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Calderón-Zygmund operators with variable kernels acting on weak Musielak-Orlicz Hardy spaces

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ABSTRACT. Let $\varphi : \mathbb{R}^n \times [0, \infty) \to [0, \infty)$ satisfy that $\varphi(x, \cdot)$ is an Orlicz function for any given $x \in \mathbb{R}^n$, and $\varphi(\cdot, t)$ is a Muckenhoupt A_∞ weight uniformly in $t \in (0, \infty)$. The weak Musielak-Orlicz Hardy space $WH^\varphi(\mathbb{R}^n)$ is defined to be the set of all tempered distributions such that their grand maximal functions belong to the weak Musielak-Orlicz space $WL^\varphi(\mathbb{R}^n)$. In this paper, we discuss the boundedness of the Calderón-Zygmund operator with variable kernel from $WH^\varphi(\mathbb{R}^n)$ to $WL^\varphi(\mathbb{R}^n)$. These results are new even for the classical weighted weak Hardy space and probably new for the classical weak Hardy space.

1. Introduction

Let S^{n-1} be the unit sphere in the n-dimensional Euclidean space $\mathbb{R}^n$ $(n \geq 2)$ with normalized Lebesgue measure $d\sigma$. A function $\Omega(x, z)$ defined on $\mathbb{R}^n \times \mathbb{R}^n$ is said to be in $L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$ with $q \geq 1$, if $\Omega(x, z)$ satisfies the following conditions:

$$\Omega(x, \lambda z) = \Omega(x, z) \qquad \text{for any } x, z \in \mathbb{R}^n \text{ and } \lambda \in (0, \infty), \tag{1}$$

$$\int_{S^{n-1}} \Omega(x, z) d\sigma(z') = 0 \qquad \text{for any } x \in \mathbb{R}^n, \tag{2}$$

$$\sup_{x \in \mathbb{R}^n} \left(\int_{S^{n-1}} |\Omega(x, z')|^q d\sigma(z') \right)^{1/q} < \infty, \tag{3}$$

where $z' := z/|z|$ for any $z \neq \mathbf{0}$. Set $K(x, z) := \Omega(x, z')/|z|^n$ for all $(x, z) \in \mathbb{R}^n \times \mathbb{R}^n$. The *Calderón-Zygmund operator with variable kernel* is defined by

$$T(f)(x) := \text{p.v.} \int_{\mathbb{R}^n} K(x, x - y) f(y) dy.$$

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In 1955 and 1956, Calderón and Zygmund [2, 3] investigated the L^p boundedness of T. They found that these operators are closely related to the problem about certain second-order linear elliptic equations with variable coefficients. In 2008, Lee et al. [19] further discussed the boundedness of T on the weighted Lebesgue space $L^p_\omega(\mathbb{R}^n)$ under the Hörmander condition assumed on kernel, where $\omega \in A_p$ and A_p denotes the Muckenhoupt weight class. Their result is the following theorem. We denote the *conjugate index* of $q > 1$ by $q' := q/(q-1)$.

Theorem A ([19, Theorem 1]). *Let $q \in (1, \infty]$. Suppose that $\Omega \in L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$ satisfies that, for any $R \in (0, \infty)$,*

$$\sup_{\substack{x \in \mathbb{R}^n \\ 0<|y|<R}} \sum_{m=1}^{\infty} (2^m R)^{n/q'} \left(\int_{2^m R \le |z| < 2^{m+1} R} |K(x, z-y) - K(x, z)|^q dz \right)^{1/q} \le C < \infty \quad (4)$$

and

$$\sup_{\substack{x, y \in \mathbb{R}^n \\ 0<|x-y|<R}} \sum_{m=1}^{\infty} (2^m R)^{n/q'} \left(\int_{2^m R \le |z| < 2^{m+1} R} |K(x, z) - K(y, z)|^q dz \right)^{1/q} \le C < \infty. \quad (5)$$

If $\omega \in A_{p/q'}$ with $p \in [q', \infty)$, then T is bounded on $L^p_\omega(\mathbb{R}^n)$.

Not only that, they also established the boundedness of T from a weighted Hardy space to a weighted Lebesgue space under an extra Dini type condition assumed on Ω.

On the other hand, the impact of the theory of Hardy space in the last forty years has been significant. The classical Hardy space on the unit circle or upper half-plane is defined with the aid of complex method. And its theory was one-dimensional. The higher dimensional Euclidean theory of the Hardy space was developed by Fefferman and Stein [8] who proved a variety of characterizations for them. As everyone knows, many important operators have better behaved on the Hardy space $H^p(\mathbb{R}^n)$ than on the Lebesgue space $L^p(\mathbb{R}^n)$ in the range $p \in (0, 1]$. For example, when $p \in (0, 1]$, the Riesz transforms are bounded on $H^p(\mathbb{R}^n)$, but not on $L^p(\mathbb{R}^n)$. Therefore, one can consider $H^p(\mathbb{R}^n)$ to be a very natural replacement for $L^p(\mathbb{R}^n)$ when $p \in (0, 1]$. Moreover, when studying the endpoint estimate for variant important operators, the weak Hardy space $WH^p(\mathbb{R}^n)$ naturally appears as a good substitute of $H^p(\mathbb{R}^n)$ with $p \in (0, 1]$. For instance, if $\delta \in (0, 1]$, T_δ is a δ-Calderón-Zygmund operator and $T^*_\delta(1) = 0$, where T^*_δ denotes the adjoint operator of T_δ, it is known that T_δ is bounded on $H^p(\mathbb{R}^n)$ for any $p \in (n/(n+\delta), 1]$ (see [1]), but T_δ may be not bounded on $H^{n/(n+\delta)}(\mathbb{R}^n)$; however, Liu [18] proved that T_δ is bounded from $H^{n/(n+\delta)}(\mathbb{R}^n)$ to $WH^{n/(n+\delta)}(\mathbb{R}^n)$.

Recently, Ky [15] introduced a new Musielak-Orlicz Hardy space $H^{\varphi}(\mathbb{R}^n)$, which unifies the classical Hardy space [8], the weighted Hardy space [26], the Orlicz Hardy space [11, 12, 13, 14], and the weighted Orlicz Hardy space. Its spatial and time variables may not be separable. Later, Liang et al. [22] further introduced a weak Musielak-Orlicz Hardy space $WH^{\varphi}(\mathbb{R}^n)$, which covers both the weak Hardy space [9], the weighted weak Hardy space [25], the weak Orlicz Hardy space and the weighted weak Orlicz Hardy space, as special cases. And various equivalent characterizations of $WH^{\varphi}(\mathbb{R}^n)$ by means of maximal functions, atoms, molecules and Littlewood-Paley functions, and the boundedness of Calderón-Zygmund operators in the critical case were obtained in [22]. Apart from interesting theoretical considerations, the motivation to study Musielak-Orlicz-type space comes from applications to elasticity, fluid dynamics, image processing, nonlinear PDEs and the calculus of variation (see, for example, [4, 5]). More Musielak-Orlicz-type spaces are referred to [10, 16, 21, 23, 24, 27, 28, 29].

