ALMOST LOCALLY CONNECTED(SO) SPACES

Shing-Seung So

Central Missouri State University

A topological space M is almost locally connected(SO) at a point p iff for any open subset U of M containing p and the component C of U containing p, $\overline{C} \cap \partial U \neq \emptyset$ and C is nonclosed [1]. In this paper, we discuss the structure and properties of almost locally connected(SO) spaces and some applications. Throughout this paper, we assume all the spaces are topological and Hausdorff.

Definition. A space M is *almost locally connected(SO)* if M is almost locally connected(SO) at every point.

Theorem 1. The following conditions are equivalent in the space M.

(1) M is almost locally connected(SO) at p.

(2) If U is an open subset of M containing p, there exists a nonclosed connected subset V of U containing p such that $\overline{V} \cap \partial U \neq \emptyset$.

(3) $\overline{C} \cap \overline{U} \neq \emptyset$.

(4) $\partial C \cap \partial U \neq \emptyset$ where U is an open subset containing p, C is the component of p in U and C is nonclosed.

Proof.

(1) $\Rightarrow$ (2) Let V be the component of U containing p. Then V is nonclosed and $\overline{V} \cap \partial U \neq \emptyset$.

(2) $\Rightarrow$ (3) If C is closed then $\overline{C} \cap \partial U = \emptyset$ and hence $\overline{V} \cap \partial U = \emptyset$ for every nonclosed connected subset V of U containing p, a contradiction.

(3) $\Rightarrow$ (4) $\overline{C} \cap \overline{U} \neq \emptyset$ and $C \subset U$ imply $\partial C \cap \partial U \neq \emptyset$.

(4) $\Rightarrow$ (1) This follows immediately from the definition.

Theorem 2. If M is almost locally connected(SO) at p and U is an open subset of M containing p, then U is almost locally connected(SO) at p.

Proof. If V is an open subset of U containing p, then V is open in M. Consequently, there exists a nonclosed connected subset G of V containing p such that $\overline{G} \cap \partial V \neq \emptyset$, and the theorem follows.

Theorem 3. M is almost locally connected(SO) iff for every subset A of M and every component C of $M - A$, $\overline{C} \cap \partial A \neq \emptyset$.

Proof. "$\Rightarrow$" Suppose $\overline{C} \cap \partial A = \emptyset$. Since C is a subset of $M - A$, $\overline{C} \cap \overline{A} = \emptyset$. Hence, $\overline{C} \subset M - \overline{A}$. Now, M is almost locally connected(SO), so according to Theorem 2 there exists a nonclosed component K of $M - \overline{A}$ such that $\overline{K} \cap \partial(M - \overline{A}) \neq \emptyset$. By the definition of component, $C \subset K$. On the other hand, C is a component of $M - A$ and $M - \overline{A} \subset M - A$. Therefore, $K \subset C$ and hence C = K. It follows that $\overline{K} \subset M - \overline{A}$ and $\overline{K} \cap \partial(M - \overline{A}) \neq \emptyset$, a contradiction. Therefore, $\overline{C} \cap \partial A \neq \emptyset$.

"$\Leftarrow$" Suppose M is not almost locally connected(SO). Then there exists an open subset U of M and a nonclosed component C of M such that $\overline{C} \cap \partial U = \emptyset$. If $A = M - U$, then C is a component of $M - A$ and $\overline{C} \cap \partial A = \emptyset$, a contradiction.

Theorem 4. Let M be almost locally connected(SO) at p and let h be a homeomorphism from M to the space N. Then N is almost locally connected(SO) at h(p).

Proof. Let U be an open subset of N containing h(p). Then there exists an open set O in M such that $p \epsilon O$ and $h(O) = U$. Since M is almost locally connected(SO) at p, there is a nonclosed connected subset G of O such that $p \epsilon G$ and $\overline{G} \cap \partial O \neq \emptyset$. Since h is a homeomorphism, h(G) is nonclosed and connected.

