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# On Riemannian foliations admitting transversal conformal fields

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**ABSTRACT.** Let $(M, g_M, \mathscr{F})$ be a closed, connected Riemannian manifold with a Riemannian foliation $\mathscr{F}$ of nonzero constant transversal scalar curvature. When $M$ admits a transversal nonisometric conformal field, we find some generalized conditions that $\mathscr{F}$ is transversally isometric to $(S^q(1/c), G)$, where $G$ is the discrete subgroup of $O(q)$ acting by isometries on the last $q$ coordinates of the sphere $S^q(1/c)$ of radius $1/c$.

## 1. Introduction

A Riemannian foliation is a foliation $\mathscr{F}$ on a smooth manifold $M$ such that the normal bundle $Q = TM/T\mathscr{F}$ may be endowed with a metric $g_Q$ whose Lie derivative is zero along leaf directions [15]. Note that we can choose a Riemannian metric $g_M$ on $M$ such that $g_M|_{T\mathscr{F}^\perp} = g_Q$; such a metric is called *bundle-like*. A Riemannian foliation $\mathscr{F}$ is *transversally isometric* to $(W, G)$, where $G$ is a discrete group acting by isometries on a Riemannian manifold $(W, g_W)$, if there exists a homeomorphism $\eta : W/G \to M/\mathscr{F}$ that is locally covered by isometries [10]. Recently, S. D. Jung and K. Richardson [6] proved the *generalized Obata theorem* which states that: $\mathscr{F}$ is transversally isometric to $(S^q(1/c), G)$, where $G$ is the discrete subgroup of $O(q)$ acting by isometries on the last $q$ coordinates of the sphere $S^q(1/c)$ of radius $1/c$ if and only if there exists a non-constant basic function $f$ such that

$$\nabla_X \nabla f = -c^2 f X$$

for all foliated normal vectors $X$, where $c$ is a positive real number and $\nabla$ is the transverse Levi-Civita connection on the normal bundle $Q$.

A *transversal conformal field* is a normal vector field with a flow preserving the conformal class of the transverse metric. That is, the infinitesimal auto-

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morphism $Y$ is transversal conformal if $L_Y g_Q = 2f_Y g_Q$ for a basic function $f_Y$ depending on $Y$, where $L_Y$ is the Lie derivative. In this case, it is trivial that

$$f_Y = \frac{1}{q}\,\mathrm{div}_{\nabla}(\pi(Y)),$$

where $\mathrm{div}_{\nabla}$ is a transversal divergence and $\pi : TM \to Q$ is the natural projection. If the transversal conformal field $Y$ satisfies $\mathrm{div}_{\nabla}(\pi(Y)) = 0$, i.e, $L_Y g_Q = 0$, then $Y$ is said to be *transversal Killing field*, that is, its flow is a transversal infinitesimal isometry. The properties of the infinitesimal automorphisms have been studied by many authors ([4], [8], [13], [14], [16]).

In this article, we study the Riemannian foliation admitting a transversal nonisometric conformal field. First, we recall the well-known theorems about the Riemannian foliations admitting a transversal nonisometric conformal field ([3], [4], [5], [6], [12]).

Let $R^Q$, $\mathrm{Ric}^Q$ and $\sigma^Q$ be the transversal curvature tensor, transversal Ricci operator and transversal scalar curvature with respect to the transversal Levi-Civita connection $\nabla$ on $Q$ [15]. Let $\kappa_B$ be the basic part of the mean curvature form $\kappa$ of the foliation $\mathscr{F}$ and $\kappa_B^\sharp$ its dual vector field (precisely, see Section 2). Then we have the following well-known theorem.

Theorem A ([6]). *Let $(M, g_M, \mathscr{F})$ be a closed, connected Riemannian manifold with a Riemannian foliation $\mathscr{F}$ of a nonzero constant transversal scalar curvature $\sigma^Q$. If $M$ admits a transversal nonisometric conformal field $Y$ satisfying one of the following conditions:*

(1) $Y = \nabla h$ *for any basic function* $h$, *or*

(2) $L_Y \mathrm{Ric}^Q = \mu g_Q$ *for some basic function* $\mu$, *or*

(3) $\mathrm{Ric}^Q(\nabla f_Y) = \frac{\sigma^Q}{q}\nabla f_Y$, $g_Q(\kappa_B^\sharp, \nabla f_Y) = 0$ *and* $g_Q(A_{\kappa_B^\sharp}\nabla f_Y, \nabla f_Y) \le 0$,