Motivated by all of the facts mentioned above, it is a natural and interesting problem to ask whether the Calderón-Zygmund operator with variable kernel T is bounded from $WH^{\varphi}(\mathbb{R}^n)$ to the weak Musielak-Orlicz space $WL^{\varphi}(\mathbb{R}^n)$. In this paper we shall answer this problem affirmatively. Our results are new even for the classical weighted weak Hardy space and probably new for the classical weak Hardy space.

This paper is organized as follows. In the next section, we recall some notions concerning Muckenhoupt weights, growth functions and weak Musielak-Orlicz Hardy spaces. Then we present the boundedness of T from $WH^{\varphi}(\mathbb{R}^n)$ to $WL^{\varphi}(\mathbb{R}^n)$ (see Theorem 1, Theorem 2, Corollary 1 and Theorem 3 below). In Section 3, with the help of some auxiliary lemmas and the atomic decomposition theory of $WH^{\varphi}(\mathbb{R}^n)$, the proofs of main results are presented.

Finally, we make some conventions on notation. Let $\mathbb{Z}_+ := \{1, 2, \dots\}$ and $\mathbb{N} := \{0\} \cup \mathbb{Z}_+$. For any $\beta := (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$, let $|\beta| := \beta_1 + \cdots + \beta_n$ and $\partial^{\beta} := \left(\frac{\partial}{\partial x_1}\right)^{\beta_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\beta_n}$. Throughout this paper the letter C will denote a *positive constant* that may vary from line to line but will remain independent of the main variables. The *symbol* $P \lesssim Q$ stands for the inequality $P \le CQ$. If $P \lesssim Q \lesssim P$, we then write $P \sim Q$. For any sets $E, F \subset \mathbb{R}^n$, we use $E^{\complement}$ to denote the set $\mathbb{R}^n \backslash E$, $|E|$ the *n-dimensional Lebesgue measure* of E, χ_E the *characteristic function* of E, and $E + F$ the *algebraic sum* $\{x + y : x \in E,\ y \in F\}$. For any $s \in \mathbb{R}$, $\lfloor s \rfloor$ denotes the unique integer such that $s - 1 < \lfloor s \rfloor \le s$. If there are no special instructions, any space $\mathscr{X}(\mathbb{R}^n)$ is denoted simply by $\mathscr{X}$. For instance, $L^2(\mathbb{R}^n)$ is simply denoted by L^2. For any set $E \subset \mathbb{R}^n$, $t \in [0, \infty)$ and measurable function $\varphi(\cdot, t)$, let $\varphi(E, t) := \int_E \varphi(x, t)dx$ and $\{|f| > t\} := \{x \in \mathbb{R}^n : |f(x)| > t\}$. For any $x \in \mathbb{R}^n$, $r \in (0, \infty)$ and $\alpha \in (0, \infty)$, we use $B(x,r)$ to denote the ball $\{y \in \mathbb{R}^n : |y-x| < r\}$ and $\alpha B(x,r)$ to denote $B(x,\alpha r)$ as usual.

2. Notions and main results

In this section, we first recall the notion concerning the weak Musielak-Orlicz Hardy space WH^{φ}, and then present the boundedness of the Calderón-Zygmund operator with variable kernel T from WH^{φ} to the weak Musielak-Orlicz space WL^{φ}.

Recall that a nonnegative function φ on $\mathbb{R}^n \times [0,\infty)$ is called a *Musielak-Orlicz function* if, for any $x \in \mathbb{R}^n$, $\varphi(x,\cdot)$ is an Orlicz function on $[0,\infty)$ and, for any $t \in [0,\infty)$, $\varphi(\cdot,t)$ is measurable on $\mathbb{R}^n$. Here a function $\phi : [0,\infty) \to [0,\infty)$ is called an *Orlicz function*, if it is nondecreasing, $\phi(0) = 0$, $\phi(t) > 0$ for any $t \in (0,\infty)$, and $\lim_{t\to\infty} \phi(t) = \infty$.

Given a Musielak-Orlicz function φ on $\mathbb{R}^n \times [0,\infty)$, φ is said to be of *uniformly lower* (resp. *upper*) *type* p with $p \in \mathbb{R}$, if there exists a positive constant C such that, for any $x \in \mathbb{R}^n$, $t \in [0,\infty)$ and $s \in (0,1]$ (resp. $s \in [1,\infty)$),

$$\varphi(x,st) \le Cs^p \varphi(x,t).$$

The *critical uniformly lower type index* of φ is defined by

$$i(\varphi) := \sup\{p \in \mathbb{R}\colon \ \varphi \text{ is of uniformly lower type } p\}. \tag{6}$$

Observe that $i(\varphi)$ may not be attainable, namely, φ may not be of uniformly lower type $i(\varphi)$ (see [20, p. 415] for more details).

Definition 1. (i) Let $q \in [1,\infty)$. A locally integrable function $\varphi(\cdot,t) : \mathbb{R}^n \to [0,\infty)$ is said to satisfy the *uniformly Muckenhoupt condition* $\mathbb{A}_q$, denoted by $\varphi \in \mathbb{A}_q$, if there exists a positive constant C such that, for any ball $B \subset \mathbb{R}^n$ and $t \in (0,\infty)$, when $q = 1$,

$$\frac{1}{|B|}\int_B \varphi(x,t)dx\left\{\operatorname*{ess\,sup}_{x\in B}[\varphi(x,t)]^{-1}\right\} \le C$$

and, when $q \in (1,\infty)$,

$$\frac{1}{|B|}\int_B \varphi(x,t)dx\left\{\frac{1}{|B|}\int_B [\varphi(x,t)]^{-1/(q-1)}dx\right\}^{q-1} \le C.$$

(ii) Let $q \in (1,\infty]$. A locally integrable function $\varphi(\cdot,t) : \mathbb{R}^n \to [0,\infty)$ is said to satisfy the *uniformly reverse Hölder condition* $\mathbb{RH}_q$, denoted by $\varphi \in \mathbb{RH}_q$, if there exists a positive constant C such that, for any ball $B \subset \mathbb{R}^n$ and $t \in (0, \infty)$, when $q \in (1, \infty)$,

$$\left\{\frac{1}{|B|}\int_B \varphi(x,t)dx\right\}^{-1}\left\{\frac{1}{|B|}\int_B [\varphi(x,t)]^q dx\right\}^{1/q} \le C$$

and, when $q = \infty$,

$$\left\{\frac{1}{|B|}\int_B \varphi(x,t)dx\right\}^{-1} \operatorname*{ess\,sup}_{x \in B} \varphi(x,t) \le C.$$