Suppose $\overline{h(G)} \cap \partial U = \emptyset$. Then $h(\overline{G}) = \overline{h(G)} \subset U = h(O)$. This implies $\overline{G} \subset O$, which contradicts $\overline{G} \cap \partial O \neq \emptyset$. Therefore, $\overline{h(G)} \cap \partial U \neq \emptyset$ and N is almost locally connected(SO) at h(p).

Theorem 5. If M is connected and locally connected at p, then M is almost locally connected(SO) at p.

Proof. Theorem 5 follows from the fact that M is locally connected at p and no proper subset of M is both open and closed.

The following example shows that connectedness is necessary in Theorem 5.

Example 1. Let M be a nonempty finite set with the discrete topology. Then M is locally connected but M is not almost locally connected(SO) at any point in M.

Theorem 6. If M is almost locally connected(SO) and U is an open subset of M such that ∂U is connected, then $\overline{U}$ is connected.

Proof. Since

$$\overline{U} = \bigcup \{\overline{C} \mid C \text{ is a component of } U \text{ and } \overline{C} \cap \partial U \neq \emptyset\},$$

$\overline{U}$ is connected.

The following theorem is well known.

Theorem 7. If C is a component of the compact set X in a topological space, then each open set containing C contains an open set containing C whose boundary does not intersect X.

Theorem 8. If M is connected and locally compact at p, then M is almost locally connected(SO) at p.

Proof. Let U be an open set containing p such that $\overline{U}$ is compact. Let C be the component of U containing p. Suppose $\overline{C} \cap \partial U = \emptyset$. Then C is closed in U. Since $\overline{C}$ and ∂U are compact, there exist disjoint sets A and B relatively open in $\overline{U}$ such that $\overline{C} \subset A$ and $\partial U \subset B$. It follows that $\overline{C} \subset A$ and $\partial U \cap A = \emptyset$ so $A \subset U$. Applying Theorem 7 to $\overline{A}$, there exists an open set V such that $C \subset V \subset A$ and $\partial V \cap \overline{A} = \emptyset$. Hence $\partial V = \emptyset$. Consequently, V is both open and closed in M, which contradicts the fact that M is connected.

Definition. A set K in a Hausdorff, topological space M is called a *continuum* if K is compact and connected. A set K is called a *generalized continuum* if it is locally compact and connected.

Corollary. Every generalized continuum and thus every continuum is almost locally connected(SO).

Example 2. Let

$$M = \{(0,0)\} \cup \{(x,y) \mid 0 < x \leq 1 \; and \; y = sin\,(1/x)\}.$$

M is connected, but M is neither locally compact nor almost locally connected(SO) at (0,0).

Examples 1 and 2 illustrate that both connectedness and local compactness are necessary for Theorem 8. The following example illustrates that the converses of both Theorems 5 and 8 are not true even if the space is connected.

Example 3. Let

$$M = \{(0,y) \mid 0 \leq y \leq 1\} \cup \{(x,y) \mid 0 < x \leq 1, \; y = sin(1/x)\}.$$

M is connected and almost locally connected(SO) at (0,0), but M is neither locally compact nor locally connected at (0,0).

The following definition can be found in [2].

Definition. The connected set M is *semilocally connected at a point p* if each open set containing p contains an open set V containing p such that $M - V$ has at most a finite number of components.

Example 1 shows that semilocal connectedness does not imply almost local connectedness(SO).

Jones [3] established the following definitions and theorem.

Definition. A space M is *aposyndetic at a point p with respect to a point q* if there is a closed connected set H such that p is in the interior of H and H is a subset of $M - \{q\}$.

Definition. The space M is *aposyndetic at a point p* if it is aposyndetic at p with respect to q for each q in $M - \{p\}$.

Definition. The space M is *aposyndetic* if it is aposyndetic at every point.

