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The $E(1)$-local Picard graded homotopy groups of the sphere spectrum at the prime two

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Abstract. Let $E(1)$ be the first Johnson-Wilson spectrum at the prime two. In this paper, we calculate the homotopy groups of the $E(1)$-localized sphere spectrum with a grading over the Picard group of the stable homotopy category of $E(1)$-local spectra.

1. Introduction

Let p be a prime number. In the stable homotopy category $\mathscr{S}_p$ of p-local spectra, we denote by $[X, Y]$ the group of morphisms from X to Y in $\mathscr{S}_p$, and $[X, Y]_* = \bigoplus_{k \in \mathbb{Z}} [S^k \wedge X, Y]$. Here, S^k is the k-dimensional sphere spectrum. For the Bousfield localization functor L_E with respect to a spectrum E, we denote $\mathscr{L}_E = L_E(\mathscr{S}_p)$. The category $\mathscr{L}_E$ is a symmetric monoidal category, whose structure is given by the E-local smash product $L_E(- \wedge -)$. A spectrum $X \in \mathscr{L}_E$ is *invertible* if there exists $Y \in \mathscr{L}_E$ such that $L_E(X \wedge Y) = L_E S^0$, and the *Picard group* $\mathrm{Pic}(\mathscr{L}_E)$ *of* $\mathscr{L}_E$ is the collection of the isomorphism classes of invertible spectra in $\mathscr{L}_E$.

In this paper, we use the following notation:

$$\pi_P^E(X) = [P, L_E X] \quad \text{for } P \in \mathrm{Pic}(\mathscr{L}_E), \qquad \text{and} \qquad \pi_\star^E(X) = \bigoplus_{P \in \mathrm{Pic}(\mathscr{L}_E)} \pi_P^E(X).$$

Let $K(n)$ be the n-th Morava K-theory spectrum. Hopkins, Mahowald and Sadofsky deeply studied the Picard group $\mathrm{Pic}(\mathscr{L}_{K(n)})$ in [3], and Westerland showed many interesting results around $\pi_\star^{K(n)}(S^0)$ in [13]. In chromatic homotopy theory, we have an important object $E(n)$, the n-th Johnson-Wilson spectrum, as well as $K(n)$. The localization functor $L_{E(n)}$ and the category $\mathscr{L}_{E(n)}$ are abbreviated as L_n and $\mathscr{L}_n$, respectively, and let $\pi_\star^n(X)$ denote $\pi_\star^{E(n)}(X)$. We consider the monomorphisms

$$\begin{aligned} i_n : \pi_*(L_n X) = \bigoplus_{k \in \mathbb{Z}} [S^k, L_n X] &= \bigoplus_{k \in \mathbb{Z}} [L_n S^k, L_n X] \\ &\xrightarrow{\subset} \bigoplus_{P \in \mathrm{Pic}(\mathscr{L}_n)} [P, L_n X] = \pi_\star^n(X) \end{aligned}$$

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for $n \geq 0$. These monomorphisms fit into the following commutative diagram:

$$\begin{array}{ccccccccc}
\pi_*(L_0X) & \longleftarrow & \pi_*(L_1X) & \longleftarrow & \cdots & \longleftarrow & \pi_*(L_nX) & \longleftarrow & \cdots \\
{\scriptstyle i_0}\big\downarrow{\scriptstyle mono.} & & {\scriptstyle i_1}\big\downarrow{\scriptstyle mono.} & & & & {\scriptstyle i_n}\big\downarrow{\scriptstyle mono.} & & \\
\pi^0_\star(X) & \longleftarrow & \pi^1_\star(X) & \longleftarrow & \cdots & \longleftarrow & \pi^n_\star(X) & \longleftarrow & \cdots
\end{array}$$

From this system, we obtain

$$\lim_n(i_n) : \lim_n \pi_*(L_nX) \xrightarrow{mono.} \lim_n \pi^n_\star(X).$$

We recall that the chromatic convergence theorem (*cf.* [10, Th. 7.5.7]) implies that if X is finite, then the universal map $\pi_*(X) \to \lim_n \pi_*(L_nX)$ is an isomorphism, and therefore we have the monomorphism

$$\pi_*(S^0) \xrightarrow{\sim} \lim_n \pi_*(L_nS^0) \xrightarrow[mono.]{\lim_n(i_n)} \lim_n \pi^n_\star(S^0). \tag{1.1}$$

Under this map, we expect that $\lim_n \pi^n_\star(S^0)$ has a new information of $\pi_*(S^0)$.

We note that $\mathrm{Pic}(\mathscr{L}_0) = \mathbb{Z}$ and the homomorphism

$$\ell_0 : \mathrm{Pic}(\mathscr{L}_n) \to \mathrm{Pic}(\mathscr{L}_0) = \mathbb{Z}$$

induced by the localization functor L_0 is a splitting epimorphism. Putting $\mathrm{Pic}^0(\mathscr{L}_n) = \ker(\ell_0)$, we have the decomposition

$$\mathrm{Pic}(\mathscr{L}_n) = \mathbb{Z} \oplus \mathrm{Pic}^0(\mathscr{L}_n).$$

Here, the summand $\mathbb{Z}$ is generated by L_nS^1. The structure of the Picard group is known as follow:

Theorem 1 ([4, Th. A. and Th. 6.1], [2, Th. 1.2]).
(1) *If* $(p-1) \nmid n$ *and* $2p-2 \geq n^2+n$, *then* $\mathrm{Pic}^0(\mathscr{L}_n) = 0$.
(2) *At* $p = 2$, $\mathrm{Pic}^0(\mathscr{L}_1) = \mathbb{Z}/2$.
(3) *At* $p = 3$, $\mathrm{Pic}^0(\mathscr{L}_2) = \mathbb{Z}/3 \oplus \mathbb{Z}/3$.