Define $\mathbb{A}_\infty := \bigcup_{q \in [1,\infty)} \mathbb{A}_q$. It is well known that if $\varphi \in \mathbb{A}_q$ with $q \in (1, \infty]$, then $\varphi^\varepsilon \in \mathbb{A}_{\varepsilon q+1-\varepsilon} \subset \mathbb{A}_q$ for any $\varepsilon \in (0,1]$ and $\varphi^\eta \in \mathbb{A}_q$ for some $\eta \in (1,\infty)$. Also, if $\varphi \in \mathbb{A}_q$ with $q \in (1,\infty)$, then $\varphi \in \mathbb{A}_r$ for any $r \in (q, \infty)$ and $\varphi \in \mathbb{A}_d$ for some $d \in (1, q)$. Thus, the *critical weight index* of $\varphi \in \mathbb{A}_\infty$ is defined as follows:

$$q(\varphi) := \inf\{q \in [1,\infty) : \varphi \in \mathbb{A}_q\}. \tag{7}$$

For the uniformly Muckenhoupt (resp. reverse Hölder) condition, we have the following property as the classical case.

Lemma 1 ([15, Lemma 4.5]). *Let $\varphi \in \mathbb{A}_q$ with $q \in [1,\infty)$. Then there exists a positive constant C such that, for any ball $B \subset \mathbb{R}^n$, $\lambda \in (1,\infty)$ and $t \in (0,\infty)$,*

$$\varphi(\lambda B, t) \le C\lambda^{nq}\varphi(B,t).$$

Lemma 2 ([17, Lemma 3.5]). *Let $r \in (1,\infty)$. Then $\varphi^r \in \mathbb{A}_\infty$ if and only if $\varphi \in \mathbb{RH}_r$.*

Definition 2 ([15, Definition 2.1]). A function $\varphi : \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ is called a *growth function* if the following conditions are satisfied:

(i) φ is a Musielak-Orlicz function;

(ii) $\varphi \in \mathbb{A}_\infty$;

(iii) φ is of uniformly lower type p for some $p \in (0,1]$ and of uniformly upper type 1.

Recall that the *weak Musielak-Orlicz space* WL^φ is defined to be the space of all measurable functions f such that, for some $\lambda \in (0,\infty)$,

$$\sup_{t \in (0,\infty)} \varphi\left(\{|f| > t\}, \frac{t}{\lambda}\right) < \infty$$

equipped with the quasi-norm

$$\|f\|_{WL^\varphi} := \inf\left\{\lambda \in (0,\infty) : \sup_{t \in (0,\infty)} \varphi\left(\{|f| > t\}, \frac{t}{\lambda}\right) \le 1\right\}.$$

In what follows, we denote by $\mathcal{S}$ the *space of all Schwartz functions* and by $\mathcal{S}'$ its *dual space* (namely, the *space of all tempered distributions*). For any $m \in \mathbb{N}$, let $\mathcal{S}_m$ be the *space* of all $\psi \in \mathcal{S}$ satisfying $\|\psi\|_{\mathcal{S}_m} \le 1$, where

$$\|\psi\|_{\mathcal{S}_m} := \sup_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| \le m+1}} \sup_{x \in \mathbb{R}^n} (1 + |x|)^{(m+2)(n+1)} |\partial^\alpha \psi(x)|.$$

Then, for any $m \in \mathbb{N}$ and $f \in \mathcal{S}'$, the *non-tangential grand maximal function* f_m^* of f is defined by setting, for all $x \in \mathbb{R}^n$,

$$f_m^*(x) := \sup_{\psi \in \mathcal{S}_m} \sup_{\substack{|y-x|<t \\ t \in (0,\infty)}} |f * \psi_t(y)|, \tag{8}$$

where, for any $t \in (0, \infty)$, $\psi_t(\cdot) := t^{-n}\psi\left(\frac{\cdot}{t}\right)$. When

$$m = m(\varphi) := \left\lfloor n\left(\frac{q(\varphi)}{i(\varphi)} - 1\right)\right\rfloor, \tag{9}$$

we denote f_m^* simply by f^*, where $q(\varphi)$ and $i(\varphi)$ are as in (7) and (6), respectively.

Definition 3 ([22, Definition 2.3]). Let φ be a growth function. The *weak Musielak-Orlicz Hardy space* WH^φ is defined as the space of all $f \in \mathcal{S}'$ such that $f^* \in WL^\varphi$ endowed with the quasi-norm

$$\|f\|_{WH^\varphi} := \|f^*\|_{WL^\varphi}.$$

Remark 1. *Let ω be a Muckenhoupt weight and ϕ an Orlicz function.*

(i) *If $\varphi(x,t) := \omega(x)\phi(t)$ for all $(x,t) \in \mathbb{R}^n \times [0,\infty)$, then WH^φ goes back to the weighted weak Orlicz Hardy space WH_ω^ϕ, and particularly, when $\omega \equiv 1$, the corresponding unweighted space is also obtained.*

(ii) *If $\varphi(x,t) := \omega(x)t^p$ for all $(x,t) \in \mathbb{R}^n \times [0,\infty)$ with $p \in (0,1]$, then WH^φ goes back to the weighted weak Hardy space WH_ω^p, and particularly, when $\omega \equiv 1$, the corresponding unweighted space is also obtained.*

Before stating our main results, we recall some notions. In 2007, Ding et al. [6, 7] introduced a notion about the variable kernel $\Omega(x,z)$ when they studied the Marcinkiewicz integral. Replacing the condition (3), they strengthened it to the condition

$$\sup_{\substack{x \in \mathbb{R}^n \\ r \in [0,\infty)}} \left(\int_{S^{n-1}} |\Omega(x + rz', z')|^q d\sigma(z')\right)^{1/q} < \infty. \tag{3'}$$

For any $q \in [1, \infty)$ and $\alpha \in (0, 1]$, a function $\Omega(x, z)$ is said to satisfy the $L^{q,\alpha}$-*Dini condition*, if (1), (2), (3′) hold and

$$\int_0^1 \frac{\omega_q(\delta)}{\delta^{1+\alpha}} \, d\delta < \infty,$$

where

$$\omega_q(\delta) := \sup_{\substack{x \in \mathbb{R}^n \\ r \in [0, \infty)}} \left(\int_{S^{n-1}} \sup_{\substack{y' \in S^{n-1} \\ |y'-z'| \le \delta}} |\Omega(x + rz', y') - \Omega(x + rz', z')|^q d\sigma(z') \right)^{1/q}.$$

For any $\alpha, \beta \in (0, 1]$ with $\beta < \alpha$, it is trivial to see that if Ω satisfies the $L^{q,\alpha}$-Dini condition, then it also satisfies the $L^{q,\beta}$-Dini condition. We thus denote by Din_α^q the class of all functions which satisfy the $L^{q,\beta}$-Dini conditions for all $\beta < \alpha$. For any $\alpha \in (0, 1]$, we define

$$\mathrm{Din}_\alpha^\infty := \bigcap_{q \ge 1} \mathrm{Din}_\alpha^q.$$

A routine computation gives rise to

$$\mathrm{Din}_\alpha^r \subset \mathrm{Din}_\alpha^q \qquad \text{if } 1 \le q < r \le \infty,$$

and

$$\mathrm{Din}_\alpha^q \subset \mathrm{Din}_\beta^q \qquad \text{if } 0 < \beta < \alpha \le 1.$$

The main results are as follows. Their proofs are given in Section 3.

