EVALUATION OF A FAMILY OF IMPROPER INTEGRALS

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It is easy to see that the improper integral

$$(1) \qquad I(q) = \int_1^\infty \frac{dx}{x(x^{q-1} + \cdots + x + 1)}$$

converges if $q \geq 2$. The purpose of this paper is to evaluate (1) when q is an integer.

It has been shown in [1], page 37, that

$$(2) \qquad \psi(a) - \psi(a - b) = \frac{\Gamma(a)}{\Gamma(b)} \sum_{n=1}^{\infty} \frac{\Gamma(b + n)}{n\Gamma(a + n)}$$

for $Re(a) > Re(b) \geq 0$. Equation (2) will be of particular interest when the parameters are specialized by letting $a = 1$ and $b = \frac{1}{q}$ for q = 2, 3, 4, ... For then, (2) becomes

$$\frac{\Gamma(1)}{\Gamma\left(\frac{1}{q}\right)} \sum_{n=1}^{\infty} \frac{\Gamma(\frac{1}{q} + n)}{n\Gamma(1 + n)} = \psi(1) - \psi\left(1 - \frac{1}{q}\right) ,$$

and so

$$(3) \qquad \sum_{n=1}^{\infty} \frac{\left(\frac{1}{q}\right)_n}{n!\, n} = -\gamma - \psi\left(1 - \frac{1}{q}\right) ,$$

where γ is the Euler-Mascheroni constant and

$$(r)_n \;=\; \frac{\Gamma(r\;+\;n)}{\Gamma(r)} \;=\; r(r\;+\;1)\;\cdots\;(r\;+\;n\;-\;1)\;.$$

Now, from page 13 of [2], it is known that

(4)

$$\psi\left(1\;-\;\frac{1}{q}\right) \;=\; -\,\gamma\;-\;ln\;q\;-\;\frac{\pi}{2}cot\frac{(q\;-\;1)\pi}{q}$$

$$+\;\sum_{n=1}^{\left[\frac{q}{2}\right]} cos\frac{2n(q\;-\;1)\pi}{q} ln\left(2\;-\;2cos\frac{2n\pi}{q}\right)\;,$$

where the prime attached to the summation index indicates that if

q is even then the last term in the sum is taken at half its weight.

Substituting (4) into (3) and simplifying gives

(5)

$$\sum_{n=1}^{\infty} \frac{1(1\;+\;q)(1\;+\;2q)\;\cdots\;(1\;+\;(n-1)q)}{n!\;q^n n} \;=\; ln\;q$$

$$+\;\frac{\pi}{2}cot\frac{(q\;-\;1)\pi}{q}\;-\;2\sum_{n=1}^{\left[\frac{q}{2}\right]} cos\frac{2n(q\;-\;1)\pi}{q} ln\left(2\left|sin\frac{n\pi}{q}\right|\right)\;.$$

Since

$$(1\;-\;u)^{-\frac{1}{q}} \;=\; \sum_{n=0}^{\infty} \binom{-\frac{1}{q}}{n}(-u)^n \quad for\;-1\;\leq\;u\;<\;1\;,$$

(6)

$$\int_0^1 \frac{(1 - u)^{-\frac{1}{q}} - 1}{u} du = \sum_{n=1}^{\infty} \binom{-\frac{1}{q}}{n} (-1)^n \int_0^1 u^{n-1} du$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n \left(-\frac{1}{q}\right)\left(-\frac{1}{q} - 1\right)\left(-\frac{1}{q} - 2\right) \cdots \left(-\frac{1}{q} - n + 1\right)}{n!\ n}$$

$$= \sum_{n=1}^{\infty} \frac{1(1 + q)(1 + 2q) \cdots (1 + (n-1)q)}{n!\ q^n n} .$$

In (6), let

$$x = (1 - u)^{-\frac{1}{q}} .$$

Then

$$u = 1 - x^{-q} , \ du = q\, x^{-q-1}\, dx ,$$

and so

(7)

$$q \int_1^{\infty} \frac{(x - 1)dx}{(1 - x^{-q})\, x^{q+1}} = \sum_{n=1}^{\infty} \frac{1\ (1 + q) \cdots (1 + (n-1)q)}{n!\ q^n\ n} .$$

Simplifying (7) and using (5) yields

$$I(q) = \frac{1}{q}\Bigg(ln\ q + \frac{\pi}{2} cot \frac{(q-1)\pi}{q}$$

$$- 2 \sum_{n=1}^{\left[\frac{q}{2}\right]} cos \frac{2n(q - 1)\pi}{q} ln \left(2 \left| sin \frac{n\pi}{q} \right| \right) \Bigg) .$$

References

1. A. McBride and G. Roach, *Fractional Calculus*, Research Notes in Mathematics, Pitman Advanced Publishing Program, 1985.
2. Y. Luke, *The Special Functions and Their Approximations*, Vol. 1, Academic Press, 1969.