*then $\mathscr{F}$ is transversally isometric to $(S^q(1/c), G)$.*

Now, we recall two tensor fields $E^Q$ and $Z^Q$ ([3], [5]) by

$$E^Q(Y) = \mathrm{Ric}^Q(Y) - \frac{\sigma^Q}{q}Y, \qquad Y \in T\mathscr{F}^\perp, \tag{1}$$

$$Z^Q(X, Y) = R^Q(X, Y) - R^Q_\sigma(X, Y), \tag{2}$$

where $R^Q_\sigma(X, Y)s = \frac{\sigma^Q}{q(q-1)}\{g_Q(\pi(Y), s)\pi(X) - g_Q(\pi(X), s)\pi(Y)\}$ for any vector field $X, Y \in TM$ and $s \in \Gamma Q$. Trivially, if $E^Q = 0$ (resp. $Z^Q = 0$), then the foliation is transversally Einsteinian (resp. transversally constant sectional curvature). The tensor $Z^Q$ is called as the transversal concircular curvature tensor, which is a generalization of the concircular curvature tensor on a

Riemannian manifold. In an ordinary manifold, the concircular curvature tensor is invariant under a concircular transformation which is a conformal transformation preserving geodesic circles [17]. Then we have the well-known theorem.

THEOREM B ([3]). *Let* $(M, g_M, \mathscr{F})$ *be as in Theorem* A. *If* $M$ *admits a transversal nonisometric conformal field* $Y$ *such that*

$$\int_M g_Q(E^Q(\nabla f_Y), \nabla f_Y) \geq 0,$$

*then* $\mathscr{F}$ *is transversally isometric to* $(S^q(1/c), G)$.

REMARK 1. *Since* $\mathrm{Ric}^Q(\nabla f_Y) = \frac{\sigma^Q}{q} \nabla f_Y$ *implies* $E^Q(\nabla f_Y) = 0$, *Theorem* B *is a generalization of Theorem* A (3) *when* $\mathscr{F}$ *is minimal.*

THEOREM C ([4], [5]). *Let* $(M, g_M, \mathscr{F})$ *be as in Theorem* A, *and suppose that* $\mathscr{F}$ *is minimal. If* $M$ *admits a transversal nonisometric conformal field* $Y$ *such that*

$$\text{(i)} \quad L_Y |E^Q|^2 = 0 \ \ ([4])$$

*or*

$$\text{(ii)} \quad L_Y |Z^Q|^2 = 0 \ \ ([5]),$$

*then* $\mathscr{F}$ *is transversally isometric to* $(S^q(1/c), G)$.

REMARK 2. *Theorem* B *and Theorem* C *have been proved in* [18] *for the point foliation, that is, for ordinary manifolds.*

In this paper, we prove the following theorems.

THEOREM 1. *Let* $(M, g_M, \mathscr{F})$ *be as in Theorem* A, *and suppose that* $\mathscr{F}$ *is minimal. If* $M$ *admits a transversal nonisometric conformal field* $Y$ *such that*

$$L_Y |E^Q|^2 = \mathit{const.} \qquad \mathit{or} \qquad L_Y |Z^Q|^2 = \mathit{const.},$$

*then* $\mathscr{F}$ *is transversally isometric to* $(S^q(1/c), G)$.

REMARK 3. *Theorem* 1 *is a generalization of Theorem* C.

THEOREM 2. *Let* $(M, g_M, \mathscr{F})$ *be as in Theorem* A, *and suppose that* $\mathscr{F}$ *is minimal. If* $M$ *admits a transversal nonisometric conformal field* $Y$ *such that*

$$L_Y g_Q(L_Y E^Q, E^Q) \leq 0,$$

*then* $\mathscr{F}$ *is transversally isometric to* $(S^q(1/c), G)$.

Remark 4. *Theorem* 2 *is a generalization of Theorem* A (2) *and* (3) *when* $\mathscr{F}$ *is minimal* (*cf. Remark* 4.3). *See also* [19] *for the ordinary manifold.*

Theorem 3. *Let* $(M, g_M, \mathscr{F})$ *be as in Theorem* A. *If* $M$ *admits a transversal conformal field* $Y$ *such that* $Y = K + \nabla h$, *where* $K$ *is a transversal Killing field and* $h$ *is a basic function, then* $\mathscr{F}$ *is transversally isometric to* $(S^q(1/c), G)$.

Remark 5. *Theorem* 3 *is a generalization of Theorem* A (1).