This implies that, if $(p-1) \nmid n$ and $2p-2 \geq n^2+n$, then $\pi^n_\star(S^0) = \pi_*(L_nS^0)$. We consider the first interesting case $(p,n) = (2,1)$ in this paper. We define

$$v(t) = \max\{i \in \mathbb{Z} : 2^i \mid t\} \qquad \text{and} \qquad a(t) = \begin{cases} 1 & 2 \nmid t \\ v(t)+2 & 2 \mid t \end{cases} \tag{1.2}$$

for a nonzero integer t. The main theorem in this paper is the following:

THEOREM 2. *At $p=2$, as a $\mathbb{Z}_{(2)}$-algebra with a grading over* $\mathrm{Pic}(\mathscr{L}_1) = \mathbb{Z} \oplus \mathbb{Z}/2$,

$$\pi_\star^1(S^0) = \mathbb{Z}_{(2)}[2_Q, A_{t/a(t)} : t \neq 0]/R$$

with

$$|2_Q| = (0,1) \qquad and \qquad |A_{t/a(t)}| = \begin{cases} (2t-1,0) & t \equiv 0,1 \bmod (4) \\ (2t-1,1) & t \equiv 2,3 \bmod (4) \end{cases}.$$

Here R is the ideal of the following relations: Put $A_t = A_{t/1}$ and $X_j = A_{2^{j-2}/j}A_{-2^{j-2}/j}$ for $j > 2$.

(1) $2_Q^2 = 4$.

(2) $2X_{j+1} = X_j$ *for* $j > 2$.

(3) $2^{a(t)}A_{t/a(t)} = \begin{cases} 0 & t \equiv 0,1,2 \bmod (4) \\ \underline{0 \text{ or } A_1 A_{t-1/3}} & t \equiv 3 \bmod (4) \end{cases}.$

(4) $2^{a(t)-1}2_Q A_{t/a(t)} = \begin{cases} \underline{0 \text{ or } A_1^2 A_{t-1}} & t \equiv 0 \bmod (4) \\ A_1^2 A_{t-1} & t \equiv 2 \bmod (4) \\ 0 & t \equiv 1,3 \bmod (4) \end{cases}.$

(5) $A_{s/a(s)}A_{t/a(t)} = \begin{cases} X_{a(s)} & s+t=0, \text{ and } s \equiv t \equiv 0 \bmod (2) \\ 0 & s+t \neq 0, \text{ and } s \equiv t \equiv 0 \bmod (2) \\ A_{-3}A_{4/4} & s+t=1 \\ A_1 A_{s+t-1/a(s+t-1)} & \text{otherwise} \end{cases}.$

(6) $A_1^3 A_{t/a(t)} = 0$ *if* $t \neq -2$, $A_1^4 A_{-2/3} = 0$, *and* $A_1^2 A_{-3} A_{4/4} = 0$.

REMARK 3. *The author conjectures that* $8A_{t/3} = 0$ *for* $t \equiv 2 \bmod (4)$, *and* $2^{a(t)-1}2_Q A_{t/a(t)} = A_1^2 A_{t-1}$ *for* $t \equiv 0 \bmod (4)$.

Consider the Brown-Peterson spectrum BP at p. The homology theory $BP_*(-)$ represented by BP satisfies that

$$BP_* = BP_*(S^0) = \mathbb{Z}_{(p)}[v_1, v_2, \ldots],$$

$$BP_*(BP) = BP_*[t_1, t_2, \ldots]$$

where $|v_i| = |t_i| = 2(p^i - 1)$. Then, for the homology theory $E(n)_*(-)$ represented by $E(n)$, we have

$$E(n)_* = E(n)_*(S^0) = v_n^{-1}BP_*/(v_{n+1}, v_{n+2}, \ldots) = \mathbb{Z}_{(p)}[v_1, v_2, \ldots, v_{n-1}, v_n^{\pm 1}],$$

$$E(n)_*(E(n)) = E(n)_* \otimes_{BP_*} BP_*(BP) \otimes_{BP_*} E(n)_*$$

with $|v_i| = 2(p^i - 1)$. The $E(n)$-based Adams spectral sequence for a spectrum A is of the form

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{E(n)_*(E(n))}(E(n)_*, E(n)_*(A)) \Rightarrow \pi_{t-s}(L_n A).$$

Hereafter, we denote by $E(n)_r^{s,t}(A)$ the E_r-term of the spectral sequence. For an $E(n)_*(E(n))$-comodule M, we abbreviate

$$H^{*,*}M = \mathrm{Ext}^{*,*}_{E(n)_*(E(n))}(E(n)_*, M).$$

Let I_k denote the ideal $(v_0, v_1, \dots, v_{k-1})$ of $E(n)_*$, where $v_0 = p$. Consider the following $E(n)_*(E(n))$-comodules:

$$N_k^0 = E(n)_*/I_k, \qquad M_k^0 = v_k^{-1}N_k^0,$$
$$N_k^{i+1} = \mathrm{Coker}(N_k^i \overset{\subset}{\to} M_k^i) \qquad \text{and} \qquad M_k^i = v_{k+i}^{-1}N_k^i \qquad \text{for } i \geq 0.$$

The short exact sequence $N_0^i \to M_0^i \to N_0^{i+1}$ gives rise to the connecting homomorphism

$$\delta_i : H^*N_0^{i+1} \to H^{*+1}N_0^i.$$

The *k-th algebraic Greek letter elements* are defined by

$$\bar{\alpha}^{(k)}_{e_k/e_{k-1},\dots,e_1,e_0} = \delta_0\delta_1\cdots\delta_{k-1}(v_k^{e_k}/p^{e_0}v_1^{e_1}\cdots v_{k-1}^{e_{k-1}}) \in H^kN_0^0 = E(n)_2^k(S^0)$$

if $v_k^{e_n}/p^{e_0}v_1^{e_1}\cdots v_{k-1}^{e_{k-1}}$ is in $H^0N_0^k$. In particular, we denote

$$\bar{\alpha}_{t/a} = \bar{\alpha}^{(1)}_{t/a}, \qquad \bar{\beta}_{t/a,b} = \bar{\alpha}^{(2)}_{t/a,b}, \qquad \bar{\beta}_{t/a} = \bar{\beta}_{t/a,1}, \qquad \text{and} \qquad \bar{\beta}_t = \bar{\beta}_{t/1}.$$

