ON UNIT GROUPS OF EXTENSION RINGS: AN EXAMPLE

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The following is a well known result of algebraic number theory [1] or [2].

Theorem (Dirichlet). Suppose $Q[\alpha]$ is a finite degree extension field over the field Q of rational numbers and R is the integral closure in $Q[\alpha]$ of the ring Z of integers. Let r and $2s$ be the numbers of real and nonreal embeddings, respectively, of $Q[\alpha]$ into the field of complex numbers. Then $U(R)$, the group of units of R, can be written as a direct product $U(R) \cong G \times H$ where G is a finite group and H is a free abelian group of rank $r + s - 1$.

One peculiarity of the above result is that while it appears to be entirely algebraic, it seems that all known proofs involve some analysis. In fact, there does not seem to exist an entirely algebraic proof of the weaker conclusion that $U(R)$ has finite rank. Following this same line of questioning, a friend of the author recently posed the following problem.

Conjecture. Let R be an integral domain with field of fractions K and suppose that $U(R)$, the unit group of R, has finite rank. Suppose that F is a finite degree extension field over K and that S is the integral closure in F of R. Then $U(S)$, the unit group of S, has finite rank.

The purpose of this note is to give a counterexample to the above conjecture, thus illustrating the difficulty of giving an algebraic proof of Dirichlet's theorem.

Example. Let Z be the ring of integers. We will define an infinite sequence of rings as follows:

$$R_0 = Z \ , \quad R_1 = R_0[x_1, \sqrt{2x_1+3}, \sqrt{3x_1+5}] \ ,$$

and for $i \geq 1$,

$$R_i = R_{i-1}[x_i, \sqrt{2x_i+3}, \sqrt{3x_i+5}] \ .$$

Now, let

$$R_\infty = \bigcup_{i=1}^{\infty} R_i \ \text{ and let } \ S = R_\infty[\sqrt{2}, \sqrt{3}] \ .$$

We make two claims.

Claim 1. $U(R_\infty) = \{1, -1\}$.

Proof. Suppose not. Then $U(R_t) \neq \{1, -1\}$ for some $t \geq 1$. Choose t to be minimal with respect to this property and let $u_1 \in U(R_t)$ with $u_1 \notin \{1, -1\}$. Then we can write

$$u_1 = f + g\sqrt{2x_t+3} + h\sqrt{3x_t+5} + l\sqrt{2x_t+3}\sqrt{3x_t+5}$$

with f, g, h, $l \in R_{t-1}[x_t]$. We are then led naturally to define several conjugates of u_1 as follows:

$$u_1 = f + g\sqrt{2x_t+3} + h\sqrt{3x_t+5} + l\sqrt{2x_t+3}\sqrt{3x_t+5}$$

$$u_2 = f - g\sqrt{2x_t+3} - h\sqrt{3x_t+5} + l\sqrt{2x_t+3}\sqrt{3x_t+5}$$

$$u_3 = f - g\sqrt{2x_t+3} + h\sqrt{3x_t+5} - l\sqrt{2x_t+3}\sqrt{3x_t+5}$$

$$u_4 = f + g\sqrt{2x_t+3} - h\sqrt{3x_t+5} - l\sqrt{2x_t+3}\sqrt{3x_t+5}$$

Clearly, $u_i \in U(R_t)$ and $u_i \notin \{1, -1\}$ for each i. Now, let $u_* = u_1u_2u_3u_4$. Then $u_* \in U(R_t)$ and tedious computation yields

$$u_* = (f^2 - g^2(2x_t+3) - h^2(3x_t+5) + l^2(2x_t+3)(3x_t+5))^2 - 4(fl-gh)^2(2x_t+3)(3x_t+5)$$

