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Some Problems of deformations on three-step nilpotent Lie groups

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ABSTRACT. Let G be an exponential solvable Lie group and H a connected Lie subgroup of G. Given any discontinuous group Γ for the homogeneous space $\mathscr{M} = G/H$ and any deformation of Γ, deformation of discrete subgroups may destroy proper discontinuity of the action on $\mathscr{M}$ as H is not compact (except the case when it is trivial). To interpret this phenomenon in the case when G is a 3-step nilpotent, we provide a layering of Kobayashi's deformation space $\mathscr{T}(\Gamma, G, H)$ into Hausdorff spaces, which depends upon the dimensions of G-adjoint orbits of the corresponding parameter space. This allows us to establish a Hausdorffness theorem for $\mathscr{T}(\Gamma, G, H)$.

1. Introduction

Our attention in this paper is focused on the explicit determination of the deformation space of discontinuous groups acting on certain nilpotent homogeneous spaces for which the group in question is 3-step nilpotent. The problem of describing deformations for general settings, is advocated by Kobayashi in [13] where he formalized the study of the deformation of Clifford-Klein forms from a theoretic point of view. See [11, 13, 14, 15] for further perspectives and basic examples. As an application of the general theory, T. Kobayashi and S. Nasrin studied in [14] properly discontinuous actions of a discrete subgroup $\Gamma \simeq \mathbb{Z}^k$ which acts on $\mathbb{R}^{k+1} \simeq G/H$ through a certain 2-step nilpotent affine transformation group G of dimension $2k+1$ when the connected subgroup H in question is $\mathbb{R}^k$. In these circumstances, the authors gave a complete description of the parameter space

$$\mathscr{R}(\Gamma, G, H) := \left\{ \varphi \in \mathrm{Hom}(\Gamma, G) \;\middle|\; \begin{array}{l} \varphi \text{ is injective, } \varphi(\Gamma) \text{ is discrete and} \\ \text{acts properly and fixed point freely} \\ \text{on } G/H \end{array} \right\} \quad (1)$$

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which is introduced in [11] in full generality. Here, $\mathrm{Hom}(\Gamma, G)$ denotes the set of all homomorphisms $\Gamma \to G$ endowed with the topology of pointwise convergence and the group G acts on $\mathrm{Hom}(\Gamma, G)$ by inner conjugation. They also determined the deformation space $\mathscr{T}(\Gamma, G, H)$ which is the quotient space of the parameters space given above through the G-action.

Later, a layering of the above parameter and deformation spaces was described in [4] for the cases of general 2-step nilpotent Lie groups. Further, the authors show in this case that the deformation space is a Hausdorff space if all the G-orbits in $\mathscr{R}(\Gamma, G, H)$ have the same dimension.

In this paper, we study the setting where the underlying group G is 3-step nilpotent. We will provide a stratification of both the parameter and deformation spaces based on the dimensions of the G-adjoint orbits on $\mathrm{Hom}(\mathfrak{l}, \mathfrak{g})$, where $\mathfrak{l}$ stands for the Lie algebra of the syndetic hull of Γ (Theorems 3 and 7). We then provide a layering of the deformation space $\mathscr{T}(\Gamma, G, H)$ into some Hausdorff subspaces. The algebraic interpretation of these spaces given in Theorem 1 appears as a fundamental ingredient in this respect. We close the paper by giving a sufficient condition on (Γ, G, H) for the Hausdorffness of $\mathscr{T}(\Gamma, G, H)$.

2. Backgrounds and notations

We begin this section with fixing some notation, terminologies and recording some basic facts about deformations of Clifford-Klein forms. The readers could consult the references [7, 8, 9, 10, 12, 13] and some references therein for broader information about the subject. Concerning the entire subject, we strongly recommend the papers [8] and [13].

2.1. Proper and fixed point actions. Let $\mathscr{M}$ be a locally compact space and K a locally compact topological group. A continuous action of the group K on $\mathscr{M}$ is said to be:

(1) Proper if, for each compact subset $S \subset \mathscr{M}$ the set $K_S = \{k \in K \mid k \cdot S \cap S \neq \varnothing\}$ is compact.

(2) Fixed point free if, for each $m \in \mathscr{M}$, the isotropy group $K_m = \{k \in K \mid k \cdot m = m\}$ is trivial.

(3) (CI) if for any $m \in \mathscr{M}$, the subset K_m of K defined above is compact. (cf. [8]).

(4) Properly discontinuous if, K is discrete and the action of K on $\mathscr{M}$ is proper.

(5) The group K is said to be discontinuous, if it is discrete and acts on $\mathscr{M}$ properly and fixed point freely.

Let G be a Lie group and H a closed subgroup of G. In the case where $\mathscr{M} = G/H$ is a homogeneous space and K is a closed subgroup of G, then it is well known that the action of K on $\mathscr{M}$ is proper if $SHS^{-1} \cap K$ is compact for any compact set S in G. Likewise the action of K on $\mathscr{M}$ is free if and only if for every $g \in G$, $K \cap gHg^{-1} = \{e\}$. In this context, the subgroup K is said to be a discontinuous group for the homogeneous space $\mathscr{M}$, if K is a discrete subgroup of G and K acts properly and fixed point freely on $\mathscr{M}$.