By [6, Th. 1.1], for any invertible spectrum $X \in \mathrm{Pic}^0(\mathscr{L}_n)$, we have

$$E(n)_2^{*,*}(X) = E(n)_2^{*,*}(S^0)\{g_X\} \qquad \text{with } |g_X| = (0,0).$$

Note that if the element

$$\bar{\alpha}^{(k)}_{e_k/e_{k-1},\dots,e_1,e_0}g_X \in E(n)_2^{*,*}(X)$$

is a permanent cycle, then we have an element of

$$\pi_*(X) = \bigoplus_k [S^k, X] = \bigoplus_k [\Sigma^k X^{-1}, L_nS^0] \subset \pi_\star^n(S^0).$$

If $\bar{\alpha}^{(k)}_{e_k/e_{k-1},\dots,e_1,e_0} \in E(n)_2^k(S^0)$ detects an element in $\pi_*(L_nS^0)$, we denote it by $\alpha^{(k)}_{e_k/e_{k-1},\dots,e_1,e_0}$. In particular, we denote

$$\alpha_{t/a} = \alpha^{(1)}_{t/a}.$$

By Theorem 10 below, at $p = 2$, $\pi_*(L_1S^0)$ is generated by $\alpha_{t/b(t)}$'s with $t \equiv 0, 1, 2 \bmod (4)$. Here, $b(t)$ is the integer in (2.4). Furthermore, the monomorphism $i_1 : \pi_*(L_1S^0) \to \pi_\star^1(S^0)$ satisfies

$$i_1(\alpha_{t/b(t)}) = \begin{cases} A_{t/a(t)} & t \equiv 0, 1 \bmod (4) \\ 2_Q A_{t/3} & t \equiv 2 \bmod (4) \end{cases}. \tag{1.3}$$

We note that $\mathrm{Pic}^0(\mathscr{L}_1) = \mathbb{Z}/2$ is generated by the question mark spectrum Q (see § 3). By Proposition 8, $E(1)_2^{*,*}(S^0)$ is generated by the algebraic alpha elements $\bar{\alpha}_{t/a(t)}$. The generator $A_{t/a(t)}$ in Theorem 2 is detected by $\bar{\alpha}_{t/a(t)} \in E(1)_2^{*,*}(S^0)$ if $t \equiv 0, 1 \bmod (4)$, and by $\bar{\alpha}_{t/a(t)} g_Q \in E(1)_2^{*,*}(Q)$ if $t \equiv 2, 3 \bmod (4)$. By this fact, at $p = 2$, for any algebraic alpha element $\bar{\alpha}_{t/a}$ with $t \neq 0$, at least one of $\bar{\alpha}_{t/a}$ and $\bar{\alpha}_{t/a} g_Q$ detects a nontrivial element in $\pi_\star^1(S^0)$. We also note the following:

(1) At $p > 2$, any algebraic alpha element in $E(1)_2^{*,*}(S^0)$ survives to $\pi_*(L_1 S^0) = \pi_\star^1(S^0)$.

(2) More general, if $(p-1) \not| \, n$ and $2p - 2 \geq n^2 + n$, then any nonzero algebraic Greek letter element in $E(n)_2^{*,*}(S^0)$ survives to $\pi_*(L_n S^0) = \pi_\star^n(S^0)$.

(3) At $p = 3$, the algebraic beta element $\bar{\beta}_t$ in $E(2)_2^2(S^0)$ survives to $\pi_*(L_2 S^0)$ if and only if $t \equiv 0, 1, 2, 3, 5, 6 \bmod (9)$ [12, Th. 2.12], and $\pi_*(L_2 S^0) \neq \pi_\star^2(S^0)$.

By these facts, we conjecture the following:

Conjecture 4. *Let p be a prime number and n an integer ≥ 0. For any algebraic Greek letter element $\bar{\alpha}^{(n)}_{t/e_{n-1}, e_{n-2}, \dots, e_0} \in E(n)_2^{*,*}(S^0)$ with $t \neq 0$, there exists an invertible spectrum $X \in \mathrm{Pic}^0(\mathscr{L}_n)$ such that $\alpha^{(n)}_{t/e_{n-1}, e_{n-2}, \dots, e_0} g_X$ survives to $\pi_\star^n(S^0)$.*

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2. The structure of $\pi_*(L_1 S^0)$ at $p = 2$, revisited

Hereafter, we consider the case $p = 2$. Ravenel determined the structure of $\pi_*(L_1 S^0)$ as [9, Th. 8.15]. In this section, we review the homotopy groups.

The homology theory $E(1)_*(-)$ represented by the first Johnson-Wilson theory spectrum $E(1)$ satisfies

$$E(1)_* = E(1)_*(S^0) = \mathbb{Z}_{(2)}[v_1^{\pm 1}],$$
$$E(1)_*(E(1)) = E(1)_* \otimes_{BP_*} BP_*(BP) \otimes_{BP_*} E(1)_*.$$

Hereafter, we denote by $E(1)_r^{*,*}(X)$ the E_r-term of the $E(1)$-based Adams spectral sequence converging to $\pi_*(L_1 X)$. This spectral sequence forms as follow:

$$E(1)_2^{*,*}(X) = \mathrm{Ext}^{*,*}_{E(1)_*(E(1))}(E(1)_*, E(1)_*(X)) \Rightarrow \pi_*(L_1 X).$$

For an $E(1)_*(E(1))$-comodule M, we abbreviate $H^{*,*}M = \mathrm{Ext}^{*,*}_{E(1)_*(E(1))}(E(1)_*, M)$. We consider the following $E(1)_*(E(1))$-comodules:

$$N_0^0 = E(1)_*, \qquad M_0^0 = 2^{-1}E(1)_*, \qquad M_1^0 = E(1)_*/(2),$$

$$\text{and} \qquad M_0^1 = \operatorname{Coker}(N_0^0 \to M_0^0).$$

Theorem 5 ([11, Th. 5.2.1, and Th. 5.2.2.]).