Theorem 1. *Let $\alpha \in (0, 1]$, $r \in (1, \infty]$ and φ be a growth function with $p \in (n/(n + \alpha), 1)$. Suppose that $\Omega \in [L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})] \cap \mathrm{Din}_\alpha^1$ satisfies* (4) *and* (5). *If $\varphi^{1/(1-p)} \in \mathbb{A}_{p\alpha/[n(1-p)]}$, then T is bounded from WH^φ to WL^φ.*

Theorem 2. *Let $\alpha \in (0, 1]$, $q \in (1, \infty)$ and φ be a growth function with $p \in (n/(n + \alpha), 1]$. Suppose that $\Omega \in \mathrm{Din}_\alpha^q$ satisfies* (4) *and* (5). *If $\varphi^{q'} \in \mathbb{A}_{(p+p\alpha/n-1/q)q'}$, then T is bounded from WH^φ to WL^φ.*

Corollary 1. *Let $\alpha \in (0, 1]$ and φ be a growth function with $p \in (n/(n + \alpha), 1]$. Suppose that $\Omega \in \mathrm{Din}_\alpha^\infty$ satisfies* (4) *and* (5). *If $\varphi \in \mathbb{A}_{p(1+\alpha/n)}$, then T is bounded from WH^φ to WL^φ.*

Theorem 3. *Let $r \in (1, \infty]$ and φ be a growth function with $p := 1$ and $\varphi \in \mathbb{A}_1$. Suppose that $\Omega \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})$ satisfies* (4) *and* (5). *If there exist two positive constants C and M such that, for any $y, h \in \mathbb{R}^n$*

and $t \in (0, \infty)$,

$$\int_{|x| \geq M|y|} |K(x+h, x-y) - K(x+h, x)| \varphi(x+h, t) dx \leq \frac{C}{M} \varphi(y+h, t), \tag{10}$$

then T *is bounded from* WH^{φ} *to* WL^{φ}.

Remark 2. (i) *It is worth noting that Corollary* 1 *can be regarded as the limiting case of Theorem* 2 *by letting* $q \to \infty$.

(ii) *Theorem* 1, *Theorem* 2 *and Corollary* 1 *jointly answer the question: when* $\Omega \in \mathrm{Din}_{\alpha}^{q}$ *with* $q = 1$, $q \in (1, \infty)$ *or* $q = \infty$, *respectively, what kind of additional conditions on* φ *and* Ω *can deduce the boundedness of* T *from* WH^{φ} *to* WL^{φ}?

(iii) *Let* ω *be a Muckenhoupt weight and* ϕ *an Orlicz function.*

(a) *When* $\varphi(x, t) := \omega(x)\phi(t)$ *for all* $(x, t) \in \mathbb{R}^n \times [0, \infty)$, *we have* $WH^{\varphi} = WH_{\omega}^{\phi}$. *In this case, Theorem* 1, *Theorem* 2, *Corollary* 1 *and Theorem* 3 *hold true for weighted weak Orlicz Hardy space. Even when* $\omega \equiv 1$, *the corresponding unweighted results are also new.*

(b) *When* $\varphi(x, t) := \omega(x)t^p$ *for all* $(x, t) \in \mathbb{R}^n \times [0, \infty)$, *we have* $WH^{\varphi} = WH_{\omega}^{p}$. *In this case, Theorem* 1, *Theorem* 2, *Corollary* 1 *and Theorem* 3 *are new for weighted weak Hardy spaces. Even when* $\omega \equiv 1$, *the corresponding unweighted results are probably new.*

3. Proofs of main results

To show the main results, we need some notions and auxiliary lemmas.

Definition 4 ([15, Definition 2.4]). Let φ be a growth function as in Definition 2.

(i) A triplet (φ, q, s) is said to be *admissible*, if $q \in (q(\varphi), \infty]$ and $s \in [m(\varphi), \infty) \cap \mathbb{N}$, where $q(\varphi)$ and $m(\varphi)$ are as in (7) and (9), respectively.

(ii) For an admissible triplet (φ, q, s), a measurable function a is called a (φ, q, s)-*atom* if there exists a ball $B \subset \mathbb{R}^n$ such that the following conditions are satisfied:

(a) a is supported in B;

(b) $\|a\|_{L_{\varphi}^{q}(B)} \leq \|\chi_B\|_{L^{\varphi}}^{-1}$, where

$$\|a\|_{L_{\varphi}^{q}(B)} := \begin{cases} \sup_{t \in (0,\infty)} \left[\frac{1}{\varphi(B, t)} \int_B |a(x)|^q \varphi(x, t) dx \right]^{1/q}, & q \in [1, \infty), \\ \|a\|_{L^{\infty}}, & q = \infty, \end{cases}$$

and

$$\|\chi_B\|_{L^\varphi} := \inf\{\lambda \in (0,\infty) : \varphi(B, \lambda^{-1}) \le 1\};$$

(c) $\int_{\mathbb{R}^n} a(x)x^\gamma\, dx = 0$ for any $\gamma \in \mathbb{N}^n$ with $|\gamma| \le s$.

Definition 5 ([22, Definition 3.2]). For an admissible triplet (φ, q, s), the *weak atomic Musielak-Orlicz Hardy space* $WH_{\mathrm{at}}^{\varphi,q,s}$ is defined as the space of all $f \in \mathscr{S}'$ satisfying that there exist a sequence of (φ, q, s)-atoms, $\{a_{i,j}\}_{i\in\mathbb{Z}, j\in\mathbb{Z}_+}$, associated with balls $\{B_{i,j}\}_{i\in\mathbb{Z}, j\in\mathbb{Z}_+}$, and a positive constant C such that $\sum_{j\in\mathbb{Z}_+} \chi_{B_{i,j}}(x) \le C$ for any $x \in \mathbb{R}^n$ and $i \in \mathbb{Z}$, and $f = \sum_{i\in\mathbb{Z}} \sum_{j\in\mathbb{Z}_+} \lambda_{i,j} a_{i,j}$ in $\mathscr{S}'$, where $\lambda_{i,j} := \tilde{C} 2^i \|\chi_{B_{i,j}}\|_{L^\varphi}$ for any $i \in \mathbb{Z}$ and $j \in \mathbb{Z}_+$, and $\tilde{C}$ is a positive constant independent of f.