Theorem 9. If M is semilocally connected, then M is aposyndetic.

Example 4. Let

$$M = \{(x,y) \mid y = sin(1/x),\ x \neq 0\}$$
$$\cup \{(0,y) \mid 0 \leq y \leq 1\} \cup \{(-1/n,\ 0) \mid n = 1,2,3,\ldots\} .$$

M is almost locally connected(SO), but not aposyndetic at (0,0).

Definition. A *cut point* of a connected set M is a point p of M such that $M - \{p\}$ is disconnected.

Definition. p is an *end point* of a connected set M if each open set containing p contains an open set containing p whose boundary is degenerate.

Definition. A point r of a connected set M *separates points p and q in M* if $M - \{r\}$ is the union of two separated sets, one containing p and the other containing q.

Definition. Two points p and q of a connected set M are said to be *conjugate* in M if no point of M separates p and q in M.

Whyburn [2] first established the following decomposition theorem in cyclic element theory.

Theorem 10. If M is a connected, locally compact metric space and p is neither a cut point nor an end point of M, then there exists a point of M other than p which is conjugate to p.

Theorems 11 and 12 are similar results obtained by B. Lehman [4] and D. John [5] respectively.

Theorem 11. If M is a connected, locally compact, locally connected topological space and p is neither a cut point nor an end point of M, then there exists a point of M other than p which is conjugate to p.

Theorem 12. If M is a connected, locally compact topological space and p is neither a cut point nor an end point of M, then there exists a point of M other than p which is conjugate to p.

The statements in Theorems 10, 11 and 12 seem to suggest that local compactness plays a crucial part in the conclusion of these theorems. However, Theorem 14 below shows that the result still holds even without local compactness.

John [5] established the following theorem.

Theorem 13. If p is a non-cut point of a connected topological space M and M is semilocally connected at p, then for every open set U containing p there exists an open subset V of U containing p such that $M - V$ is connected.

Theorem 14. If M is a connected, almost locally connected(SO), semilocally connected topological space and p is neither a cut point nor an end point of M, then there exists a point of M other than p which is conjugate to p in M.

Proof. Suppose there is no point q in $M - \{p\}$ such that q is conjugate to p in M. Let U be an open set containing p. Then according to Theorem 13 there exists an open subset V of U containing p such that $M - V$ is connected. Let C be the component of V containing p. Since M is almost locally connected(SO), we have $\partial V \cap \overline{C} \neq \emptyset$. Let $z \in \partial V \cap \overline{C}$. Since z is not conjugate to p in M, there exists y in M such that $M - \{y\} = H \cup K$, where H and K are separated and p is in H and z is in K. If y is not in C, then C is connected in $M - \{y\}$, which implies $C \cup \{z\}$ is a connected subset of $M - \{y\}$. It follows that $C \cup \{z\} \subset H$ or $C \cup \{z\} \subset K$, which implies $z \in H$ or $p \in K$. Either case is impossible.

Therefore, $y \in C$. That is $M - V \subset K$. This implies $H \subset V$. Since H is open and $\partial H = \{y\}$, p is an end point and this contradicts the assumption.

References

1. L. W. Rodgers, Some Recent Developments in Connected and Locally Connected Spaces, Master's Thesis, Central Missouri State University, 1988.

2. G. T. Whyburn, *Analytic Topology*, First Edition, American Mathematical Society, Colloquium Publications, Vol. 28, 1942.

3. F. B. Jones, “Aposyndetic Continua and Certain Boundary Problems,” *American Journal of Mathematics*, 63(1941) 545-553.

4. B. Lehman, “Cyclic Element Theory in Connected and Locally Connected Hausdorff Spaces,” *Canadian Journal of Mathematics*, Vol. 27, 5(1976) 1032-1050.

5. D. A. John, Cyclic Element Theory in Paraseparable Connected Hausdorff Spaces, Doctoral Dissertation, University of Missouri-Kansas City, 1979.