(1) $H^{s,t}M_0^0 = \begin{cases} \mathbb{Q} & s = t = 0 \\ 0 & \textit{otherwise} \end{cases}.$

(2) $H^{*,*}M_1^0 = K(1)_*[h_0, \rho_1]/(\rho_1^2 = 0)$ *where* $h_0 \in H^{1,2}M_1^0$ *and* $\rho_1 \in H^{1,0}M_1^0$, *which are represented by* t_1 *and* $v_1^{-3}(t_2 + t_1^3)$ *in* $E(1)_*(E(1))/(2)$, *respectively. Here,* $K(1)_* = E(1)_*/(2) = \mathbb{Z}/2[v_1^{\pm 1}]$.

For an element in $H^*M_0^1$, we use the notation of Behrens' type (see [1]) defined as follows. Consider the short exact sequence

$$0 \to M_1^0 \xrightarrow{\varphi} M_0^1 \xrightarrow{2} M_0^1 \to 0$$

where $\varphi(x) = x/2$. For an element $x \in H^*M_1^0$, we define $x_{t/s} \in H^*M_0^1$ by

$$2^{s-1}x_{t/s} = \varphi_*(v_1^t x) = v_1^t x/2.$$

Theorem 6 ([7, Th. 4.16]).

$$H^sM_0^1 = \begin{cases} \mathbb{Q}/\mathbb{Z}_{(2)} \oplus \langle 1_{t/a(t)} : t \neq 0 \rangle & s = 0 \\ \mathbb{Q}/\mathbb{Z}_{(2)} \oplus \langle (h_0)_{t/1}, (\rho_1)_{t/1} : 2 \nmid t \rangle & s = 1 \\ \langle (h_0^s)_{t/1}, (\rho_1 h_0^{s-1})_{t/1} : 2 \nmid t \rangle & s > 1 \end{cases}.$$

Here, $\langle - \rangle$ *is an exterior algebra, the summand* $\mathbb{Q}/\mathbb{Z}_{(2)}$ *at* $s = 0$ *(resp.* $s = 1$*) is generated by the elements* $1_{0/j}$ *(resp.* $(\rho_1)_{0/j}$*) for* $j > 0$, *and* $a(t)$ *is the integer in* (1.2).

The short exact sequence

$$0 \to N_0^0 \to M_0^0 \to M_0^1 \to 0 \tag{2.1}$$

gives rise to the exact sequence

$$\begin{gathered} 0 \to H^0N_0^0 \to H^0M_0^0 \to H^0M_0^1 \xrightarrow{\delta} H^1N_0^0 \to 0 \qquad \text{and} \\ H^{s-1}M_0^1 \underset{\sim}{\xrightarrow{\delta}} H^sN_0^0 \qquad \text{for } s \neq 0, 1 \end{gathered} \tag{2.2}$$

by Theorem 5. Here δ is the connecting homomorphism associated with (2.1). In $H^*N_0^0$, we denote

$$\bar{\alpha}_{t/s} = \delta(1_{t/s}) \qquad \text{for } t \neq 0 \text{ and } 1 \le s \le a(t),$$

$$\bar{\alpha}_t = \bar{\alpha}_{t/1}, \qquad \text{and} \qquad \bar{\xi}_j = \delta((\rho_1)_{0/j}).$$

Then, by Theorem 6, we have the following:

$$H^s N_0^0 = \begin{cases} \mathbf{Z}_{(2)} & s = 0 \\ \langle \bar{\alpha}_{t/a(t)} : t \neq 0 \rangle & s = 1 \\ \mathbf{Q}/\mathbf{Z}_{(2)} \oplus \langle \delta((h_0)_{t/1}), \delta((\rho_1)_{t/1}) : 2 \nmid t \rangle & s = 2 \\ \langle \delta((h_0^s)_{t/1}), \delta((\rho_1 h_0^{s-1})_{t/1}) : 2 \nmid t \rangle & s > 2 \end{cases}. \tag{2.3}$$

Here the summand $\mathbf{Q}/\mathbf{Z}_{(2)}$ at $s = 2$ is generated by the elements $\bar{\bar{\xi}}_j$ for $j > 0$.

Proposition 7. *In $H^* N_0^0$, the following hold:*

(1) $\delta((h_0)_{t/1}) = \bar{\alpha}_1 \bar{\alpha}_t$ *for odd t, and* $\delta((\rho_1)_{t/1}) = \bar{\alpha}_1 \bar{\alpha}_{t-1/a(t-1)}$ *for odd $t \neq 1$. In addition,* $\delta((\rho_1)_{1/1}) = \bar{\alpha}_{-3} \bar{\alpha}_{4/4}$.

(2) *Suppose that s is odd. Then* $\bar{\alpha}_s \bar{\alpha}_{t/a(t)} = \bar{\alpha}_1 \bar{\alpha}_{s+t-1/a(s+t-1)}$ *if $s + t \neq 1$, and* $\bar{\alpha}_s \bar{\alpha}_{-s+1/a(-s+1)} = \bar{\alpha}_{-3} \bar{\alpha}_{4/4}$.