Moreover, define

$$\|f\|_{WH_{\mathrm{at}}^{\varphi,q,s}} := \inf\left\{\inf\left\{\lambda \in (0,\infty) : \sup_{i\in\mathbb{Z}}\left\{\sum_{j\in\mathbb{Z}_+} \varphi\left(B_{i,j}, \frac{2^i}{\lambda}\right)\right\} \le 1\right\}\right\},$$

where the first infimum is taken over all decompositions of f as above.

Lemma 3 ([22, Theorem 3.5]). *Let (φ, q, s) be an admissible triplet. Then*

$$WH^\varphi = WH_{\mathrm{at}}^{\varphi,q,s}$$

with equivalent quasi-norms.

Lemma 4 ([7]). *Let $q \in [1,\infty)$. Suppose $\Omega \in L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$ satisfies $(3')$. If there exists a constant $\beta \in (0, 1/2)$ such that $|y| < \beta R$, then, for any $h \in \mathbb{R}^n$,*

$$\left(\int_{R\le|x|<2R} |K(x+h, x-y) - K(x+h, x)|^q dx\right)^{1/q} \le CR^{-n/q'}\left(\frac{|y|}{R} + \int_{2|y|/R}^{4|y|/R} \frac{\omega_q(\delta)}{\delta}\, d\delta\right),$$

where the positive constant C is independent of R and y.

Lemma 5. *Suppose Ω satisfies the $L^{q,\alpha}$-Dini condition with $q \in [1,\infty)$ and $\alpha \in (0,1]$. Let b be a multiple of a (φ, ∞, s)-atom associated with some ball $B(x_0, r) \subset \mathbb{R}^n$.*

(i) *If $q = 1$, then there exists a positive constant C independent of b such that, for any $R \in [8r, \infty)$,*

$$\int_{R\le|x-x_0|<2R} |T(b)(x)|dx \le C\|b\|_{L^\infty} R^n \left(\frac{r}{R}\right)^{n+\alpha}.$$

(ii) *If* $q \in (1, \infty)$, *then there exists a positive constant* C *independent of* b *such that, for any* $R \in [8r, \infty)$ *and* $t \in (0, \infty)$,

$$\int_{R \le |x-x_0| < 2R} |T(b)(x)|\varphi(x,t)dx \le C\|b\|_{L^\infty}[\varphi^{q'}(B(x_0,2R),t)]^{1/q'} R^{n/q}\left(\frac{r}{R}\right)^{n+\alpha}.$$

Proof. We only prove (ii), since the proof of (i) is analogous to that of (ii). From the vanishing moments of b, Fubini's theorem, Hölder's inequality and Lemma 4, we deduce that, for any $R \in [8r, \infty)$ and $t \in (0, \infty)$,

$$\begin{aligned}
&\int_{R \le |x-x_0| < 2R} |T(b)(x)|\varphi(x,t)dx \\
&\quad = \int_{R \le |x-x_0| < 2R} \left|\int_{|y-x_0|<r} K(x, x-y)b(y)dy\right| \varphi(x,t)dx \\
&\quad = \int_{R \le |x-x_0| < 2R} \left|\int_{|y-x_0|<r} [K(x, x-y) - K(x, x-x_0)]b(y)dy\right| \varphi(x,t)dx \\
&\quad \le \int_{|y-x_0|<r} |b(y)| \left(\int_{R \le |x-x_0| < 2R} |K(x, x-y) - K(x, x-x_0)|\varphi(x,t)dx\right) dy \\
&\quad \le \int_{|y-x_0|<r} |b(y)| \left(\int_{R \le |x-x_0| < 2R} |\varphi(x,t)|^{q'} dx\right)^{1/q'} \\
&\qquad \times \left(\int_{R \le |x-x_0| < 2R} |K(x, x-y) - K(x, x-x_0)|^q dx\right)^{1/q} dy \\
&\quad \le \|b\|_{L^\infty}[\varphi^{q'}(B(x_0,2R),t)]^{1/q'} \\
&\qquad \times \int_{|y|<r} \left(\int_{R \le |x| < 2R} |K(x+x_0, x-y) - K(x+x_0, x)|^q dx\right)^{1/q} dy \\
&\quad \lesssim \|b\|_{L^\infty}[\varphi^{q'}(B(x_0,2R),t)]^{1/q'} \int_{|y|<r} R^{-n/q'} \left(\frac{|y|}{R} + \int_{2|y|/R}^{4|y|/R} \frac{\omega_q(\delta)}{\delta}\, d\delta\right) dy \\
&\quad \lesssim \|b\|_{L^\infty}[\varphi^{q'}(B(x_0,2R),t)]^{1/q'} \int_{|y|<r} R^{-n/q'} \left[\frac{|y|}{R} + \left(\frac{|y|}{R}\right)^{\alpha}\right] dy \\
&\quad \lesssim \|b\|_{L^\infty}[\varphi^{q'}(B(x_0,2R),t)]^{1/q'} R^{n/q}\left(\frac{r}{R}\right)^{n+\alpha}.
\end{aligned}$$

Hence the statement in Lemma 5(ii) is proved.

Proof of Theorem 1. We only consider the case $r \in (1, \infty)$, since the case $r = \infty$ can be derived from the case $r = 2$. Indeed, when $r = \infty$, by $L^\infty(\mathbb{R}^n) \times L^\infty(S^{n-1}) \subset L^\infty(\mathbb{R}^n) \times L^2(S^{n-1})$ and $\varphi^{1/(1-p)} \in \mathbb{A}_{p\alpha/[n(1-p)]}$, we know that Theorem 1 holds true for $r = \infty$. We are now turning to the proof of Theorem 1 under the case $r \in (1, \infty)$. Let (φ, ∞, s) be an admissible triplet. By Lemma 3, we know that, for any $f \in WH^\varphi = WH_{\mathrm{at}}^{\varphi,\infty,s}$, there exists a sequence of multiples of (φ, ∞, s)-atoms, $\{b_{i,j}\}_{i\in\mathbb{Z}, j\in\mathbb{Z}_+}$, associated with balls $\{B_{i,j}\}_{i\in\mathbb{Z}, j\in\mathbb{Z}_+}$, such that

$$f = \sum_{i\in\mathbb{Z}} \sum_{j\in\mathbb{Z}_+} b_{i,j} \qquad \text{in } \mathcal{S}',$$

$\sum_{j\in\mathbb{Z}_+} \chi_{B_{i,j}}(x) \lesssim 1$ for any $x \in \mathbb{R}^n$ and $i \in \mathbb{Z}$, $\|b_{i,j}\|_{L^\infty} \lesssim 2^i$ for any $i \in \mathbb{Z}$ and $j \in \mathbb{Z}_+$, and

$$\|f\|_{WH^\varphi} \sim \inf\left\{\lambda \in (0,\infty) : \sup_{i\in\mathbb{Z}}\left\{\sum_{j\in\mathbb{Z}_+} \varphi\left(B_{i,j}, \frac{2^i}{\lambda}\right)\right\} \le 1\right\}.$$