## 2. Preliminaries

Let $(M, g_M, \mathscr{F})$ be a $(p+q)$-dimensional Riemannian manifold with a foliation $\mathscr{F}$ of codimension $q$ and a bundle-like metric $g_M$ with respect to $\mathscr{F}$ [15]. Let $TM$ be the tangent bundle of $M$, $T\mathscr{F}$ its integrable subbundle given by $\mathscr{F}$, and $Q = TM/T\mathscr{F}$ the corresponding normal bundle. Then there exists an exact sequence of vector bundles

$$0 \to T\mathscr{F} \to TM \underset{\sigma}{\overset{\pi}{\rightleftarrows}} Q \to 0,$$

where $\pi : TM \to Q$ is a natural projection and $\sigma : Q \to T\mathscr{F}^{\perp}$ is a bundle map satisfying $\pi \circ \sigma = \mathrm{id}$. Let $g_Q$ be the holonomy invariant metric on $Q$ induced by $g_M$, that is, $L_X g_Q = 0$ for any $X \in T\mathscr{F}$, where $L_X$ is the transversal Lie derivative, which is defined by $L_X s = \pi[X, \sigma(s)]$ for any $s \in \Gamma Q$. Let $\nabla$ be the transverse Levi-Civita connection in $Q$ [7]. The transversal curvature tensor $R^Q$ of $\nabla$ is defined by $R^Q(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ for any vector fields $X, Y \in \Gamma TM$. Let $\mathrm{Ric}^Q$ and $\sigma^Q$ be the transversal Ricci operator and the transversal scalar curvature of $\mathscr{F}$, respectively. The foliation $\mathscr{F}$ is said to be (transversally) *Einsteinian* if $\mathrm{Ric}^Q = \frac{1}{q}\sigma^Q \cdot \mathrm{id}$ with constant transversal scalar curvature $\sigma^Q$. The mean curvature vector field $\tau$ is defined by

$$\tau = \sum_{i=1}^{p} \pi(\nabla^M_{f_i} f_i),$$

where $\{f_i\}$ $(i = 1, \dots, p)$ is a local orthonormal frame field on $T\mathscr{F}$. The foliation $\mathscr{F}$ is said to be *minimal* if the mean curvature vector field $\tau$ vanishes. Let $\{e_a\}$ $(a = 1, \dots, q)$ be a local orthonormal frame field on $Q$. For any $s \in \Gamma Q$, the transversal divergence $\mathrm{div}_\nabla(s)$ is given by

$$\mathrm{div}_\nabla(s) = \sum_{a=1}^{q} g_Q(\nabla_{e_a} s, e_a).$$

For the later use, we recall the transversal divergence theorem [20] on a foliated Riemannian manifold.

Theorem 1 ([20]). *Let $(M, g_M, \mathscr{F})$ be a closed, connected Riemannian manifold with a foliation $\mathscr{F}$ and a bundle-like metric $g_M$ with respect to $\mathscr{F}$. Then*

$$\int_M \operatorname{div}_\nabla(s) = \int_M g_Q(s, \tau)$$

*for all $s \in \Gamma Q$.*

A differential form $\omega \in \Omega^r(M)$ is *basic* if $i(X)\omega = 0$ and $i(X)d\omega = 0$ for all $X \in T\mathscr{F}$, where $i(X)$ is the interior product. Let $\Omega_B^r(\mathscr{F})$ be the set of all basic r-forms on $M$. Then $\Omega^*(M) = \Omega_B^*(\mathscr{F}) \oplus \Omega_B^*(\mathscr{F})^\perp$ [1]. Let $\kappa$ be the mean curvature form of $\mathscr{F}$, which is given by

$$\kappa(s) = g_Q(\tau, s)$$

for any $s \in Q$. Then the basic part $\kappa_B$ of the mean curvature form is closed, i.e., $d\kappa_B = 0$ [1]. Let $d_B$ be the restriction of $d$ on $\Omega_B(\mathscr{F})$ and $\delta_B$ its formal adjoint operator of $d_B$ with respect to the global inner product $\langle\langle \cdot , \cdot \rangle\rangle$, which is given by

$$\langle\langle \phi, \psi \rangle\rangle = \int_M \phi \wedge \bar{*}\psi \wedge \chi_{\mathscr{F}}$$

for any basic $r$-forms $\phi$ and $\psi$, where $\bar{*}$ is the star operator on $\Omega_B^*(\mathscr{F})$ and $\chi_{\mathscr{F}}$ is the characteristic form of $\mathscr{F}$ [15]. The operator $\delta_B$ is given by

$$\delta_B\phi = (\delta_T + i(\kappa_B^\sharp))\phi, \qquad \delta_T\phi = (-1)^{q(r+1)+1}\bar{*}d_B\bar{*}\phi.$$