(3) *Suppose that the both s and t are even. Then* $\bar{\alpha}_{s/a(s)} \bar{\alpha}_{t/a(t)} = 0$ *or* $\bar{\alpha}_1 \bar{\alpha}_{s+t-1}$ *if $s + t \neq 0$, and* $\bar{\alpha}_{s/a(s)} \bar{\alpha}_{-s/a(s)} = x_s \bar{\bar{\xi}}_{a(s)} + y_s \bar{\alpha}_{-1} \bar{\alpha}_1$ *for an odd integer x_s and $y_s \in \{0, 1\}$.*

Proof. (1): By [7, Lem. 4.12], for any nonzero $t \in \mathbf{Z}$,

$$\begin{aligned} \bar{\alpha}_1 \bar{\alpha}_{t/a(t)} &= \delta(1_{1/1}) \bar{\alpha}_{t/a(t)} \\ &= \delta(v_1(\bar{\alpha}_{t/a(t)})/2) \\ &= \begin{cases} \delta((h_0)_{t/1}) & 2 \nmid t \\ \delta((\rho_1)_{t+1/1}) & 2 \mid t \end{cases}. \end{aligned}$$

We also have $\delta((\rho_1)_{1/1}) = \delta((v_1^4 \rho_1)_{-3/1}) = \delta(v_1^{-3}(\bar{\alpha}_{4/4})/2) = \delta(1_{-3/1}) \bar{\alpha}_{4/4} = \bar{\alpha}_{-3} \bar{\alpha}_{4/4}$ by [7, Lem. 4.12].

(2): By [7, Lem. 4.12] and (1),

$$\begin{aligned} \bar{\alpha}_s \bar{\alpha}_{t/a(t)} &= \delta(1_{s/1}) \bar{\alpha}_{t/a(t)} \\ &= \delta(v_1^s(\bar{\alpha}_{t/a(t)})/2) \\ &= \begin{cases} \delta((h_0)_{s+t-1/1}) & 2 \nmid t \\ \delta((\rho_1)_{s+t/1}) & 2 \mid t \end{cases} \\ &= \begin{cases} \bar{\alpha}_1 \delta(1_{s+t-1/1}) & 2 \nmid t \\ \delta((\rho_1)_{s+t/1}) & 2 \mid t \end{cases} \\ &= \begin{cases} \bar{\alpha}_1 \bar{\alpha}_{s+t-1/a(s+t-1)} & s + t \neq 1 \\ \bar{\alpha}_{-3} \bar{\alpha}_{4/4} & s + t = 1 \end{cases}. \end{aligned}$$

(3): By (2.2), the connecting homomorphism $\delta : H^{1,2(s+t)} M_0^1 \to H^{2,2(s+t)} N_0^0$ is an isomorphism. Assume $s + t \neq 0$. If both s and t are even, then, by Theorem 6, we have $H^{1,2(s+t)} M_0^1 = \mathbf{Z}/2\{(h_0)_{s+t-1/1}\}$. Hence, if $\bar{\alpha}_{s/a(s)}\bar{\alpha}_{t/a(t)} \neq 0$ in $H^{2,2(s+t)}N_0^0$, then $\bar{\alpha}_{s/a(s)}\bar{\alpha}_{t/a(t)} = \delta((h_0)_{s+t-1/1}) = \bar{\alpha}_1\delta(1_{s+t-1/1}) = \bar{\alpha}_1\bar{\alpha}_{s+t-1}$. If $s+t=0$, then, by [7, Lem. 4.12], $2^{a(s)-1}\bar{\alpha}_{s/a(s)}\bar{\alpha}_{-s/a(s)} = 2^{a(s)-1}\delta(1_{s/a(s)})\bar{\alpha}_{-s/a(s)} = \delta(1_{s/1})\bar{\alpha}_{-s/a(s)} = \delta(v_1^s(\bar{\alpha}_{-s/a(s)})/2) = \delta((v_1^{-s}\rho_1)_{s/1}) = \delta((\rho_1)_{0/1}) = \bar{\xi}_1$. This implies our claim by (2.3).

For $s, t \in \mathbb{Z}\backslash\{0\}$, we denote

$$\underline{v}(s,t) = \min\{v(s), v(t)\}.$$

Proposition 8. *As a bigraded $\mathbb{Z}_{(2)}$-algebra,*

$$E(1)_2^{*,*}(S^0) = H^{*,*}N_0^0 = \mathbb{Z}_{(2)}[\bar{\alpha}_{t/a(t)} : t \neq 0]/R$$

with $|\bar{\alpha}_{t/a(t)}| = (1, 2t)$, where R is an ideal of the following relations:

(1) $2^{a(t)}\bar{\alpha}_{t/a(t)} = 0$.

(2) $$\bar{\alpha}_{s/a(s)}\bar{\alpha}_{t/a(t)} = \begin{cases} \bar{\alpha}_1\bar{\alpha}_{s+t-1/a(s+t-1)} & \underline{v}(s,t) = 0 \text{ and } s+t \neq 1 \\ \bar{\alpha}_{-3}\bar{\alpha}_{4/4} & \underline{v}(s,t) = 0 \text{ and } s+t = 1 \\ 0 \text{ or } \bar{\alpha}_1\bar{\alpha}_{s+t-1} & \underline{v}(s,t) > 0 \text{ and } s+t \neq 0 \\ x_s\bar{\xi}_{a(s)} + y_s\bar{\alpha}_{-1}\bar{\alpha}_1 & \underline{v}(s,t) > 0 \text{ and } s+t = 0 \end{cases}.$$

Here, x_s is an odd integer and y_s is in $\{0,1\}$.

Proof. We note that $\bar{\alpha}_1/2 = (h_0)_{0/1}$ in $H^1M_0^1$. By (2.3) and Proposition 7, $H^*N_0^0$ is generated by the elements $\bar{\alpha}_{t/a(t)}$ as a $\mathbb{Z}_{(2)}$-algebra. By the definition of the generators, the first relation is immediately given. The second relation is shown by Proposition 7.