2.2. Clifford-Klein forms. For any given discontinuous subgroup Γ of a Lie group G for the homogeneous space G/H, the quotient space $\Gamma \backslash G/H$ is said to be a *Clifford-Klein form* for the homogeneous space G/H. The following point was emphasized in [11]. Any Clifford-Klein form is endowed with a smooth manifold structure for which the quotient canonical surjection $\pi : G/H \to \Gamma \backslash G/H$ turns out to be an open covering and particularly a local diffeomorphism. On the other hand, any Clifford-Klein form $\Gamma \backslash G/H$ inherits any G-invariant local geometric structure (e.g. complex structure, pseudo-Riemanian structure, conformal structure, symplectic structure, ...) on the homogeneous space G/H through the covering map π.

2.3. Parameter and deformation spaces. The material dealt with in this subsection comes from [13]. The reader could also consult the references [9] and [12] for precise definitions. Throughout this paper, we only consider the case where Γ is finitely generated. As in the first introductory section, we designate by $\mathrm{Hom}(\Gamma, G)$ the set of group homomorphisms from Γ to G endowed with the point wise convergence topology. The same topology is obtained by taking generators $\gamma_1, \dots, \gamma_k$ of Γ, then using the injective map

$$\mathrm{Hom}(\Gamma, G) \to G \times \cdots \times G, \qquad \varphi \mapsto (\varphi(\gamma_1), \dots, \varphi(\gamma_k))$$

to equip $\mathrm{Hom}(\Gamma, G)$ with the relative topology induced from the direct product $G \times \cdots \times G$. The topology of the parameter space $\mathscr{R}(\Gamma, G, H)$ in Section 1 is defined as the relative topology in $\mathrm{Hom}(\Gamma, G)$. For each $\varphi \in \mathscr{R}(\Gamma, G, H)$, the space $\varphi(\Gamma) \backslash G/H$ is a Clifford-Klein form which is a Hausdorff topological space and even equipped with a structure of a smooth manifold for which, the quotient canonical map is a smooth open covering. Let now $\varphi \in \mathscr{R}(\Gamma, G, H)$ and $g \in G$, we consider the element φ^g of $\mathrm{Hom}(\Gamma, G)$ defined by $\varphi^g(\gamma) = g^{-1}\varphi(\gamma)g$ for $\gamma \in \Gamma$. It is then clear that the element φ^g is in $\mathscr{R}(\Gamma, G, H)$ and that the map:

$$\varphi(\Gamma) \backslash G/H \to \varphi^g(\Gamma) \backslash G/H, \qquad \varphi(\Gamma)xH \mapsto \varphi^g(\Gamma)g^{-1}xH$$

is a natural diffeomorphism. Following [13], we then consider the orbit space

$$\mathscr{T}(\Gamma, G, H) = \mathscr{R}(\Gamma, G, H)/G$$

instead of $\mathscr{R}(\Gamma, G, H)$ in order to avoid the unessential part of deformations arising inner automorphisms and to be quite precise on parameters. The quotient space $\mathscr{T}(\Gamma, G, H)$ is called the deformation space of the discontinuous action of Γ on the homogeneous space G/H.

2.4. Algebraic description of the parameter and deformation spaces. Let $\mathfrak{g}$ be a finite dimensional real exponential solvable Lie algebra and G its associated Lie group. This means that the exponential map $\exp : \mathfrak{g} \to G$ is a global C^∞-diffeomorphism from $\mathfrak{g}$ into G. That is, G is connected and simply connected. Let $\log : G \to \mathfrak{g}$ denote the inverse map of $\exp : \mathfrak{g} \to G$. The Lie algebra $\mathfrak{g}$ acts on $\mathfrak{g}$ by the adjoint representation ad, that is $\mathrm{ad}_T(Y) = [T, Y]$ for $T, Y \in \mathfrak{g}$. The group G acts on $\mathfrak{g}$ by the adjoint representation Ad, defined by

$$\mathrm{Ad}_g = \mathrm{Exp}(\mathrm{ad}_T) = \sum_{k=0}^{\infty} \frac{1}{k!} (\mathrm{ad}_T)^k$$

for $g = \exp T \in G$. Let H be a closed connected subgroup of G and denote by $\mathfrak{h}$ the Lie algebra of H. Let Γ be a discrete subgroup of G of rank k and define the parameter space $\mathscr{R}(\Gamma, G, H)$ as given in (1) in Section 1. Let L be the syndetic hull of Γ which is the smallest (and hence the unique) connected closed subgroup of G which contains Γ co-compactly (see [3]). Recall that the Lie subalgebra $\mathfrak{l}$ of L is the real span of the lattice $\log \Gamma$ in $\mathfrak{g}$, which is generated by $\{\log \gamma_1, \dots, \log \gamma_k\}$ where $\{\gamma_1, \dots, \gamma_k\}$ is a set of generators of Γ. The group G also acts on $\mathrm{Hom}(\mathfrak{l}, \mathfrak{g})$ by:

$$g \cdot \psi = \mathrm{Ad}_g \circ \psi. \tag{2}$$

Here $\mathrm{Hom}(\mathfrak{l}, \mathfrak{g})$ is the set of homomorphisms of Lie algebras from $\mathfrak{l}$ to $\mathfrak{g}$ endowed with the trace topology of $\mathscr{L}(\mathfrak{l}, \mathfrak{g})$, the set of linear maps from $\mathfrak{l}$ to $\mathfrak{g}$. Let $\mathrm{Hom}^{\mathrm{inj}}(\mathfrak{l}, \mathfrak{g})$ be the set of injective homomorphisms from $\mathfrak{l}$ to $\mathfrak{g}$. The following useful result was originated in [14] and obtained in [3].

THEOREM 1. *Let $G = \exp \mathfrak{g}$ be a completely solvable Lie group, $H = \exp \mathfrak{h}$ a closed connected subgroup of G, Γ a discontinuous group for the homogeneous space G/H and $L = \exp \mathfrak{l}$ its syndetic hull. Then up to a homeomorphism, the parameter space $\mathscr{R}(\Gamma, G, H)$ is given by:*

$$\mathscr{R}(\mathfrak{l}, \mathfrak{g}, \mathfrak{h}) = \{\psi \in Hom^{inj}(\mathfrak{l}, \mathfrak{g}) \mid \exp(\psi(\mathfrak{l})) \text{ acts properly on } G/H\}.$$

The deformation space $\mathscr{T}(\Gamma, G, H)$ is likewise homeomorphic to the space

$$\mathscr{T}(\mathfrak{l}, \mathfrak{g}, \mathfrak{h}) = \mathscr{R}(\mathfrak{l}, \mathfrak{g}, \mathfrak{h})/Ad,$$

where the action Ad of G is given as in (2).

2.5. On the structure of the parameter space. Let L be a closed subgroup of G. We remark that if L acts on G/H properly, then the L-action is (CI) by definition. The converse claim is not true in general (see [17]).

Definition 1. Let G be an exponential solvable Lie group and H a connected and closed subgroup of G. A pair (G, H) is said to have the Lipsman property if for any connected closed subgroup L of G acting on G/H with the property (CI), the L-action on G/H is proper.

When for instance G is nilpotent and of n-step $n \leq 3$, any pair (G, H) has the Lipsman property (cf. [1, 16, 18]).

Definition 2 (cf. [5]). A subset V of $\mathbb{R}^n$ is called semi-algebraic if there exist some polynomial functions $\mathrm{P}_{i,j}$ $(i = 1, \ldots, s,\ j = 1, \ldots, r_i)$ and binary relations $\varsigma_{ij} \in \{>, =, <\}$ such that

$$V = \bigcup_{i=1}^{s} \{x \in \mathbb{R}^n \mid \mathrm{P}_{i,j}(x) \varsigma_{ij} 0 \text{ for } j = 1, \ldots, r_i\}.$$

The following proposition shows that the parameter space is semi-algebraic whenever the pair (G, H) has Lipsman's property with G a connected and simply connected nilpotent Lie group.

Proposition 1 (cf. [4]). *Let (G, H) be a pair having the Lipsman property with G a connected simply connected nilpotent Lie group, Γ a discontinuous subgroup for G/H, and $\mathfrak{l}$ the Lie algebra of the syndetic hull of Γ. Then the parameter space $\mathscr{R}(\Gamma, G, H)$ is semi-algebraic in $\mathscr{L}(\mathfrak{l}, \mathfrak{g})$.*

2.6. Some preliminary results.

Fact 2.1. *Let V be a vector space, E and F two subspaces of V such that $V = E \oplus F$. Then for any $v \in V$*

$$(v + E) \cap F = P(v),$$

where P is the projection of V on F parallel to E.

Proof. Let $v \in V$ and write $v = v_1 + v_2$ with $v_1 \in E$ and $v_2 \in F$. Then $P(v) = v_2$ and $v + E = v_2 + E$. Let $u \in (v + E) \cap F$, then there exists $w \in E$ such that $u = v_2 + w$ and we have

$$u \in F \Rightarrow v_2 + w \in F \Rightarrow w \in F \Rightarrow w \in E \cap F = \{0\}.$$

Thus $u = v_2$. □

FACT 2.2. *Let F, K be two finite dimensional vector spaces and $\mathscr{B} = \{e_1, \dots, e_m\}$ a basis of F.*

(1) *If $\varphi \in \mathscr{L}(F, K)$ of rank $t > 0$, then there exists $\{e_{j_1}, \dots, e_{j_t}\}$ a subset of $\mathscr{B}$ such that $Im\ \varphi = \mathbb{R}$-span$\{\varphi(e_{j_1}), \dots, \varphi(e_{j_t})\}$.*

(2) *Let $S = \{e_{j_1}, \dots, e_{j_t}\}$ be a subset of $\mathscr{B}$, then the set*

$$A(S) = \{\varphi \in \mathscr{L}(F, K) \mid dim\ \mathbb{R}\text{-}span\{\varphi(e_{j_1}), \dots, \varphi(e_{j_t})\} = t\}$$

is open in $\mathscr{L}(F, K)$.