Note that the induced connection $\nabla$ on $\Omega_B^*(\mathscr{F})$ from the connection $\nabla$ on $Q$ and Riemannian connection $\nabla^M$ on $M$ extends the partial Bott connection, which satisfies $\nabla_X\omega = L_X\omega$ for any $X \in T\mathscr{F}$ [9]. Then the operator $\delta_T$ is given by

$$\delta_T\phi = -\sum_{a=1}^{q} i(e_a)\nabla_{e_a}\phi. \tag{3}$$

The *basic Laplacian* $\Delta_B$ acting on $\Omega_B^*(\mathscr{F})$ is defined by

$$\Delta_B = d_B\delta_B + \delta_B d_B.$$

Then for any basic function $f$, we have

$$\Delta_B f = \delta_B d_B f = -\sum_a \nabla_{e_a}\nabla_{e_a} f + \kappa_B^\sharp(f). \tag{4}$$

REMARK 6. *Note that for any basic form $\omega$, the relation between $\delta_B$ and the ordinary operator $\delta$ is given by*

$$\delta\omega = \delta_B\omega + *\gamma(\omega),$$

*where* $\gamma(\omega) = \pm\bar{*}\omega \wedge \varphi_0$ *and* $\varphi_0 = d\chi_{\mathscr{F}} + \kappa \wedge \chi_{\mathscr{F}}$ *with* $\varphi_0 \wedge \chi_{\mathscr{F}} = 0$ [15]. *If* $\omega \in \Omega_B^r$ $(r = 0, 1)$, *then we easily have*

$$\gamma(\omega) = 0,$$

*which implies that*

$$\delta\omega = \delta_B\omega, \qquad \Delta^M\omega = \Delta_B\omega,$$

*where* $\Delta^M = d\delta + \delta d$ *is the ordinary Laplacian.*

For later use, we recall the generalized maximum principle for foliation ([6]).

THEOREM 2 ([6]). *Let* $(M, g_M, \mathscr{F})$ *be a closed, connected Riemannian manifold with a foliation* $\mathscr{F}$ *and a bundle-like metric* $g_M$. *For any basic function* $f$, *the condition* $(\Delta_B - \kappa_B^\sharp)f \geq 0$ *implies that* $f$ *is constant.*

And we review some theorems for transversal nonisometric conformal field ([4]).

THEOREM 3 ([4]). *Let* $(M, g_M, \mathscr{F})$ *be a closed, connected Riemannian manifold with a foliation* $\mathscr{F}$ *of codimension* $q$ *and bundle-like metric* $g_M$ *such that* $\delta_B\kappa_B = 0$. *Assume that the transversal scalar curvature* $\sigma^Q$ *is nonzero constant. Then for any transversal nonisometric conformal field* $Y$ *such that* $L_Y g_Q = 2f_Y g_Q$ $(f_Y \neq 0)$,

$$(\Delta_B - \kappa_B^\sharp)f_Y = \frac{\sigma^Q}{q-1}f_Y \qquad \textit{and} \qquad \int_M f_Y = 0.$$

## 3. Tensors $E^Q$ and $Z^Q$

In this section, we give the properties of tensors $E^Q$ and $Z^Q$ on a Riemannian foliation. From (1) and (2), we have

$$\sum_a Z^Q(s, e_a)e_a = E^Q(s)$$

for any $s \in \Gamma Q$. Also, we have the following ([4], [5]).

$$\operatorname{tr}_Q E^Q = 0, \qquad \operatorname{div}_\nabla(E^Q) = \frac{q-2}{2q}\nabla\sigma^Q, \tag{5}$$

$$|E^Q|^2 = |\operatorname{Ric}^Q|^2 - \frac{(\sigma^Q)^2}{q}, \qquad |Z^Q|^2 = |R^Q|^2 - \frac{2(\sigma^Q)^2}{q(q-1)} \qquad \text{if } q \geq 2. \tag{6}$$

Now, we recall the Lie derivatives of tensors along the transversal conformal field.

Lemma 1 ([3], [4], [5]). *Let $Y$ be a transversal conformal field such that $L_Y g_Q = 2f_Y g_Q$. Then*

$$g_Q((L_Y R^Q)(e_a, e_b)e_c, e_d) = \delta_b^d \nabla_a f_c - \delta_b^c \nabla_a f_d - \delta_a^d \nabla_b f_c + \delta_a^c \nabla_b f_d, \tag{7}$$

$$(L_Y \operatorname{Ric}^Q)(e_a, e_b) = -(q-2)\nabla_a f_b + (\Delta_B f_Y - \kappa_B^\sharp(f_Y))\delta_a^b, \tag{8}$$

$$L_Y \sigma^Q = 2(q-1)(\Delta_B f_Y - \kappa_B^\sharp(f_Y)) - 2f_Y \sigma^Q, \tag{9}$$