Proposition 9. *In the $E(1)$-based Adams spectral sequence converging to $\pi_*(L_1S^0)$, the following hold:*

(1) *If $t \equiv 0, 1 \bmod (4)$, then $\bar{\alpha}_{t/a(t)}$ is permanent.*

(2) *If $2 \neq t \equiv 2, 3 \bmod (4)$, then $d_3(\bar{\alpha}_{t/a(t)}) = \bar{\alpha}_1^3\bar{\alpha}_{t-2/a(t-2)}$, and also $d_3(\bar{\alpha}_{2/3}) = \bar{\alpha}_1^2\bar{\alpha}_{-3}\bar{\alpha}_{4/4}$.*

Proof. (1): By [8, Th. 5.8], for $s \geq 0$, the elements $\bar{\alpha}_{4s+4/a(4s+4)}$ and $\bar{\alpha}_{4s+1}$ are permanent cycles. This fact is immediately extended to any $s \in \mathbb{Z}$.

(2): By [8, Th. 5.8], for $s \geq 0$, we have $d_3(\bar{\alpha}_{4s+3}) = \bar{\alpha}_1^3\bar{\alpha}_{4s+1}$ and $d_3(\bar{\alpha}_{4s+6/3}) = \bar{\alpha}_1^3\bar{\alpha}_{4s+4/a(4s+4)}$ in the spectral sequence. It is easy to extend these differentials to any $s \in \mathbb{Z}$, except for $d_3(\bar{\alpha}_{2/3})$. We also have $d_3(\bar{\alpha}_{2/3}) = d_3(v_1^{-4}\bar{\alpha}_{6/3}) = \bar{\alpha}_1^2(v_1^{-4}\bar{\alpha}_1)\bar{\alpha}_{4/4} = \bar{\alpha}_1^2\bar{\alpha}_{-3}\bar{\alpha}_{4/4}$.

For a nonzero integer t, we define

$$b(t) = \begin{cases} a(t) - 1 & v(t) = 1 \\ a(t) & \text{othewise} \end{cases} = \begin{cases} v(t) + 1 & v(t) = 0, 1 \\ v(t) + 2 & v(t) > 1 \end{cases}. \tag{2.4}$$

We then have the following:

THEOREM 10. *As a graded* $\mathbb{Z}_{(2)}$*-algebra,*

$$\pi_*(L_1S^0) = \mathbb{Z}_{(2)}[\alpha_{t/b(t)} : 0 \neq t \equiv 0, 1, 2 \bmod (4)]/R$$

with $|\alpha_{t/b(t)}| = 2t - 1$, *where* R *is the ideal of the following relations: Put* $\alpha_t = \alpha_{t/1}$ *and* $\xi_j = \alpha_{2^{j-2}/j}\alpha_{-2^{j-2}/j}$ *for* $j > 3$.

(1) $2\xi_{j+1} = \xi_j$ *for* $j > 3$.

(2) $2^{b(t)}\alpha_{t/b(t)} = \begin{cases} 0 & t \equiv 0, 1 \bmod (4) \\ \alpha_1^2\alpha_{t-1} & t \equiv 2 \bmod (4) \end{cases}.$

(3) $\alpha_{s/b(s)}\alpha_{t/b(t)} = \begin{cases} \xi_{a(s)} & s + t = 0 \text{ and } s \equiv t \equiv 0 \bmod (4) \\ 8\xi_4 = 8\alpha_{4/4}\alpha_{-4/4} & s + t = 0 \text{ and } s \equiv t \equiv 2 \bmod (4) \\ 0 & s + t \neq 0 \text{ and } s \equiv t \equiv 0 \bmod (2), \\ & \quad \text{or } st \equiv 2 \bmod (4) \\ \alpha_{-3}\alpha_{4/4} & s + t = 1 \\ \alpha_1\alpha_{s+t-1/b(s+t-1)} & \text{otherwise} \end{cases}.$

(4) $\alpha_1^{n(t)}\alpha_{t/b(t)} = 0$ *for* $n(t) = \begin{cases} 3 & t \equiv 0, 1 \bmod (4) \\ 1 & t \equiv 2 \bmod (4) \end{cases}$ *and* $\alpha_1^2\alpha_{-3}\alpha_{4/4} = 0$.

PROOF. By Proposition 8 and Proposition 9, for the $E(1)$-based Adams spectral sequence

$$E(1)_2^{a,b}(S^0) \Rightarrow \pi_{b-a}(L_1S^0),$$

we have the following tables for the E_4-term:

3						$\bar{\alpha}_1\bar{\alpha}_{-3}\bar{\alpha}_{4/4}$		$\bar{\alpha}_1^3$
2			$\bar{\xi}_j$		$\bar{\alpha}_{-3}\bar{\alpha}_{4/4}$		$\bar{\alpha}_1^2$	
1						$\bar{\alpha}_1$		$\bar{\alpha}_{2/2}$
0					1			
	-4	-3	-2	-1	0	1	2	3

(2.5)

and

3						$\bar{\alpha}_1^2\bar{\alpha}_{4s/a(4s)}$		$\bar{\alpha}_1^2\bar{\alpha}_{4s+1}$
2					$\bar{\alpha}_1\bar{\alpha}_{4s/a(4s)}$		$\bar{\alpha}_1\bar{\alpha}_{4s+1}$	
1				$\bar{\alpha}_{4s/a(4s)}$		$\bar{\alpha}_{4s+1}$		$\bar{\alpha}_{4s+2/2}$
0								
	$8s-4$	$8s-3$	$8s-2$	$8s-1$	$8s$	$8s+1$	$8s+2$	$8s+3$

(2.6)

for $s \neq 0$. Here, $b-a$ is the horizontal coordinate and a is the vertical coordinate. By degree reason, this spectral sequence collapses at E_4. If $j > 3$, then $\bar{\bar{\xi}}_j$ detects ξ_j in the statement. If $j \leq 3$, then $\bar{\bar{\xi}}_j = 2^{4-j}\bar{\bar{\xi}}_4$ detects $2^{4-j}\xi_4 = 2^{4-j}\alpha_{4/4}\alpha_{-4/4}$.