PROOF. (1) As $\{\varphi(e_1), \dots, \varphi(e_m)\}$ is a generating family of $\mathrm{Im}(\varphi)$, we can extract from this family a basis of $\mathrm{Im}(\varphi)$. (2) Let $r = \dim K$, fix a basis $\mathscr{B}'$ of K and identify $\mathscr{L}(F, K)$ to the set of matrices $M_{r,m}(\mathbb{R})$ as a topological space. In this context, the set $A(S)$ is identified to

$$A'(S) = \{M \in M_{r,m}(\mathbb{R}) \mid \mathrm{rk}(M') = t\}$$

where $M' \in M_{r,t}(\mathbb{R})$ is the matrix obtained from M by deleting all the columns of index $k \notin \{j_1, \dots, j_t\}$. Let now

$$J(t, r) = \{(k_1, \dots, k_t) \in \mathbb{N}^t \mid 1 \le k_1 < \cdots < k_t \le r\}.$$

For $\alpha = (k_1, \dots, k_t)$ and $M' \in M_{r,t}(\mathbb{R})$, we denote by M'_α the square matrix obtained from M' by deleting all the lines of index $k \notin \{k_1, \dots, k_t\}$. Then the condition $\mathrm{rk}(M') = t$ is equivalent to

$$\sum_{\alpha \in J(t,r)} [\det(M'_\alpha)]^2 \neq 0$$

which proves that $A'(S)$ is open and therefore $A(S)$ is also open. □

FACT 2.3. *Let F be a finite dimensional vector space, V a subspace of F and $t = \dim F - \dim V$. For all integer n, let $S_n = \{u_{1,n}, \dots, u_{t,n}\}$ be a family of linearly independent vectors in F such that*

(1) *$F = \mathbb{R}$-span$(S_n) \oplus V$.*

(2) *For all $1 \le i \le t$, the sequence $(u_{i,n})_n$ converges to some vector u_i.*

(3) *$S = \{u_1, \dots, u_t\}$ is formed by linearly independent vectors and $F = \mathbb{R}$-span$(S) \oplus V$.*

Let P_n denote the projection of F on V parallel to $\mathbb{R}$-span(S_n) and P the projection of F on V parallel to $\mathbb{R}$-span(S). Then the sequence $(P_n)_n$ converges in $\mathscr{L}(V)$ to P.

PROOF. Let $m = \dim F$ and $\mathscr{B} = \{e_1, \dots, e_m\}$ a basis of F such that $\{e_{t+1}, \dots, e_m\}$ is a basis of V. By hypothesis (1), the set

$$\mathscr{B}_n = \{u_{1,n}, \dots, u_{t,n}, e_{t+1}, \dots, e_m\}$$

is a basis of F for all n and the matrix of P_n in the basis $\mathscr{B}_n$ is

$$Q = \begin{pmatrix} 0_{\mathbb{R}\text{-span}(S_n)} & 0 \\ 0 & \mathrm{id}_V \end{pmatrix}.$$

If $P_{\mathscr{B}\mathscr{B}_n}$ is the transition base matrix, then the matrix of P_n in $\mathscr{B}$ is

$$Q_n = P_{\mathscr{B}\mathscr{B}_n} Q P^{-1}_{\mathscr{B}\mathscr{B}_n}.$$

Now by (2) and (3), $(P_{\mathscr{B}\mathscr{B}_n})_n$ converges to $P_{\mathscr{B}\mathscr{B}'}$, where $\mathscr{B}' = \{u_1, \dots, u_t, e_{t+1}, \dots, e_n\}$. Then $(Q_n)_n$ converges to the matrix $Q' = P_{\mathscr{B}\mathscr{B}'} Q P^{-1}_{\mathscr{B}\mathscr{B}'}$, which is the matrix of P in $\mathscr{B}$. □

FACT 2.4. *Let V, W be two finite dimensional vector spaces, $\mathscr{B} = \{e_1, \dots, e_n\}$ a basis of V and $f : V \to W$ a linear map such that $\{f(e_1), \dots, f(e_k)\}$ is a basis of Im f. Assume that*

$$f(e_{k+j}) = \alpha_{1,j} f(e_1) + \cdots + \alpha_{k,j} f(e_k), \qquad 1 \le j \le n-k.$$

Then the family of vectors

$$u_j = e_{k+j} - \alpha_{1,j} e_1 - \cdots - \alpha_{k,j} e_k, \qquad 1 \le j \le n-k$$

is a basis of $\ker f$.