$$(L_Y E^Q)(e_a, e_b) = -(q-2)\left\{\nabla_a f_b + \frac{1}{q}(\Delta_B f - \kappa_B^\sharp(f))\delta_a^b\right\}, \tag{10}$$

$$L_Y |E^Q|^2 = -2(q-2)g_Q(\nabla\nabla f_Y, E^Q) - 4f_Y |E^Q|^2, \tag{11}$$

$$L_Y |Z^Q|^2 = -8g_Q(\nabla\nabla f_Y, E^Q) - 4f_Y |Z^Q|^2. \tag{12}$$

*where $\nabla_a = \nabla_{e_a}$ and $f_a = \nabla_a f_Y$.*

Lemma 2. *If a transversal conformal field $Y$ satisfies $L_Y \operatorname{Ric}^Q = \mu g_Q$ for some basic function $\mu$, then*

$$L_Y E^Q = 0.$$

Proof. Let $Y$ be the transversal conformal field such that $L_Y g_Q = 2f_Y g_Q$. From (3.4), we have

$$-(q-2)\nabla_a f_b + (\Delta_B f_Y - \kappa_B^\sharp(f_Y))\delta_a^b = \mu\delta_a^b. \tag{13}$$

From (3) and (13), we have

$$\mu = \frac{2(q-1)}{q}(\Delta_B f_Y - \kappa_B^\sharp(f_Y)). \tag{14}$$

From (13) and (14), we have

$$-(q-2)\left\{\nabla_a f_b + \frac{1}{q}(\Delta_B f_Y - \kappa_B^\sharp(f_Y))\delta_a^b\right\} = 0.$$

Therefore, the proof follows from (10).

Lemma 3. *If $Y$ is a transversal conformal field, then*

$$L_Y|E^Q|^2 = 2g_Q(L_Y E^Q, E^Q).$$

Proof. Let $\{e_a\}$ be a local orthonormal basis on $Q$ such that $(\nabla e_a)_x = 0$ at a point $x$. Let $Y$ be the transversal conformal field $Y$ such that $L_Y g_Q = 2f_Y g_Q$. Then at $x$, we have

$$\begin{aligned} L_Y|E^Q|^2 &= \sum_a L_Y g_Q(E^Q(e_a), E^Q(E_a)) \\ &= \sum_a (L_Y g_Q)(E^Q(e_a), E^Q(e_a)) + 2\sum_a g_Q((L_Y E^Q)(e_a), E^Q(e_a)) \\ &\quad + 2\sum_a g_Q(E^Q(L_Y e_a), E^Q(e_a)) \\ &= 2f_Y|E^Q|^2 + 2g_Q(L_Y E^Q, E^Q) + 2\sum_a g_Q(E^Q(L_Y e_a), E^Q(e_a)). \end{aligned} \tag{15}$$

Now, we calculate the last term in the above equation. That is,

$$\begin{aligned} &\sum_a g_Q(E^Q(L_Y e_a), E^Q(e_a)) \\ &\quad = \sum_{a,b} g_Q(E^Q(L_Y e_a), e_b) g_Q(E^Q(e_a), e_b) \\ &\quad = \sum_{a,b} g_Q(E^Q(e_b), L_Y e_a) g_Q(E^Q(e_b), e_a) \\ &\quad = \frac{1}{2}\sum_{a,b} L_Y\{g_Q(E^Q(e_b), e_a) g_Q(E^Q(e_b), e_a)\} - 2f_Y|E^Q|^2 \\ &\qquad - \sum_a g_Q((L_Y E^Q)(e_a), E^Q(e_a)) - \sum_a g_Q(E^Q(L_Y e_a), E^Q(e_a)). \end{aligned}$$

Hence we have

$$\begin{aligned} 2\sum_a g_Q(E^Q(L_Y e_a), E^Q(e_a)) = \frac{1}{2} L_Y|E^Q|^2 - 2f_Y|E^Q|^2 \\ - g_Q(L_Y E^Q, E^Q). \end{aligned} \tag{16}$$

From (15) and (16), the proof is completed.

Lemma 4. *Let $Y$ be a transversal conformal field such that $L_Y g_Q = 2f_Y g_Q$. Then*

$$L_Y|Z^Q|^2 = 2g_Q(L_Y Z^Q, Z^Q) - 4f_Y|Z^Q|^2 \tag{17}$$

$$(q-2)g_Q(L_Y Z^Q, Z^Q) = 4g_Q(L_Y E^Q, E^Q) + 8f_Y|E^Q|^2. \tag{18}$$

Proof. Note that $g_Q(L_Y Z^Q, Z^Q) = -4g_Q(\nabla\nabla f_Y, E^Q)$ [5]. So (17) follows from (12). For the proof of (18), from (11) and (12),

$$4L_Y|E^Q|^2 = (q-2)L_Y|Z^Q|^2 + 4(q-2)f_Y|Z^Q|^2 - 16f_Y|E^Q|^2.$$

Hence from Lemma 3.3 and (17), the equation (18) is proved. □

From (6) and Theorem C, we have the following.