Thus, our problem reduces to prove that, for any $\beta, \lambda \in (0, \infty)$ and $f \in WH^\varphi$,

$$\varphi\left(\{|T(f)| > \beta\}, \frac{\beta}{\lambda}\right) \lesssim \sup_{i\in\mathbb{Z}}\left\{\sum_{j\in\mathbb{Z}_+} \varphi\left(B_{i,j}, \frac{2^i}{\lambda}\right)\right\}.$$

To show this inequality, without loss of generality, we may assume that there exists $i_0 \in \mathbb{Z}$ such that $\beta = 2^{i_0}$. Let us write

$$f = \sum_{i=-\infty}^{i_0-1} \sum_{j\in\mathbb{Z}_+} b_{i,j} + \sum_{i=i_0}^{\infty} \sum_{j\in\mathbb{Z}_+} b_{i,j} =: F_1 + F_2.$$

We estimate F_1 first. A tedious calculation gives $p\alpha r' > n(1-p)$. For simplicity, denote $p\alpha r'/[n(1-p)]$ by u. By Theorem A with $\varphi \in \mathbb{A}_{u/r'}$, Minkowski's inequality, $\sum_{j\in\mathbb{Z}_+} \chi_{B_{i,j}}(x) \lesssim 1$ for any $x \in \mathbb{R}^n$ and $i \in \mathbb{Z}$, and the uniformly upper type 1 property of φ, we know that, for any $\lambda \in (0, \infty)$,

$$\begin{aligned}\varphi\left(\{|T(F_1)| > 2^{i_0}\}, \frac{2^{i_0}}{\lambda}\right) &= \int_{\{|T(F_1)|>2^{i_0}\}} \varphi\left(x, \frac{2^{i_0}}{\lambda}\right) dx \\ &\le 2^{-ui_0} \int_{\mathbb{R}^n} |T(F_1)(x)|^u \varphi\left(x, \frac{2^{i_0}}{\lambda}\right) dx\end{aligned}$$

$$
\begin{aligned}
&\lesssim 2^{-ui_0}\int_{\mathbb{R}^n}|F_1(x)|^u\varphi\left(x,\frac{2^{i_0}}{\lambda}\right)dx \\
&\lesssim 2^{-ui_0}\left\{\sum_{i=-\infty}^{i_0-1}\left[\int_{\mathbb{R}^n}\left|\sum_{j\in\mathbb{Z}_+}b_{i,j}(x)\right|^u\varphi\left(x,\frac{2^{i_0}}{\lambda}\right)dx\right]^{1/u}\right\}^u \\
&\lesssim 2^{-ui_0}\left\{\sum_{i=-\infty}^{i_0-1}2^i\left[\sum_{j\in\mathbb{Z}_+}\varphi\left(B_{i,j},\frac{2^{i_0}}{\lambda}\right)\right]^{1/u}\right\}^u \\
&\lesssim 2^{-ui_0}\left\{\sum_{i=-\infty}^{i_0-1}2^i\left[2^{i_0-i}\sum_{j\in\mathbb{Z}_+}\varphi\left(B_{i,j},\frac{2^{i}}{\lambda}\right)\right]^{1/u}\right\}^u \\
&\lesssim 2^{-ui_0}\left(\sum_{i=-\infty}^{i_0-1}2^i2^{(i_0-i)/u}\right)^u\sup_{i\in\mathbb{Z}}\left\{\sum_{j\in\mathbb{Z}_+}\varphi\left(B_{i,j},\frac{2^{i}}{\lambda}\right)\right\} \\
&\sim \sup_{i\in\mathbb{Z}}\left\{\sum_{j\in\mathbb{Z}_+}\varphi\left(B_{i,j},\frac{2^{i}}{\lambda}\right)\right\}, \qquad (11)
\end{aligned}
$$

which is wished.

Next let us deal with F_2. Denote the center of $B_{i,j}$ by $x_{i,j}$ and the radius by $r_{i,j}$. Set

$$
A_{i_0} := \bigcup_{i=i_0}^{\infty}\bigcup_{j\in\mathbb{Z}_+}\widetilde{B_{i,j}},
$$

where $\widetilde{B_{i,j}} := B(x_{i,j}, 8(3/2)^{(i-i_0)/(n+\alpha)}r_{i,j})$. To show that

$$
\varphi\left(\{|T(F_2)|>2^{i_0}\},\frac{2^{i_0}}{\lambda}\right)\lesssim\sup_{i\in\mathbb{Z}}\left\{\sum_{j\in\mathbb{Z}_+}\varphi\left(B_{i,j},\frac{2^{i}}{\lambda}\right)\right\},
$$

we cut $\{|T(F_2)|>2^{i_0}\}$ into A_{i_0} and $\{x\in(A_{i_0})^{\complement}:|T(F_2)(x)|>2^{i_0}\}$.

For A_{i_0}, from Lemma 1 with $\varphi\in\mathbb{A}_{p(1+\alpha/n)}$ (since $\varphi^{1/(1-p)}\in\mathbb{A}_{p\alpha/[n(1-p)]}$), and the uniformly lower type p property of φ, it follows that, for any $\lambda\in(0,\infty)$,

$$
\begin{aligned}
\varphi\left(A_{i_0},\frac{2^{i_0}}{\lambda}\right) &\le \sum_{i=i_0}^{\infty}\sum_{j\in\mathbb{Z}_+}\varphi\left(\widetilde{B_{i,j}},\frac{2^{i_0}}{\lambda}\right) \\
&\lesssim \sum_{i=i_0}^{\infty}\sum_{j\in\mathbb{Z}_+}\left(\frac{3}{2}\right)^{(i-i_0)p}\varphi\left(B_{i,j},\frac{2^{i_0}}{\lambda}\right)
\end{aligned}
$$

$$\lesssim \sum_{i=i_0}^{\infty} \sum_{j\in\mathbb{Z}_+} \left(\frac{3}{4}\right)^{(i-i_0)p} \varphi\left(B_{i,j}, \frac{2^i}{\lambda}\right)$$

$$\lesssim \sup_{i\in\mathbb{Z}} \left\{ \sum_{j\in\mathbb{Z}_+} \varphi\left(B_{i,j}, \frac{2^i}{\lambda}\right) \right\}, \tag{12}$$

which is also wished.