PROOF. Clearly the family $\{u_j, 1 \le j \le n-k\}$ is a family of linearly independent vectors and we have $f(u_j) = 0$ for all $1 \le j \le n-k$. As $\dim \ker f = n-k$, the result follows. □

FACT 2.5. *Let V, W be two finite dimensional vector spaces, $\mathscr{B} = \{e_1, \dots, e_m\}$ a basis of V and $(f_n : V \to W)_n$ a sequence of linear maps such that:*

(1) *$(f_n)_n$ converges to a linear map $f : V \to W$.*

(2) *The family $\{f_n(e_1), \dots, f_n(e_k)\}$ is a basis of Im f_n for all n.*

(3) *The family $\{f(e_1), \dots, f(e_k)\}$ is a basis of Im f.*

Assume that for all $n \ge 0$ and $1 \le j \le m-k$, we have

$$f_n(e_{k+j}) = \alpha^n_{1,j} f_n(e_1) + \cdots + \alpha^n_{k,j} f_n(e_k)$$

and

$$f(e_{k+j}) = \alpha_{1,j} f(e_1) + \cdots + \alpha_{k,j} f(e_k).$$

Then for all $1 \le j \le n-k$ and $1 \le l \le k$, the sequence $(\alpha^n_{l,j})_n$ converges to $\alpha_{l,j}$.

PROOF. As $(f_n)_n$ converges to f, we have $(f_n(e_{k+j}) - f(e_{k+j}))_n$ converges to zero. Let now $\dim W = r$ and let $u_1, \dots, u_{r-k} \in W$ be such that $\mathscr{B}' = \{f(e_1), \dots, f(e_k), u_1, \dots, u_{r-k}\}$ is a basis of W. As $(f_n(e_i))_n$ converges to

$f(e_i)$ for all $1 \le i \le k$, there exists $N > 0$ such that for all $n > N$ the family $\{f_n(e_1), \dots, f_n(e_k), u_1, \dots, u_{r-k}\}$ is also a basis of W. Let

$$S_n = \{f_n(e_2), \dots, f_n(e_k), u_1, \dots, u_{r-k}\},$$

$F = \mathbb{R}\text{-span}\{f(e_1)\}$ and q_n the projection of W on F parallel to $\mathbb{R}\text{-span}(S_n)$. Then by Fact 2.3, $(q_n)_n$ converges to the projection q of W on F parallel to $\mathbb{R}\text{-span}(S)$, where $S = \{f(e_2), \dots, f(e_k), u_1, \dots, u_{r-k}\}$ and $(q_n(f_n(e_{k+j})))_n$ converges to $q(f(e_{k+j}))$. Note that $q_n(f_n(e_{k+j})) = q_n(\alpha_{1,j}^n f_n(e_1)) = \alpha_{1,j}^n q_n(f_n(e_1))$ and $q(f(e_{k+j})) = \alpha_{1,j} f(e_1)$. As $(\alpha_{1,j} q_n(f_n(e_1)))_n$ converges to $\alpha_{1,j} f(e_1)$, the sequence $((\alpha_{1,j}^n - \alpha_{1,j}) q_n(f_n(e_1)))_n$ converges to zero. But $(q_n(f_n(e_1)))_n$ converges to the non-zero vector $f(e_1)$, then $(\alpha_{1,j}^n)_n$ converges to $\alpha_{1,j}$. Using the same argument, we can show that $(\alpha_{l,j}^n)_n$ converges to $\alpha_{l,j}$ for all $1 \le l \le k$ and $1 \le j \le n-k$. □

FACT 2.6. *Let $(x_n)_n$ be a sequence in $\mathbb{R}^q$ such that*

(1) *Any subsequence of $(x_n)_n$ contains a convergent subsequence.*

(2) *Two convergent subsequences of $(x_n)_n$ converge to the same element.*

Then $(x_n)_n$ is convergent.

PROOF. Suppose that $(x_n)_n$ is not bounded, then for every integer k there exists $(x_{n_k})_{n_k}$ such that $\|x_{n_k}\| > k$. Then obviously $(x_n)_n$ contains a subsequence $(x_{n_k})_{n_k}$ such that $\lim \|x_{n_k}\| = +\infty$, then $(x_{n_k})_{n_k}$ does not have convergent subsequence. Thus by (1), $(x_n)_n$ is bounded. Let $(x_{n_k})_{n_k}$ be a convergent subsequence of $(x_n)_n$ which converges to y and let $A > 0$ such that $\|x_n - y\| < A$ for all n that is $(x_n)_n$ belongs to the closed ball $B(y, A)$ of center y and radius A. Let U be a neighborhood of y. If $B(y, A) \backslash U$ contains an infinite terms of $(x_n)_n$, then we can find a subsequence which converges to $y' \neq y$ and then by (2) only finite terms of $(x_n)_n$ are not in U, thus $(x_n)_n$ converges to y. □

FACT 2.7. *Let $(f_n : \mathbb{R}^q \to \mathbb{R}^m)_n$ be a sequence of linear maps and $(x_n)_n$ a sequence in $\mathbb{R}^q$ such that:*

(1) *$(f_n)_n$ converges to a linear map $f : \mathbb{R}^q \to \mathbb{R}^m$.*

(2) *For all n, f_n and f are injective.*

(3) *There exists $x \in \mathbb{R}^q$ such that $(f_n(x_n))_n$ converges to $f(x)$.*

Then $(x_n)_n$ converges to x.

