A SIMPLE PROOF OF CAUCHY'S INTEGRAL FORMULA FOR CIRCLES

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Let U be an open subset of the complex plane C. We say that a function $f : U \rightarrow C$ is analytic if f is continuously differentiable on U. Using Green's theorem, one may obtain a simple version of the Cauchy integral theorem and then show that an analytic function is representable by power series. In most elementary complex variable textbooks, a function on U is called analytic if it is differentiable on U and the basic theory for analytic functions is developed in a manner in which representation of analytic functions by power series appears quite late. This approach is not pedagogically satisfying for undergraduate students.

Our purpose is to present a simple proof of the Cauchy integral formula for circular paths without appealing to Green's theorem. From this result, one immediately derives the fact that an analytic function on an open disk is representable by power series. An independence of path result for circular paths is first established using integration by parts.

Given $a \in C$ and $r > 0$, let $D(a,r)$ and $\Delta(a,r)$ denote the open and closed disks with center a and radius r, respectively. Define the integral over $\partial\Delta(a,r)$ by

$$\int_{\partial\Delta(a,r)} f(z)\,dz \;=\; \int_0^{2\pi} f(\gamma(\theta))\gamma'(\theta)\,d\theta$$

where $\gamma(\theta) \;=\; a + re^{i\theta}$.

Theorem. Let $f : U \;\to\; C$ be continuously differentiable.

If $\Delta(a_0, r_0) \subset \Delta(a_1, r_1)$ and $\Delta(a_1, r_1) - D(a_0, r_0) \subset U$, then

$$\int_{\partial\Delta(a_0,r_0)} f(z)\,dz \;=\; \int_{\partial\Delta(a_1,r_1)} f(z)\,dz.$$

Proof. Set $a(t) = (1-t)a_0 \;+\; ta_1$ and $r(t) = (1-t)r_0 \;+\; tr_1$ for $0 \le t \le 1$. Let $\gamma(t)(\theta) \;=\; \gamma(\theta, t) \;=\; a(t) \;+\; r(t)e^{i\theta}$ for each $0 \le t \le 1$ and $0 \le \theta \le 2\pi$. The correspondence $t \to \gamma(t)$ defines a parameterized family of circular paths inside of U. Set $F(t) \;=\; \int_{\gamma(t)} f(z)\,dz$ for each $0 \le t \le 1$. We must show that $F(0) = F(1)$. First note that

$$F(t) = \int_0^{2\pi} f(\gamma(\theta,t))\frac{\partial\gamma}{\partial\theta}(\theta,t)\,d\theta.$$

Differentiating with respect to t and applying differentiation under the integral and the product rule, we find that $F'(t)$ equals

$$\int_0^{2\pi} f'(\gamma(\theta,t))\frac{\partial\gamma}{\partial t}(\theta,t)\frac{\partial\gamma}{\partial\theta}(\theta,t)d\theta+\int_0^{2\pi} f(\gamma(\theta,t))\frac{\partial}{\partial t}\left(\frac{\partial\gamma}{\partial\theta}\right)(\theta,t)d\theta.$$

Let I_1 denote the first integral and let I_2 denote the second integral. Integrating by parts in I_1, we obtain

$$I_1 = \left[f(\gamma(\theta,t))\frac{\partial\gamma}{\partial t}(\theta,t)\right]_{\theta=0}^{\theta=2\pi} - \int_0^{2\pi} f(\gamma(\theta,t))\frac{\partial}{\partial\theta}\left(\frac{\partial\gamma}{\partial t}\right)(\theta,t)d\theta.$$

Thus, $I_1 = -I_2$ since the bracketed term vanishes and the mixed partials of γ are equal. Hence, $F'(t) = 0$ for each $0 \le t \le 1$ and so F is constant.

Corollary. If $f : U \to C$ is continuously differentiable and if $\Delta(a,r) \subset U$, then

$$f(w) = (2\pi i)^{-1}\int_{\partial\Delta(a,r)} \frac{f(z)}{z-w}\,dz$$

for each $w \in D(a,r)$.

Proof. Assume that $\Delta(a,r) \subset U$ and let $w \in D(a,r)$. If $0 < \epsilon < r- \mid w-a \mid$, then by the above independence of path theorem we have

$$\int_{\partial\Delta(a,r)} \frac{f(z)}{z-w}\,dz = \int_{\partial\Delta(w,\epsilon)} \frac{f(z)}{z-w}\,dz$$
$$= i\int_0^{2\pi} f(w+\epsilon\, e^{i\theta})d\theta.$$

Letting $\epsilon \to 0$, we obtain

$$\int_{\partial\Delta(a,r)} \frac{f(z)}{z-w}\,dz \;=\; 2\pi i f(w).$$

Using the corollary and standard techniques [2, p. 145], one may show that if f is continuously differentiable on an open disk D, then f is representable by power series on D. Our proof of the theorem can be used to establish the following homotopy version of the Cauchy integral theorem. If γ is a twice continuously differentiable homotopy with fixed end points between the paths α and β in U and if f is continuously differentiable on U, then $\int_\alpha f(z)\,dz = \int_\beta f(z)\,dz$. In [1, pp. 70-72], Conway presents a different simple proof of the Cauchy integral theorem for circles but his proof does not yield the above homotopy version. One may develop a general version of the Cauchy integral formula for closed chains homologous to zero by using Dixon's proof [2, pp. 162-164].

The Cauchy-Goursat theorem may still be presented without retracing any steps in the development of the basic properties of analytic functions.

References

1. John Conway, *Functions of One Complex Variable*, Springer-Verlag, New York, 1973.
2. Serge Lang, *Complex Analysis*, 2nd ed., Springer-Verlag, New York, 1985.