It remains to estimate $(A_{i_0})^\complement$. Applying the inequality $\|\cdot\|_{\ell^1} \le \|\cdot\|_{\ell^p}$ with $p \in (0,1)$, we conclude that, for any $\lambda \in (0,\infty)$,

$$\varphi\left(\{x \in (A_{i_0})^\complement : |T(F_2)(x)| > 2^{i_0}\}, \frac{2^{i_0}}{\lambda}\right)$$

$$\le 2^{-i_0 p} \int_{(A_{i_0})^\complement} |T(F_2)(x)|^p \varphi\left(x, \frac{2^{i_0}}{\lambda}\right) dx$$

$$\le 2^{-i_0 p} \sum_{i=i_0}^{\infty} \sum_{j\in\mathbb{Z}_+} \int_{(\widetilde{B_{i,j}})^\complement} |T(b_{i,j})(x)|^p \varphi\left(x, \frac{2^{i_0}}{\lambda}\right) dx. \tag{13}$$

Below, we will give the estimate of integral

$$\mathrm{I} := \int_{(\widetilde{B_{i,j}})^\complement} |T(b_{i,j})(x)|^p \varphi\left(x, \frac{2^{i_0}}{\lambda}\right) dx.$$

For any $k \in \mathbb{N}$, let

$$E_k := (2^{k+1}\widetilde{B_{i,j}}) \backslash (2^k \widetilde{B_{i,j}}).$$

It follows from Hölder's inequality that

$$\mathrm{I} \le \sum_{k=0}^{\infty} \left[\int_{E_k} |T(b_{i,j})(x)| dx\right]^p \left\{ \int_{E_k} \left[\varphi\left(x, \frac{2^{i_0}}{\lambda}\right)\right]^{1/(1-p)} dx \right\}^{1-p}.$$

On the one hand, noticing that $\varphi^{1/(1-p)} \in \mathbb{A}_{p\alpha/[n(1-p)]}$, there exists a constant $d \in (1, p\alpha/[n(1-p)])$ such that $\varphi^{1/(1-p)} \in \mathbb{A}_d$. By Lemma 2, we have $\varphi \in \mathbb{RH}_{1/(1-p)}$. Thus, thanks to Lemma 1 with $\varphi^{1/(1-p)} \in \mathbb{A}_d$, and $\varphi \in \mathbb{RH}_{1/(1-p)}$, it follows that, for any $\lambda \in (0,\infty)$,

$$\left\{ \int_{E_k} \left[\varphi\left(x, \frac{2^{i_0}}{\lambda}\right)\right]^{1/(1-p)} dx \right\}^{1-p}$$

$$\le \left[\varphi^{1/(1-p)}\left(2^{k+1}\widetilde{B_{i,j}}, \frac{2^{i_0}}{\lambda}\right)\right]^{1-p}$$

$$\lesssim \left[\varphi^{1/(1-p)}\left(B_{i,j}, \frac{2^{i_0}}{\lambda}\right)\right]^{1-p}\left[2^k\left(\frac{3}{2}\right)^{(i-i_0)/(n+\alpha)}\right]^{nd(1-p)}$$

$$\lesssim (r_{i,j})^{-np}\varphi\left(B_{i,j}, \frac{2^{i_0}}{\lambda}\right)\left[2^k\left(\frac{3}{2}\right)^{(i-i_0)/(n+\alpha)}\right]^{nd(1-p)}.$$

On the other hand, since $d < p\alpha/[n(1-p)]$, we may choose $\tilde{\alpha} \in (0, \alpha)$ such that $d < p\tilde{\alpha}/[n(1-p)]$. By the assumption $\Omega \in \mathrm{Din}^1_\alpha$, we know that Ω satisfies the $L^{1,\tilde{\alpha}}$-Dini condition. Then Lemma 5(i) yields that

$$\int_{E_k} |T(b_{i,j})(x)|dx \lesssim 2^i(r_{i,j})^n\left[2^k\left(\frac{3}{2}\right)^{(i-i_0)/(n+\alpha)}\right]^{-\tilde{\alpha}}.$$

The above three inequalities give us that, for any $\lambda \in (0, \infty)$,

$$\mathrm{I} \lesssim 2^{ip}\varphi\left(B_{i,j}, \frac{2^{i_0}}{\lambda}\right)\sum_{k=0}^{\infty}\left[2^k\left(\frac{3}{2}\right)^{(i-i_0)/(n+\alpha)}\right]^{nd-ndp-p\tilde{\alpha}}.$$

Substituting the above inequality into (13) and using the uniformly lower type p property of φ, we obtain that, for any $\lambda \in (0, \infty)$,

$$\begin{aligned}
&\varphi\left(\{x \in (A_{i_0})^{\complement} : |T(F_2)(x)| > 2^{i_0}\}, \frac{2^{i_0}}{\lambda}\right) \\
&\quad\lesssim 2^{-i_0 p}\sum_{i=i_0}^{\infty}\sum_{j\in\mathbb{Z}_+} 2^{ip}\varphi\left(B_{i,j}, \frac{2^{i_0}}{\lambda}\right)\sum_{k=0}^{\infty}\left[2^k\left(\frac{3}{2}\right)^{(i-i_0)/(n+\alpha)}\right]^{nd-ndp-p\tilde{\alpha}} \\
&\quad\lesssim \sup_{i\in\mathbb{Z}}\left\{\sum_{j\in\mathbb{Z}_+}\varphi\left(B_{i,j}, \frac{2^i}{\lambda}\right)\right\}\sum_{i=i_0}^{\infty}\sum_{k=0}^{\infty}\left[2^k\left(\frac{3}{2}\right)^{(i-i_0)/(n+\alpha)}\right]^{nd-ndp-p\tilde{\alpha}} \\
&\quad\sim \sup_{i\in\mathbb{Z}}\left\{\sum_{j\in\mathbb{Z}_+}\varphi\left(B_{i,j}, \frac{2^i}{\lambda}\right)\right\}, \qquad (14)
\end{aligned}$$

where the last "$\sim$" is due to $d < p\tilde{\alpha}/[n(1-p)]$.

Finally, combining (11), (12) and (14), we obtain the desired inequality. This finishes the proof of Theorem 1.

PROOF OF THEOREM 2. We only consider the case $p < 1$. The proof of the case $p = 1$ is similar and easier. Once we prove Lemma 5(ii), the proof of this theorem is quite similar to that of Theorem 1, the major change being the substitution of

$$\mathrm{I} \le \sum_{k=0}^{\infty} \left[\int_{E_k} |T(b_{i,j})(x)| \varphi\left(x, \frac{2^{i_0}}{\lambda}\right) dx \right]^p \left[\int_{E_k} \varphi\left(x, \frac{2^{i_0}}{\lambda}\right) dx \right]^{1-p}$$

for

$$\mathrm{I} \le \sum_{k=0}^{\infty} \left[\int_{E_k} |T(b_{i,j})(x)| dx \right]^p \left\{ \int_{E_k} \left[\varphi\left(x, \frac{2^{i_0}}{\lambda}\right) \right]^{1/(1-p)} dx \right\}^{1-p}.$$

We leave the details to the reader.