PROOF. If for all $N > 0$, there exists $n > N$ such that $x_n = 0$ then $(x_n)_n$ contains a subsequence $(x_{n_k})_{n_k}$ such that $x_{n_k} = 0$ for all k and we have $f_{n_k}(x_{n_k}) = 0$ then $(f_n(x_n))_n$ converges to zero. Let $(x_{n_{k_1}})_{n_{k_1}}$ be a subsequence of $(x_n)_n$ such that $x_{n_{k_1}} \neq 0$ for all k_1 and we have to prove that $(x_{n_{k_1}})_{n_{k_1}}$ converges to zero. Note that $f_{n_{k_1}}(x_{n_{k_1}}) = \|x_{n_{k_1}}\| f_{n_{k_1}}\left(\frac{x_{n_{k_1}}}{\|x_{n_{k_1}}\|}\right)$. As the sequence

$\left(\frac{x_{n_{k_1}}}{\|x_{n_{k_1}}\|}\right)_{n_{k_1}}$ is bounded, we can assume that it converges to some $y \neq 0$. Now suppose that $(\|x_{n_{k_1}}\|)_{n_{k_1}}$ does not converge to zero, then up to the choice of a subsequence, we can assume that $(\|x_{n_{k_1}}\|)_{n_{k_1}}$ converges to some $a \in]0, +\infty]$ then $(f_{n_{k_1}}(x_{n_{k_1}}))_{n_{k_1}}$ converges to $af(y) \neq 0$ because $y \neq 0$ and f is injective which is a contradiction. Now assume that there exists $N > 0$ such that $x_n \neq 0$ for all $n > N$. In this case we have to show that $(x_n)_n$ satisfies the conditions (1) and (2) of Fact 2.6. Let $(x_{n_1})_{n_1}$ be a subsequence of $(x_n)_n$, then we can find a subsequence $(x_{n_2})_{n_2}$ of $(x_{n_1})_{n_1}$ such that $\left(\frac{x_{n_2}}{\|x_{n_2}\|}\right)_{n_2}$ converges to some $y \neq 0$ and using (1), we see that $\left(f_{n_2}\left(\frac{x_{n_2}}{\|x_{n_2}\|}\right)\right)_{n_2}$ converges to $f(y)$. As $(f_{n_2}(x_{n_2}))_{n_2} = \left(\|x_{n_2}\| f_{n_2}\left(\frac{x_{n_2}}{\|x_{n_2}\|}\right)\right)_{n_2}$ converges to $f(x)$, we deduce that $(\|x_{n_2}\|)_{n_2}$ converges to $\frac{\|f(x)\|}{\|f(y)\|}$. In particular, the sequence $(x_{n_2})_{n_2}$ is bounded and contains a convergent subsequence. This shows that any subsequence of $(x_n)_n$ contains a convergent subsequence. Let $(x_{n,1})_{n,1}$ and $(x_{n,2})_{n,2}$ be two convergent subsequences of $(x_n)_n$ such that $(x_{n,1})_{n,1}$ converges to y_1 and $(x_{n,2})_{n,2}$ converges to y_2. Then $(f_{n,1}(x_{n,1}))_{n,1}$ and $(f_{n,2}(x_{n,2}))_{n,2}$ converge to $f(y_1)$ and $f(y_2)$ respectively. By (3), $f(y_1) = f(y_2)$. Then $f(y_1 - y_2) = 0$ and using (2) we deduce that $y_1 = y_2$. Then two convergent subsequences of $(x_n)_n$ converge to the same element. Using Fact 2.6, we conclude that $(x_n)_n$ is convergent. □

3. On the quotient space $\mathrm{Hom}(\mathfrak{l}, \mathfrak{g})/G$

3.1. Describing $\mathrm{Hom}(\mathfrak{l}, \mathfrak{g})$**.** We assume henceforth that $\mathfrak{g}$ is a 3-step nilpotent Lie algebra and $\mathfrak{l}$ a subalgebra of $\mathfrak{g}$. We consider the decompositions

$$\mathfrak{g} = [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \qquad \text{and} \qquad \mathfrak{l} = [\mathfrak{l}, [\mathfrak{l}, \mathfrak{l}]] \oplus \mathfrak{l}_1 \oplus \mathfrak{l}_2 \tag{3}$$