Note that $u_* \in R_{t-1}[x_t]$. Since each $u_i \in U(R_t)$ then $u_i^{-1} \in U(R_t)$ for each i and a similar computation to that done above will yield $u_*^{-1} \in R_{t-1}[x_t]$. Since R_{t-1} is a domain and u_* is a unit in $R_{t-1}[x_t]$ then the degree of u_* as a polynomial in x_t must be zero. Thus, $u_* \in R_{t-1}$ and then by the minimality of t we must have either $u_* = 1$ or $u_* = -1$. We now define

$$\phi = f^2 - g^2(2x_t+3) - h^2(3x_t+5) + l^2(2x_t+3)(3x_t+5)$$

and

$$\psi = 2(fl - gh)\ .$$

Then

$$u_* = \phi^2 - \psi^2(2x_t + 3)(3x_t + 5)$$

with

$$\phi, \psi \in R_{t-1}[x_t] \ .$$

We now have two cases to consider. Either $\psi = 0$ or $\psi \neq 0$. We will see that both cases lead to contradictions.

Suppose $\psi = 0$. Then either $\phi = 1$ or $\phi = -1$ and so

$$f^2 + l^2(2x_t + 3)(3x_t + 5) = g^2(2x_t + 3) + h^2(3x_t + 5) + a$$

where

$$a \in \{1, -1\} \ .$$

By comparing degrees of the polynomials in x_t on the right and on the left hand sides of this equation, we see that $f^2 = 1$ and $g = h = l = 0$. This implies that $u_1 = 1$ which is a contradiction.

Suppose $\psi \neq 0$. Since u_* has degree zero in x_t, then ϕ^2 and $\psi^2(2x_t + 3)(3x_t + 5)$ have the same degree in x_t. Let $2n$ be this degree. Then we can write $\phi = \phi_n x_t^n + \cdots + \phi_0$ and $\psi = \psi_{n-1} x_t^{n-1} + \cdots + \psi_0$ where each ϕ_i and ψ_i is in R_{t-1}. Then $\phi_n / \psi_{n-1} = \sqrt{6}$ is in the field of fractions of R_{t-1}. Let F_{t-1} and K be the fields of fractions of R_{t-1} and $Z[x_1, x_2, \ldots, x_{t-1}]$, respectively. Since F_{t-1} is the composition of $2t-2$ quadratic extensions of K, the Galois group of F_{t-1} over K is an elementary abelian 2-group of order 2^{2t-2}. Such a group contains $2^{2t-2}-1$ maximal subgroups and so F_{t-1} contains exactly $2^{2t-2}-1$ quadratic extensions of K. We can construct $2^{2t-2}-1$ such extensions by adjoining to K expressions of the form

$$\prod_{i=1}^{t-1}(\sqrt{2x_i+3})^{d_i}(\sqrt{3x_i+5})^{e_i}$$

where $d_i, e_i = 0$ or 1 for each i. Clearly, none of these extensions is equal to $K[\sqrt{6}]$. Thus, $\sqrt{6} \notin F_{t-1}$ and we have reached a contradiction.

Since neither $\psi = 0$ nor $\psi \neq 0$ is possible, we conclude that $U(R_\infty) = \{1, -1\}$.

Claim 2. $U(S)$ has infinite rank.

Proof. The equation

$$(\sqrt{2}\sqrt{3x_i+5}+\sqrt{3}\sqrt{2x_i+3})(\sqrt{2}\sqrt{3x_i+5}-\sqrt{3}\sqrt{2x_i+3}) = 1$$

demonstrates that $u_i = (\sqrt{2}\sqrt{3x_i+5}+\sqrt{3}\sqrt{2x_i+3})$ is a unit in S for each i. Then the set $\{u_i | i \geq 1\}$ generates a group of units in S with infinite rank. Thus $U(S)$ has infinite rank.

References

1. D. A. Marcus, *Number Fields*, Springer-Verlag, New York/Berlin, 1977.
2. G. Karpilovsky, *Unit Groups of Classical Rings*, Oxford University Press, New York, 1988.