AN ANALYTICAL APPROACH TO A TRIGONOMETRIC INTEGRAL

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The integral of $\sin^2 x$ or $\cos^2 x$ between 0 and $\frac{\pi}{2}$ (or 0 and π) is usually calculated by changing the integrand according to the well-known half-angle formula. However, students with a phobia for trigonometry would prefer a simple geometric derivation of the answer suggested in [1]. The result is obtained immediately by noticing the equality of the areas of the regions under the graphs of $\sin^2 x$ and $\cos^2 x$ and by integrating the basic trigonometric identity $\sin^2 x + \cos^2 x = 1$. This idea is equivalent to the following analytical approach. Let

$$C = \int_0^{\frac{\pi}{2}} \cos^2 x \, dx, \quad S = \int_0^{\frac{\pi}{2}} \sin^2 x \, dx.$$

Since the substitution $x = \frac{\pi}{2} - u$ transforms either integral to the other, then $C = S$. Furthermore, integrating the basic identity between 0 and $\frac{\pi}{2}$ yields $C + S = \frac{\pi}{2}$, which implies $C = S = \frac{\pi}{4}$.

Some students even try to use this method for integrals of $\sin^4 x$ and $\cos^4 x$ between 0 and $\frac{\pi}{2}$. From the start, things evolve as expected, the above substitution again shows the equality of the corresponding integrals, and a few students decide to integrate the "identity" $\sin^4 x + \cos^4 x = 1$. Doubts and questions quickly reveal the error, gradually lead to serious work, and raise new questions on further generalizations. Of course, Wallis's integrals

$$C_n = \int_0^{\frac{\pi}{2}} \cos^{2n} x \, dx, \quad S_n = \int_0^{\frac{\pi}{2}} \sin^{2n} x \, dx$$

are too hard for most students but may be suggested as a challenging exercise. The following method of evaluating C_n is new and unusual in that it reduces to a minimum the amount of trigonometric transformations and illustrates the interaction between integration and differentiation. It is easy to calculate the improper integral

$$\int_0^{\infty} \frac{dy}{y^2 + p} = \frac{\pi}{2} \, p^{-\frac{1}{2}}, \quad p > 0.$$

Successively differentiating this relation with respect to p yields

$$\int_0^\infty \frac{dy}{(y^2+p)^{n+1}} = \frac{1}{2}\,\frac{(2n)!}{4^n(n!)^2}\,\pi p^{-\frac{(2n+1)}{2}}.$$

Letting $p = a^2, y = a\tan x$, we obtain the remarkable Wallis's formulas

$$C_n = S_n = \frac{(2n)!}{4^n(n!)^2}\left(\frac{\pi}{2}\right).$$

Reference

1. R. Euler, "A Geometric Approach to a Trigonometric Integral," *Missouri J. Math. Sci.,* Winter 1989, Vol. 1. No. 1, 28–29.