where $\mathfrak{g}_1$ (respectively $\mathfrak{l}_1$) designates a subspace of $\mathfrak{g}$ (of $\mathfrak{l}$ respectively) such that $[\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] \oplus \mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}]$ (respectively $[\mathfrak{l}, [\mathfrak{l}, \mathfrak{l}]] \oplus \mathfrak{l}_1 = [\mathfrak{l}, \mathfrak{l}]$). $\mathfrak{g}_2$ (respectively $\mathfrak{l}_2$) is a subspace of $\mathfrak{g}$ (of $\mathfrak{l}$ respectively) supplementary to $[\mathfrak{g}, \mathfrak{g}]$ (to $[\mathfrak{l}, \mathfrak{l}]$ respectively) in $\mathfrak{g}$ (in $\mathfrak{l}$ respectively). Denote by $\mathfrak{g}_0 = [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]]$ and $\mathfrak{l}_0 = [\mathfrak{l}, [\mathfrak{l}, \mathfrak{l}]]$. Obviously we can see that $\mathfrak{g}_0$ (respectively $\mathfrak{l}_0$) lies in the center of $\mathfrak{g}$ (of $\mathfrak{l}$ respectively). Any $\varphi \in \mathscr{L}(\mathfrak{l}, \mathfrak{g})$ can be written as

$$\varphi = \begin{pmatrix} A_\varphi & B_\varphi & C_\varphi \\ I_\varphi & D_\varphi & E_\varphi \\ J_\varphi & K_\varphi & F_\varphi \end{pmatrix}, \tag{4}$$

where $A_\varphi \in \mathscr{L}(\mathfrak{l}_0, \mathfrak{g}_0)$, $B_\varphi \in \mathscr{L}(\mathfrak{l}_1, \mathfrak{g}_0)$, $C_\varphi \in \mathscr{L}(\mathfrak{l}_2, \mathfrak{g}_0)$, $I_\varphi \in \mathscr{L}(\mathfrak{l}_0, \mathfrak{g}_1)$, $D_\varphi \in \mathscr{L}(\mathfrak{l}_1, \mathfrak{g}_1)$, $E_\varphi \in \mathscr{L}(\mathfrak{l}_2, \mathfrak{g}_1)$, $J_\varphi \in \mathscr{L}(\mathfrak{l}_0, \mathfrak{g}_2)$, $K_\varphi \in \mathscr{L}(\mathfrak{l}_1, \mathfrak{g}_2)$ and $F_\varphi \in \mathscr{L}(\mathfrak{l}_2, \mathfrak{g}_2)$. For each $\varphi \in \mathscr{L}(\mathfrak{l}, \mathfrak{g})$, we define $\varphi_1 \in \mathscr{L}(\mathfrak{l}, \mathfrak{g})$ by

$$\varphi_1 = \begin{pmatrix} A_\varphi & B_\varphi & 0 \\ 0 & D_\varphi & E_\varphi \\ 0 & 0 & F_\varphi \end{pmatrix}. \tag{5}$$

We first remark the following assertion.

LEMMA 1. *An element* $\varphi \in \mathscr{L}(\mathfrak{l}, \mathfrak{g})$ *is a Lie algebra homomorphism if and only if* $I_\varphi = 0$, $J_\varphi = 0$, $K_\varphi = 0$ *and* $\varphi_1 \in Hom(\mathfrak{l}, \mathfrak{g})$.

PROOF. We point out first that if $\varphi \in \mathrm{Hom}(\mathfrak{l}, \mathfrak{g})$, then $\varphi(\mathfrak{l}_0) = \varphi([\mathfrak{l}, [\mathfrak{l}, \mathfrak{l}]]) = [\varphi(\mathfrak{l}), [\varphi(\mathfrak{l}), \varphi(\mathfrak{l})]] \subset \mathfrak{g}_0$, in particular $I_\varphi = 0$ and $J_\varphi = 0$. Now $\varphi(\mathfrak{l}_1) \subset \varphi([\mathfrak{l}, \mathfrak{l}]) = [\varphi(\mathfrak{l}), \varphi(\mathfrak{l})] \subset [\mathfrak{g}, \mathfrak{g}]$, so $K_\varphi = 0$ and

$$\varphi = \begin{pmatrix} A_\varphi & B_\varphi & C_\varphi \\ 0 & D_\varphi & E_\varphi \\ 0 & 0 & F_\varphi \end{pmatrix}. \tag{6}$$