The relations in the statement are immediately shown by Proposition 8 and the above tables, except for

$$4\alpha_{4s+2/2} = \alpha_1^2\alpha_{4s+1} \qquad \text{and} \qquad 2\alpha_{4s+1} = 0. \tag{2.7}$$

They are immediately shown by [8, Th. 5.8 (b)].

3. The question mark spectrum Q

We recall the following theorem:

THEOREM 11 ([6, Th. 1.1]). *$L_nX \in \mathrm{Pic}^0(\mathscr{L}_n)$ if and only if $E(n)_*(X) = E(n)_*$ as an $E(n)_*(E(n))$-comodule.*

Consider the cofiber sequence

$$S^0 \xrightarrow{2} S^0 \xrightarrow{i} V(0) \xrightarrow{j} S^1. \tag{3.1}$$

We notice that $\pi_1(S^0) = \mathbb{Z}/2$, which is generated by the stable complex Hopf map η. Since $2\eta = 0$, there exists $\tilde{\eta} \in \pi_2(V(0))$ such that $j\tilde{\eta} = \eta$. The *question mark spectrum* Q is defined by the following cofiber sequence:

$$\Sigma^2 Q \xrightarrow{i_Q} S^2 \xrightarrow{\tilde{\eta}} V(0) \xrightarrow{j_Q} \Sigma^3 Q. \tag{3.2}$$

Since $\tilde{\eta} : S^2 \to V(0)$ induces $v_1 : E(1)_* \to E(1)_{*+2}/(2)$, we have the following commutative diagram.

$$\begin{array}{ccccc} E(1)_*(Q) & \xrightarrow{(i_Q)_*} & E(1)_* & \xrightarrow{v_1 = \tilde{\eta}_*} & E(1)_*/(2) \\ \uparrow & & \| & & \uparrow{\scriptstyle v_1}\,\wr \\ E(1)_* & \xrightarrow{2} & E(1)_* & \xrightarrow{i_*} & E(1)_*/(2). \end{array} \tag{3.3}$$

Hence $E(1)_*(Q)$ is isomorphic to $E(1)_*$, and so L_1Q is in $\mathrm{Pic}^0(\mathscr{L}_1)$ by Theorem 11. From [4, Th. 6.1], we obtain the isomophism

$$L_1(Q \wedge Q) = L_1S^0 \tag{3.4}$$

and $\mathrm{Pic}^0(\mathscr{L}_1) = \mathbb{Z}/2$ is generated by L_1Q.

4. The structure of $\pi^1_\star(S^0)$

We note that $E(1)_*(Q) = E(1)_*\{g_Q\}$ as $E(1)_*(E(1))$-comodules, where g_Q is an element in $E(1)_0(Q)$ which is corresponding to $1 \in \mathbb{Z}_{(2)} = E(1)_0$. This implies that

$$E(1)_2^{*,*}(Q) = E(1)_2^{*,*}(S^0)\{g_Q\} \qquad \text{with } |g_Q| = (0,0). \tag{4.1}$$

Lemma 12. *$d_3(g_Q) = \bar{\alpha}_{-1}\bar{\alpha}_1^2 g_Q$ in the $E(1)$-based Adams spectral sequence converging to $\pi_*(L_1 Q)$.*

Proof. The cofiber sequence (3.2) gives rise to the long exact sequence

$$\cdots \xrightarrow{\delta_Q} E(1)_2^{s,t}(Q) \xrightarrow{(i_Q)_*} E(1)_2^{s,t}(S^0) \xrightarrow{v_1} E(1)_2^{s,t+2}(V(0))$$
$$\xrightarrow{\delta_Q} E(1)_2^{s+1,t}(Q) \longrightarrow \cdots.$$

By the diagram (3.3), the element $g_Q \in E(1)_2^{0,0}(Q)$ satisfies $(i_Q)_*(g_Q) = 2$, and so $(i_Q)_*(g_Q)$ survives to $2 \in \pi_0(L_1 S^0)$. Recall that the diagram

$$\begin{array}{ccc} V(0) & \xrightarrow{2} & V(0) \\ {\scriptstyle j}\downarrow & & \uparrow{\scriptstyle i} \\ S^1 & \xrightarrow{\eta} & S^0 \end{array}$$

is commutative. Hence, in $\pi_2(V(0))$, we have $2\tilde{\eta} = i\eta j\tilde{\eta} = i\eta^2$. Therefore, since $\alpha_1 \in \pi_1(L_1 S^0)$ is the $E(1)$-localization of $\eta \in \pi_1(S^0)$, the generator $(L_1 i)\alpha_1^2 \in \pi_2(L_1 V(0))$ is detected by $i_*(\bar{\alpha}_1^2) \in E(1)_2^{2,4}(V(0))$, where i_* is the map induced by i in (3.1). By an easy calculation in the cobar complex, we have $\delta_Q(i_*(\bar{\alpha}_1^2)) = \bar{\alpha}_{-1}\bar{\alpha}_1^2 g_Q$. This implies $d_3(g_Q) = \bar{\alpha}_{-1}\bar{\alpha}_1^2 g_Q$.