Proposition 1. *Let $(M, g_M, \mathscr{F})$ be a closed, connected Riemannian manifold with a minimal foliation $\mathscr{F}$ of codimension $q \geq 2$ and a bundle-like metric $g_M$. Assume that the transversal scalar curvature is nonzero constant and either $|\mathrm{Ric}^Q|$ or $|R^Q|$ is constant. If $M$ admits a transversal nonisometric conformal field, then $\mathscr{F}$ is transversally isometric to $(S^q(1/c), G)$.*

Remark 7. *For the ordinary manifold, Proposition* 3.5 *has been proved in* [2] *and* [11], *respectively.*

## 4. The proofs of Theorems

First, we recall the integral formulas for the tensor $E^Q$ and $Z^Q$.

Proposition 2 ([3], [5]). *Let $(M, g_M, \mathscr{F})$ be a closed, connected Riemannian manifold with a foliation $\mathscr{F}$ of codimension $q$ and a bundle-like metric $g_M$ with respect to $\mathscr{F}$. Assume that the transversal scalar curvature $\sigma^Q$ is nonzero constant. Then for any transversal nonisometric conformal field $Y$ such that $L_Y g_Q = 2f_Y g_Q$ $(f_Y \neq 0)$, we have*

$$2(q-2)\int_M g_Q(E^Q(\nabla f_Y), \nabla f_Y) = \int_M \{4f_Y^2|E^Q|^2 + f_Y L_Y|E^Q|^2\} + 2(q-2)\int_M g_Q(E^Q(f_Y\nabla f_Y), \kappa_B^\sharp)$$

*and*

$$\int_M g_Q(E^Q(\nabla f_Y), \nabla f_Y) = \frac{1}{2}\int_M \left\{f_Y^2|Z^Q|^2 + \frac{1}{4}f_Y L_Y|Z^Q|^2\right\}$$

$$\int_M g_Q(\mathrm{Ric}^Q(f_Y\nabla f_Y), \kappa_B^\sharp)$$

Proof of Theorem 1. Let $Y$ be the transversal nonisometric conformal field such that $L_Y g_Q = 2f_Y g_Q$. From Theorem 2.3, we have

$$\int_M f_Y = 0. \tag{19}$$

Assume that $\mathscr{F}$ is minimal. Since $L_Y|E^Q|^2 = const$ or $L_Y|Z^Q|^2 = const$, from (19) and Proposition 4.1, we have

$$2(q-2)\int_M g_Q(E^Q(\nabla f_Y), \nabla f_Y) = 4\int_M f_Y^2|E^Q|^2$$

or

$$\int_M g_Q(E^Q(\nabla f_Y), \nabla f_Y) = \frac{1}{2}\int_M f_Y^2|Z^Q|^2,$$

respectively. Hence from Theorem B, the proof is completed.

Lemma 5. *Let $Y$ be a transversal conformal field such that $L_Y g_Q = 2f_Y g_Q$. Then for any basic function $h$,*

$$\int_M hf_Y = -\frac{1}{q}\int_M L_Y h + \frac{1}{q}\int_M \operatorname{div}_\nabla(hY).$$

Proof. Let $\omega = Y^b$ be the dual basic 1-form of the transversal conformal form $Y$. Then

$$\int_M h(\delta_B\omega) = \int_M g_Q(\omega, d_B h) = \int_M i(Y)d_B h = \int_M L_Y h.$$

Since $\delta_B = \delta_T + i(\kappa_B^\sharp)$ and $\delta_T\omega = -\operatorname{div}_\nabla(Y) = -qf_Y$, we have

$$\begin{aligned} q\int_M hf_Y &= -\int_M h(\delta_T\omega) \\ &= -\int_M h(\delta_B\omega) + \int_M hi(\kappa_B^\sharp)\omega \\ &= -\int_M L_Y h + \int_M g_Q(hY, \kappa_B^\sharp) \\ &= -\int_M L_Y h + \int_M \operatorname{div}_\nabla(hY). \end{aligned}$$

Last equality in above follows from the transversal divergence theorem (Theorem 2.1). Therefore, the proof is completed. □