Proof of Corollary 1. By $\varphi \in \mathbb{A}_{p(1+\alpha/n)}$, we see that there exists $d \in (1, \infty)$ such that $\varphi^d \in \mathbb{A}_{p(1+\alpha/n)}$. For any $q \in (1, \infty)$, by $p > n/(n+\alpha)$, some tedious manipulation yields $(p + p\alpha/n - 1/q)q' > p(1+\alpha/n)$ and hence $\varphi^d \in \mathbb{A}_{(p+p\alpha/n-1/q)q'}$. Thus, we may choose $q := d/(d-1)$ such that

$$\varphi^{q'} = \varphi^d \in \mathbb{A}_{(p+\alpha/n-1/q)q'}$$

and hence Corollary 1 follows from Theorem 2.

Proof of Theorem 3. Since the proof of Theorem 3 is similar to that of Theorem 1, we use the same notation as in the proof of Theorem 1. Rather than giving a complete proof, we just give out the necessary modifications with respect to the estimate of $(A_{i_0})^{\complement}$. Reset

$$A_{i_0} := \bigcup_{i=i_0}^{\infty} \bigcup_{j \in \mathbb{Z}_+} \widetilde{B_{i,j}},$$

where $\widetilde{B_{i,j}} := B(x_{i,j}, (3/2)^{(i-i_0)/n} r_{i,j})$. For any $\lambda \in (0, \infty)$, we have

$$\begin{aligned} &\varphi\left(\{x \in (A_{i_0})^{\complement} : |T(F_2)(x)| > 2^{i_0}\}, \frac{2^{i_0}}{\lambda}\right) \\ &\quad \le 2^{-i_0} \int_{(A_{i_0})^{\complement}} |T(F_2)(x)| \varphi\left(x, \frac{2^{i_0}}{\lambda}\right) dx \\ &\quad \le 2^{-i_0} \sum_{i=i_0}^{\infty} \sum_{j \in \mathbb{Z}_+} \int_{(\widetilde{B_{i,j}})^{\complement}} |T(b_{i,j})(x)| \varphi\left(x, \frac{2^{i_0}}{\lambda}\right) dx. \end{aligned} \tag{15}$$

Below, we will give the estimate of integral

$$\mathrm{I} := \int_{(\widetilde{B_{i,j}})^{\complement}} |T(b_{i,j})(x)| \varphi\left(x, \frac{2^{i_0}}{\lambda}\right) dx.$$

By the vanishing moments of $b_{i,j}$, Fubini's theorem and (10), we obtain that, for any $\lambda \in (0,\infty)$,

$$
\begin{aligned}
\mathrm{I} &= \int_{(\widetilde{B_{i,j}})^{\complement}} \left| \int_{B_{i,j}} [K(x, x-y) - K(x, x - x_{i,j})] b_{i,j}(y)\,dy \right| \varphi\left(x, \frac{2^{i_0}}{\lambda}\right) dx \\
&\le \int_{B_{i,j}} |b_{i,j}(y)| \left[\int_{(\widetilde{B_{i,j}})^{\complement}} |K(x, x-y) - K(x, x-x_{i,j})| \varphi\left(x, \frac{2^{i_0}}{\lambda}\right) dx \right] dy \\
&= \int_{|y|<r_{i,j}} |b_{i,j}(y + x_{i,j})| \left[\int_{|x| \ge (3/2)^{(i-i_0)/n} r_{i,j}} \right. \\
&\qquad \left. |K(x + x_{i,j}, x-y) - K(x + x_{i,j}, x)| \varphi\left(x + x_{i,j}, \frac{2^{i_0}}{\lambda}\right) dx \right] dy \\
&\lesssim \int_{|y|<r_{i,j}} |b_{i,j}(y + x_{i,j})| \left(\frac{2}{3}\right)^{(i-i_0)/n} \varphi\left(y + x_{i,j}, \frac{2^{i_0}}{\lambda}\right) dy \\
&\lesssim \|b_{i,j}\|_{L^\infty} \left(\frac{2}{3}\right)^{(i-i_0)/n} \varphi\left(B_{i,j}, \frac{2^{i_0}}{\lambda}\right).
\end{aligned}
$$

Substituting the above inequality into (15) and using the uniformly lower type 1 property of φ, we obtain that, for any $\lambda \in (0,\infty)$,

$$
\begin{aligned}
&\varphi\left(\{x \in (A_{i_0})^{\complement} : |T(F_2)(x)| > 2^{i_0}\}, \frac{2^{i_0}}{\lambda}\right) \\
&\quad \lesssim 2^{-i_0} \sum_{i=i_0}^{\infty} \sum_{j \in \mathbb{Z}_+} \|b_{i,j}\|_{L^\infty} \left(\frac{2}{3}\right)^{(i-i_0)/n} \varphi\left(B_{i,j}, \frac{2^{i_0}}{\lambda}\right) \\
&\quad \lesssim 2^{-i_0} \sum_{i=i_0}^{\infty} \sum_{j \in \mathbb{Z}_+} 2^i \left(\frac{2}{3}\right)^{(i-i_0)/n} 2^{i_0 - i} \varphi\left(B_{i,j}, \frac{2^i}{\lambda}\right) \\
&\quad \lesssim \sum_{i=i_0}^{\infty} \left(\frac{2}{3}\right)^{(i-i_0)/n} \sup_{i \in \mathbb{Z}} \left\{ \sum_{j \in \mathbb{Z}_+} \varphi\left(B_{i,j}, \frac{2^i}{\lambda}\right) \right\} \\
&\quad \sim \sup_{i \in \mathbb{Z}} \left\{ \sum_{j \in \mathbb{Z}_+} \varphi\left(B_{i,j}, \frac{2^i}{\lambda}\right) \right\}.
\end{aligned}
$$

This finishes the proof of Theorem 3.

REMARK 3. *We should point out that, if φ is a growth function of uniformly lower type 1 and of uniformly upper type 1, then $WH^\varphi = WH^1_{\varphi(\cdot,1)}$ and $WL^\varphi =$*

$WL^1_{\varphi(\cdot,1)}$. *In fact, there exists a positive constant* C *such that, for any* $x \in \mathbb{R}^n$ *and* $t \in (0, \infty)$,

$$C^{-1}t\varphi(x,1) = C^{-1}t\varphi(x,t/t) \le \varphi(x,t) \le Ct\varphi(x,1),$$

which implies that

$$\sup_{t\in(0,\infty)} \varphi(\{|f| > t\}, t) \sim \sup_{t\in(0,\infty)} \varphi(\{|f| > t\}, 1)t.$$

Hence, we have $WL^\varphi = WL^1_{\varphi(\cdot,1)}$. *Analogously,* $WH^\varphi = WH^1_{\varphi(\cdot,1)}$.

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