Proposition 13. *In the $E(1)$-based Adams spectral sequence converging to $\pi_*(L_1 Q)$, the following hold:*

(1) *$d_3(g_Q) = \bar{\alpha}_{-1}\bar{\alpha}_1^2 g_Q$, and $2g_Q$ is permanent.*

(2) *If $t \equiv 0, 1 \bmod (4)$, then $d_3(\bar{\alpha}_{t/a(t)} g_Q) = \bar{\alpha}_1^3 \bar{\alpha}_{t-2/a(t-2)} g_Q$.*

(3) *If $t \equiv 2, 3 \bmod (4)$, then $\bar{\alpha}_{t/a(t)} g_Q$ is permanent.*

Proof. In the spectral sequence, we have the following by Theorem 8, Proposition 9 and Lemma 12:

$$d_3(\bar{\alpha}_{t/a(t)} g_Q) = \begin{cases} \bar{\alpha}_{t/a(t)}\bar{\alpha}_{-1}\bar{\alpha}_1^2 g_Q & t \equiv 0, 1 \bmod (4) \\ \bar{\alpha}_1^3 \bar{\alpha}_{t-2/a(t-2)} g_Q + \bar{\alpha}_{t/a(t)}\bar{\alpha}_{-1}\bar{\alpha}_1^2 g_Q & 2 \neq t \equiv 2, 3 \bmod (4) \\ \bar{\alpha}_1^2 \bar{\alpha}_{-3}\bar{\alpha}_{4/4} g_Q + \bar{\alpha}_{2/3}\bar{\alpha}_{-1}\bar{\alpha}_1^2 g_Q & t = 2 \end{cases}$$

$$= \begin{cases} \bar{\alpha}_1^3 \bar{\alpha}_{t-2/a(t-2)} g_Q & t \equiv 0, 1 \bmod (4) \\ 0 & t \equiv 2, 3 \bmod (4) \end{cases}.$$

We also remark that $d_3(\bar{\alpha}_{-3}\bar{\alpha}_{4/4}g_Q) = \bar{\alpha}_1^4 \bar{\alpha}_{-2/3}g_Q$. Hence, for the $E(1)$-based Adams spectral sequence

$$E(1)_2^{a,b}(Q) \Rightarrow \pi_{b-a}(L_1 Q),$$

we have the following tables of the E_4-term:

4			$\bar{\alpha}_1^3(\bar{\alpha}_{-2/3}g_Q)$					
3		$\bar{\alpha}_1^2(\bar{\alpha}_{-2/3}g_Q)$						
2	$\bar{\alpha}_1(\bar{\alpha}_{-2/3}g_Q)$		$\bar{\alpha}_1(\bar{\alpha}_{-1}g_Q)$ $\bar{\xi}_j g_Q$					
1		$\bar{\alpha}_{-1}g_Q$						$\bar{\alpha}_{2/3}g_Q$
0					$2g_Q$			
	-4	-3	-2	-1	0	1	2	3

(4.2)

and

3		$\bar{\alpha}_1^2(\bar{\alpha}_{4s-2/3}g_Q)$		$\bar{\alpha}_1^2(\bar{\alpha}_{4s-1}g_Q)$				
2	$\bar{\alpha}_1(\bar{\alpha}_{4s-2/3}g_Q)$		$\bar{\alpha}_1(\bar{\alpha}_{4s-1}g_Q)$					
1		$\bar{\alpha}_{4s-1}g_Q$		$\bar{\alpha}_{4s/a(4s)}(2g_Q)$				$\bar{\alpha}_{4s+2/3}g_Q$
0								
	$8s-4$	$8s-3$	$8s-2$	$8s-1$	$8s$	$8s+1$	$8s+2$	$8s+3$

(4.3)

for $s \neq 0$. Here $b-a$ is the horizontal coordinate and a is the vertical coordinate. By degree reason, this spectral sequence collapses at E_4, and our claim is shown.

Proof (Proof of Theorem 2). By (3.4), we have the pairing $E(1)_*(Q) \otimes E(1)_*(Q) \to E(1)_*$, and

$$\operatorname{gsq} g_Q^2 = 1. \tag{4.4}$$

We note that $\pi_\star^1(S^0) = \pi_*(L_1 S^0) \oplus [L_1 Q, L_1 S^0]_* = \pi_*(L_1 S^0) \oplus \pi_*(L_1 Q)$ as a graded $\mathbb{Z}_{(2)}$-module. Consider the two spectral sequences

$$E(1)_2^{*,*}(S^0) \Rightarrow \pi_*(L_1 S^0) \quad \text{and} \quad E(1)_2^{*,*}(Q) \Rightarrow \pi_*(L_1 Q) = [L_1 Q, L_1 S^0]_*.$$

We define $\bar{A}_{t/a(t)} \in E(1)_2^{*,*}(S^0)[g_Q]/(g_Q^2 = 1)$ by

$$\bar{A}_{t/a(t)} = \begin{cases} \bar{\alpha}_{t/a(t)} & t \equiv 0, 1 \bmod (4) \\ \bar{\alpha}_{t/a(t)} g_Q & t \equiv 2, 3 \bmod (4) \end{cases}.$$

By Lemma 9 and Lemma 13, the element $\bar{A}_{t/a(t)}$ survives to $\pi_\star^1(S^0) = \pi_*(L_1 S^0) \oplus [L_1 Q, L_1 S^0]_*$ for any nonzero t, which is denoted by $A_{t/a(t)}$. We also denote by $2_Q \in [L_1 Q, L_1 S^0] = \pi_0(L_1 Q)$ an element detected by $2g_Q \in E(1)_2^{0,0}(Q)$. The relations in the statement are given by Proposition 7, Lemma 9, Theorem 10, Lemma 13, (4.4), and the tables (2.5), (2.6), (4.2) and (4.3), except

$$4 \cdot 2_Q A_{t/3} = A_1^2 A_{t-1} \quad \text{for } t \equiv 2 \bmod (4), \qquad \text{and}$$

$$2A_t = 0 \quad \text{for } t \equiv 1 \bmod (4).$$

By (1.3) and (2.7), these relations are given by $4 \cdot 2_Q A_{t/3} = i_1(4\alpha_{t/2}) = i_1(\alpha_1^2 \alpha_{t-1}) = A_1^2 A_{t-1}$ for $t \equiv 2 \bmod (4)$, and $2A_t = i_1(2\alpha_t) = i_1(0) = 0$ for $t \equiv 1 \bmod (4)$.

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