A NOTE ON A DIFFERENTIAL EQUATION

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If $a \neq b$, then two linearly independent solutions of the differential equation

$$y'' - (a+b)y' + aby = 0 \tag{1}$$

are e^{ax} and e^{bx} and the general solution of (1) is

$$y = c_1 e^{ax} + c_2 e^{bx} . \tag{2}$$

When $a = b$, the two fundamental solutions are $y_1 = e^{ax}$ and $y_2 = xe^{ax}$. This is easy to check but not so easy to motivate, especially y_2. The motivation of the form of the general solution in the case of equal roots of the characteristic equation can be accomplished by considering the case when $a \neq b$, renaming the constants in equation (2), and considering the limit as b approaches a.

Letting

$$c_1 = c_3 - \frac{c_4}{b-a} \quad \text{and} \quad c_2 = \frac{c_4}{b-a}$$

in (2) leads to

$$y = c_3 e^{ax} + c_4 \frac{e^{bx} - e^{ax}}{b-a} \tag{3}$$

as the general solution of (1).

Notice that when $a = b$ the second term in equation (3) is of the indeterminate form $\frac{0}{0}$. So, employing L'Hospital's rule to compute the limit as b approaches a in (3) yields

$$y = c_3 e^{ax} + c_4 x e^{ax} \ ,$$

which is the general solution of (1) when $a = b$ and the technique utilized clearly shows how $y_2 = xe^{ax}$ arises.