ is connected then $p$ is a *non-cutpoint* of $M$. The space $M$ is *paraseparable* if and only if $M$ does not contain uncountably many disjoint open sets. A space $M$ is said to be *strongly connected* if for each two points $a$ and $b$ in $M$, there exists a continuum $L$ in $M$ such that $L$ contains $a$ and $b$. $p$ is an *end point* of the connected space $M$ if each open set containing $p$ contains an open set containing $p$ whose boundary is degenerate.

The following theorem is known and can be found in Moore [2].

Theorem 1. If $H$ is an uncountable set of cut points of the paraseparable connected set $M$, then some two points of $H$ are separated in $M$ by a third point of $H$.

Theorem 2. If $M$ is a paraseparable strongly connected space and each continuum in $M$ contains uncountable many cut points of $M$, then $M$ is dendritic.

Proof. Let $a$ and $b$ be any two points of $M$. Since $M$ is strongly connected, there exists a continum in $M$ containing $a$ and $b$. Let $L$ be an irreducible continuum in $M$ containing

$a$ and $b$. Then $L$ contains uncountably many cut points of $M$. Let $N$ be the set of all cut points of $M$ in $L$. By Theorem 1, some two points $p$ and $q$ of $N$ are separated in $M$ by a third point $y$ of $N$. Then there exist two separated sets $H$ and $K$ such that $p \in H$, $q \in K$, and

$$M - \{y\} = H \cup K \ .$$

Since

$$L - \{y\} = (H \cap L) \cup (K \cap L) \ ,$$

$y$ is a cut point of $L$. Since $L - \{a\}$ and $L - \{b\}$ are connected, $y \neq a$ and $y \neq b$. If $a$ and $b$ are in $H$, then

$$(L \cap H) \cup \{y\}$$

is a proper subcontinuum of $L$ containing $a$ and $b$ which involves a contradiction. Similarly $a$ and $b$ are not both in $K$. Hence $y$ separates $a$ and $b$ in $M$ and $M$ is dendritic.

Definition. The space $M$ is *connected im kleinen* at the point $p$ if and only if each open set $U$ containing $p$ contains an open set $V$ containing $p$ such that each point of $V$ belongs to a connected set containing $p$ and lying in $U$.

Definitions. If $\{M_a : a \in A\}$ is a net of sets, the $\liminf M_a$ is the set of all points $p$ such that for each open set $U$ containing $p$, there exists $b \in A$ such that for each $a > b$, $a \in A$, $M_a$ intersects $U$. The $\limsup M_a$ is the set of all points $p$ such that for each open set $U$ containing $p$ and for each $b \in A$, there exists $a \in A$ such that $a > b$ and $M_a$ intersects $U$. If

$$\liminf M_a = \limsup M_a = L \ ,$$

then $L$ is denoted by $\lim M_a$ and $\{M_a : a \in A\}$ is said to *converge* to $L$. A nondegenerate continuum $K$ in a space $M$ is a *continuum of convergence* if and only if there is a net $\{K_a : a \in A\}$ of continua such that for each $a$ in $A$,

$$K \cap K_a = \emptyset \quad \text{and} \quad K = \lim K_a \ .$$

Theorem 3. If $M$ is a paraseparable locally compact connected space and each continuum in $M$ contains uncountably many cut points of $M$, then $M$ is dendritic.

Proof. Let $p$ be a point of $M$ and assume $M$ is not connected im kleinen at $p$. Then $M$ contains a continuum of convergence $L$ of $M$ containing $p$. $L$ contains uncountably many cut points of $M$. By Theorem 1, some two points of $L$ are separated in $M$ by a third point of $L$. But this is impossible, since $L$ is a continuum of convergence of $M$. Hence $M$ is connected im kleinen at $p$, and therefore $M$ is a locally connected space. It follows that $M$ is strongly connected. Thus by Theorem 2, $M$ is dendritic.

Theorem 4. If $M$ is a connected space, $a$ and $b$ are points of $M$, and $L$ is an irreducible continuum in $M$ from $a$ to $b$, then $L - \{a, b\}$ contains no end point of $M$.

Proof. Let $L$ be an irreducible continuum in $M$ from $a$ to $b$. Assume $L$ contains an end point $p$ of $M$. Let $U$ be an open set containing $p$ such that $a$ and $b$ are not in $U$. There exists an open set $V$ such that $p \in V$, $V \subseteq U$, and $\partial V$ is degenerate. Let $\partial V = \{q\}$. $L \cap V$ is separated from $L \cap (M - V)$, and therefore $q$ is a cut point of $L$. Now

$$L \cap \big((M - V) \cup \{q\}\big)$$

is a proper subcontinuum of $L$ from $a$ to $b$. This is a contradiction. Hence $L$ contains no end point of $M$.

Theorem 5. If $M$ is a connected space and $U$ is an open subset of $M$ such that $U$ contains at most a countable number of non-cut points of $M$ that are non-end points of $M$, then each nondegenerate continuum in $U$ contains uncountably many cut points of $M$.