Proof of Theorem 2. Let $Y$ be a transversal nonisometric conformal field, i.e., $L_Y g_Q = 2f_Y g_Q$. From (4), Lemma 3.4 and Proposition 4.1, if we put $h = g_Q(L_Y E^Q, E^Q)$, then from Lemma 4.2, we have

$$\begin{aligned}
&(q-2)\int_M g_Q(E(\nabla f_Y), \nabla f_Y) \\
&\quad = 2\int_M f_Y^2 |E^Q|^2 + \int_M hf_Y + (q-2)\int_M g_Q(E(f_Y \nabla f_Y), \kappa_B^\sharp) \\
&\quad = 2\int_M f_Y^2 |E^Q|^2 - \frac{1}{q}\int_M L_Y h + \frac{1}{q}\int_M g_Q(hY, \kappa_B^\sharp) \\
&\qquad + (q-2)\int_M g_Q(E^Q(f_Y \nabla f_Y), \kappa_B^\sharp).
\end{aligned}$$

Since $\mathscr{F}$ is minimal, we have

$$(q-2)\int_M g_Q(E^Q(\nabla f_Y), \nabla f_Y) = 2\int_M f_Y^2 |E^Q|^2 - \frac{1}{q}\int_M L_Y g_Q(L_Y E^Q, E^Q).$$

Hence by the condition $L_Y g_Q(L_Y E^Q, E^Q) \le 0$, we have

$$\int_M g_Q(E^Q(\nabla f_Y), \nabla f_Y) \ge 0.$$

From Theorem B, the proof of Theorem 2 is completed.

Remark 8. *Let $\mathscr{F}$ be minimal. Then the following holds.*

(1) *From Lemma* 3.2, *Theorem* 2 *yields Theorem* A (2).

(2) *Theorem* 2 *is also a generalization of Theorem* A (3). *In fact, assume that* $\mathrm{Ric}^Q(\nabla f_Y) = \frac{\sigma^Q}{q}\nabla f_Y$, *that is,* $E^Q(\nabla f_Y) = 0$. *By differentiation, we have*

$$(\nabla_{e_a} E^Q)(\nabla f_Y) + E^Q(\nabla_a \nabla f_Y) = 0. \tag{20}$$

*From* (20), *we have*

$$\begin{aligned}
0 &= \sum_a g_Q((\nabla_{e_a} E^Q)(\nabla f_Y) + E^Q(\nabla_a \nabla f_Y), e_a) \\
&= g_Q(\nabla f_Y, \mathrm{div}_\nabla(E^Q)) + \sum_a g_Q(E^Q(\nabla_a \nabla f_Y), e_a) \\
&= \sum_a g_Q(\nabla_a \nabla f_Y, E^Q(e_a)).
\end{aligned} \tag{21}$$

*From* (5), $\mathrm{div}_\nabla E^Q = 0$ *and so the last equality in the above follows. Hence from* (10) *and* (21), *we have*

$$\begin{aligned} g_Q(L_Y E^Q, E^Q) &= \sum_a g_Q((L_Y E^Q)(e_a), E^Q(e_a)) \\ &= -(q-2)\sum_a g_Q(\nabla_a \nabla f_Y, E^Q(e_a)) \\ &\quad - \frac{q-2}{q}(\Delta_B f_Y)\sum_a g_Q(e_a, E^Q(e_a)) \\ &= -(q-2)\sum_a g_Q(\nabla_a \nabla f_Y, E^Q(e_a)) - \frac{q-2}{q}(\Delta_B f_Y)\,\mathrm{tr}_Q\, E^Q \\ &= 0. \end{aligned}$$

*The last equality follows from* $\mathrm{tr}_Q\, E^Q = 0$. *Hence the conditions of Theorem* A (3) *implies that* $g_Q(L_Y E^Q, E^Q) = 0$. *That is, by Theorem* 2, $\mathscr{F}$ *is transversally isometric to the sphere.*

Proof of Theorem 3. Let $Y$ be a transversal conformal field such that $L_Y g_Q = 2f_Y g_Q$ and $Y = K + \nabla h$, where $K$ is a transversal Killing field and $h$ is a basic function. Then

$$g_Q(\nabla_X Y, Z) + g_Q(\nabla_Z Y, X) = 2f_Y g_Q(X, Z)$$

for any normal vector field $X, Z \in \Gamma Q$. On the other hand, since the transversal scalar curvature $\sigma^Q$ is constant, from Theorem 2.4, we have

$$(\Delta_B - \kappa_B^\sharp) f_Y = \frac{\sigma^Q}{q-1} f_Y. \tag{22}$$

Since $Y = K + \nabla h$, we have $L_Y g_Q = L_{\nabla h} g_Q = 2f_Y g_Q$. That is,

$$g_Q(\nabla_X \nabla h, Z) + g_Q(\nabla_Z \nabla h, X) = 2f_Y g_Q(X, Z). \tag{23}$$

On the other hand, $(\nabla\nabla h)(X, Z) = g_Q(\nabla_X \nabla h, Z)$ is symmetric. Therefore, from (23)

$$(\nabla\nabla h)(X, Z) = f_Y g_Q(X, Z). \tag{24}$$

Hence from (3) and (24), we have

$$(\Delta_B - \kappa_B^\sharp) h = -q f_Y. \tag{25}$$

From (22) and (25), we get

$$(\Delta_B - \kappa_B^\sharp)\left(f_Y + \frac{\sigma^Q}{q(q-1)} h\right) = 0.$$

By the generalized maximum principle (Theorem 2.3), we have

$$f_Y + \frac{\sigma^Q}{q(q-1)} h = const,$$

which implies

$$\nabla\nabla f_Y + \frac{\sigma^Q}{q(q-1)} \nabla\nabla h = 0. \tag{26}$$

From (24) and (26), we have

$$\nabla\nabla f_Y = -\frac{\sigma^Q}{q(q-1)} f_Y.$$

By the generalized Obata theorem [6], $\mathscr{F}$ is transversally isometric to $(S^q(1/c), G)$, where $c^2 = \frac{\sigma^Q}{q(q-1)}$.

REMARK 9. Theorem 3 is a generalization of Theorem A (1).

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## References

[1] J. A. Alvarez López, The basic component of the mean curvature of Riemannian foliations, Ann. Global Anal. Geom. 10 (1992), 179–194.
[2] C. C. Hsiung, On the group of conformal transformations of a compact Riemannian manifold, Proc. Nat. Acad. Sci. U.S.A. 56 (1965), 1509–1513.
[3] S. D. Jung, Riemannian foliations admitting transversal conformal fields II, Geom. Dedicata 175 (2015), 257–266.
[4] M. J. Jung and S. D. Jung, Riemannian foliations admitting transversal conformal fields, Geom. Dedicata 133 (2008), 155–168.
[5] S. D. Jung and K. R. Lee, The properties of Riemannian foliations admitting transversal conformal fields, Bull. Korean Math. Soc. 55 (2018), 1273–1283.
[6] S. D. Jung, K. R. Lee and K. Richardson, Generalized Obata theorem and its applications on foliations, J. Math. Anal. Appl. 376 (2011), 129–135.
[7] F. W. Kamber and Ph. Tondeur, Harmonic foliations, Proc. National Science Foundation Conference on Harmonic Maps, Tulane, Dec. 1980, Lecture Notes in Math. 949, Springer-Verlag, New-York, 1982, 87–121.
[8] F. W. Kamber and Ph. Tondeur, Infinitesimal automorphisms and second variation of the energy for harmonic foliations, Tohoku Math. J. 34 (1982), 525–538.
[9] F. W. Kamber and Ph. Tondeur, De Rham-Hodge theory for Riemannian foliations, Math. Ann. 277 (1987), 415–431.

[10] J. Lee and K. Richardson, Lichnerowicz and Obata theorems for foliations, Pacific J. Math. 206 (2002), 339–357.

[11] A. Lichnerowicz, Sur les transformations conformes d'une variété riemannienne compacte, C. R. Acad. Sci. Paris 259 (1964), 697–700.

[12] H. K. Pak and J. H. Park, A note on generalized Lichnerowica-Obata theorems for Riemannian foliations, Bull. Korean Math. Soc. 48 (2011), 769–777.

[13] J. S. Pak and S. Yorozu, Transverse fields on foliated Riemannian manifolds, J. Korean Math. Soc. 25 (1988), 83–92.

[14] J. H. Park and S. Yorozu, Transversal conformal fields of foliations, Nihonkai Math. J. 4 (1993), 73–85.

[15] Ph. Tondeur, Geometry of foliations, Birkhäuser-Verlag, Basel; Boston: Berlin, 1997.

[16] Ph. Tondeur and G. Toth, On transversal infinitesimal automorphisms for harmonic foliations, Geom. Dedicata 24 (1987), 229–236.

[17] K. Yano, Concircular geometry I, Concircular transformations, Proc. Imp. Acad. Tokyo 16 (1940), 195–200.

[18] K. Yano, On Riemannian manifolds with constant scalar curvature admitting a conformal transformation group, Proc. Nat. Acad. Sci. U.S. 55 (1966), 472–476.

[19] K. Yano and S. Sawaki, Riemannioan manifolds admitting a conformal transformation group, J. Differential Geometry 2 (1968), 161–184.

[20] S. Yorozu and T. Tanemura, Green's theorem on a foliated Riemannian manifold and its applications, Acta Math. Hungar. 56 (1990), 239–245.

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