Proof. Let $K$ be a nondegenerate continuum in $U$, and let $p$ and $q$ be points in $K$. Let $L$ be an irreducible subcontinuum of $K$ from $p$ to $q$. By Theorem 4, $L - \{p, q\}$ contains no end point of $M$. Hence $L$ contains uncountably many cut points of $M$, and therefore $K$ contains uncountably many cut points of $M$.

Theorem 6 follows from Theorem 5 and Theorem 3.

Theorem 6. If $M$ is a paraseparable locally compact connected space and each point of $M$ is a cut point or an end point of $M$, then $M$ is dendritic.

Definitions. Two points $a$ and $b$ of a connected space $M$ are said to be *conjugate* if no point of $M$ separates $a$ from $b$ in $M$. If $p$ is neither a cut point nor an end point of a connected space $M$ and $p \in M$, then the set consisting of $p$ and all points of $M$ conjugate to $p$ is called the *simple link of $M$ generated by $p$*.

The following theorem was established by John in [3].

Theorem 7. Every simple link of a locally compact connected space $M$ is nondegenerate, and every point of $M$ is a cut point, an end point, or a point of a simple link of $M$.

The following theorem is an immediate result of Theorem 6 and Theorem 7.

Theorem 8. If $M$ is a paraseparable locally compact connected space and $M$ contains no simple link, then $M$ is dendritic.

Theorem 9. If $M$ is a paraseparable locally compact connected space and $N$ is the set of all non-cut points of $M$ that are non-end points of $M$, then $N$ is empty or uncountable.

Proof. Assume $N$ is countable. By Theorem 5, each continuum in $M$ contains uncountably many cut points of $M$. By Theorem 3, $M$ is dendritic. Now each point of $M$ is a cut point or an end point of $N$, and hence $N = \emptyset$.

Theorem 10. If $M$ is a paraseparable locally compact connected space, $E$ is the set of all cut points of $M$, and for each point $p$ in $M - E$, $E \cup \{p\}$ is an open connected set, then $M$ is dendritic.

Proof. If $E$ is finite, then $E \cup \{p\}$ is both open and closed in $M$. Thus $E$ is infinite. Let $a$ and $b$ be any pair of distinct points in $M$. Let $p \in M - E$. Then $E \cup \{a, b, p\}$ is an open connected set. Assume $E \cup \{a, b, p\}$ is not connected im kleinen. Then $E \cup \{a, b, p\}$ contains a continuum of convergence $K$. Now $K$ contains uncountably many cut points of $M$. By Theorem 1, some two points in $K$ are separated in $M$ by a third point of $K$. Since $K$ is a continuum of convergence, this is a contradiction. Hence $E \cup \{a, b, p\}$ is connected im kleinen, and therefore $E \cup \{a, b, p\}$ is locally connected. Since $E \cup \{a, b, p\}$ is open, $E \cup \{a, b, p\}$ is locally compact. It follows that $E \cup \{a, b, p\}$ is strongly connected. Let $L$ be an irreducible continuum in $E \cup \{a, b, p\}$ from $a$ to $b$. $L$ contains uncountably many cut points of $M$. By Theorem 1, some two points $x$ and $y$ of $L$ are separated in $M$ by a third point $z$ of $L$. By the irreducibility of $L$, $z$ must also separate $a$ and $b$ in $M$. Hence $M$ is dendritic.

The following theorem was established in Moore [2] for metric continua.

Theorem 11. If $M$ is a paraseparable connected space, $N$ is the set of non-cut points of $M$, and for each subset $T$ of $N$, $M - T$ is strongly connected, then $M$ is dendritic.

Proof. Let $a$ and $b$ be distinct points of $M$, and let $T = N - \{a, b\}$. Then

$$M - T = (M - N) \cup \{a, b\}$$

is strongly connected. Thus there exists a continuum $K$ such that $K$ contains $a$ and $b$ and $K \subseteq M - T$. Let $L$ be an irreducible continuum in $K$ containing $a$ and $b$. Since each point of $L - \{a, b\}$ is a cut point of $M$, $L$ contains uncountably many cut points of $M$. By Theorem 1, some two points $x$ and $y$ in $L$ are separated in $M$ by a third point $z$ in $L$. Now by the irreducibility of $L$, $z$ must also separate $a$ and $b$ in $M$. Hence $M$ is dendritic.

## *References*


1. L. E. Ward, Jr. *Recent Developments in Dendritic Spaces and Related Topics,* Studies in Topology, Academic Press, New York, New York, 1975.
2. R. L. Moore, *Foundations of Point Set Theory,* revised edition, American Society Colloquium Publications, Volume 14, American Mathematical Society, Providence, R.I., MR 27 #709.
3. D. John, "Simple Links in Locally Compact Connected Hausdorff Spaces are Nondegenerate," *Canadian Journal of Mathematics,* 34 (1982), 349–355.