Let us take $\varphi \in \mathscr{L}(\mathfrak{l}, \mathfrak{g})$ with $I_\varphi = 0$, $J_\varphi = 0$, $K_\varphi = 0$. Then for each $x = x_0 + x_1 + x_2$ and $x' = x_0' + x_1' + x_2' \in \mathfrak{l}$ where $x_i, x_i' \in \mathfrak{l}_i$, $i = 0, 1, 2$, we have the following:

$$\begin{aligned} [\varphi(x), \varphi(x')] &= [A_\varphi(x_0) + B_\varphi(x_1) + D_\varphi(x_1) + C_\varphi(x_2) + E_\varphi(x_2) + F_\varphi(x_2), A_\varphi(x_0') \\ &\quad + B_\varphi(x_1') + D_\varphi(x_1') + C_\varphi(x_2') + E_\varphi(x_2') + F_\varphi(x_2')] \\ &= [D_\varphi(x_1) + E_\varphi(x_2) + F_\varphi(x_2), D_\varphi(x_1') + E_\varphi(x_2') + F_\varphi(x_2')] \\ &= [\varphi_1(x), \varphi_1(x')]. \end{aligned} \tag{7}$$

On the other hand:

$$\begin{aligned} \varphi([x, x']) &= \varphi([x_0 + x_1 + x_2, x_0' + x_1' + x_2']) \\ &= \varphi([x_1 + x_2, x_1' + x_2']) \\ &= \varphi([x_1, x_2']) + \varphi([x_2, x_1']) + \varphi([x_2, x_2']) \\ &= A_\varphi([x_1, x_2']) + A_\varphi([x_2, x_1']) + (A_\varphi + B_\varphi + D_\varphi)([x_2, x_2']) \\ &= \varphi_1([x, x']). \end{aligned} \tag{8}$$

Conversely, let $I_\varphi = 0$, $J_\varphi = 0$, $K_\varphi = 0$ and $\varphi_1 \in \mathrm{Hom}(\mathfrak{l}, \mathfrak{g})$, then φ is as in (6). Hence,

$$\begin{aligned} \varphi([x, x']) &= \varphi_1([x, x']) \qquad \text{by (8)} \\ &= [\varphi_1(x), \varphi_1(x')] \\ &= [\varphi(x), \varphi(x')] \qquad \text{by (7)}. \end{aligned}$$

Then for each $\varphi \in \mathscr{L}(\mathfrak{l}, \mathfrak{g})$ with $I_\varphi = 0$, $J_\varphi = 0$, $K_\varphi = 0$, we have $\varphi \in \mathrm{Hom}(\mathfrak{l}, \mathfrak{g})$ if and only if $\varphi_1 \in \mathrm{Hom}(\mathfrak{l}, \mathfrak{g})$. $\square$

3.2. The G-action on $\mathrm{Hom}(\mathfrak{l}, \mathfrak{g})$**.** For any $X \in \mathfrak{g}$, the adjoint representation ad_X can be written making use the decomposition (3) as

$$\mathrm{ad}_X = \begin{pmatrix} 0 & \Sigma_{1,2}(X) & \Sigma_{1,3}(X) \\ 0 & 0 & \Sigma_{2,3}(X) \\ 0 & 0 & 0 \end{pmatrix} \tag{9}$$

for some maps $\Sigma_{1,2} : \mathfrak{g} \to \mathscr{L}(\mathfrak{g}_1, \mathfrak{g}_0)$, $\Sigma_{1,3} : \mathfrak{g} \to \mathscr{L}(\mathfrak{g}_2, \mathfrak{g}_0)$ and $\Sigma_{2,3} : \mathfrak{g} \to \mathscr{L}(\mathfrak{g}_2, \mathfrak{g}_1)$. The adjoint representation $\mathrm{Ad}_{\exp(X)}$ reads therefore

$$\begin{aligned} \mathrm{Ad}_{\exp(X)} &= \mathrm{Id} + \mathrm{ad}_X + \frac{1}{2}\,\mathrm{ad}_X^2 \\ &= \begin{pmatrix} I_{\mathfrak{g}_0} & \Sigma_{1,2}(X) & \Sigma_{1,3}(X) + \frac{1}{2}\Sigma_{1,2}(X)\Sigma_{2,3}(X) \\ 0 & I_{\mathfrak{g}_1} & \Sigma_{2,3}(X) \\ 0 & 0 & I_{\mathfrak{g}_2} \end{pmatrix}. \end{aligned}$$

Here $I_{\mathfrak{g}_0}$, $I_{\mathfrak{g}_1}$ and $I_{\mathfrak{g}_2}$ denote the identity maps of $\mathfrak{g}_0$, $\mathfrak{g}_1$ and $\mathfrak{g}_2$ respectively.

The group G acts on $\mathrm{Hom}(\mathfrak{l}, \mathfrak{g})$ through the following law:

$$\begin{aligned} g \cdot \varphi &= \mathrm{Ad}_g \circ \varphi \\ &= \begin{pmatrix} A_\varphi & B_\varphi + \Sigma_{1,2}(X)D_\varphi & C_\varphi + \Sigma_{1,2}(X)E_\varphi + \Sigma_{1,3}(X)F_\varphi \\ & & \quad + \frac{1}{2}\Sigma_{1,2}(X)\Sigma_{2,3}(X)F_\varphi \\ 0 & D_\varphi & E_\varphi + \Sigma_{2,3}(X)F_\varphi \\ 0 & 0 & F_\varphi \end{pmatrix}, \end{aligned}$$

where $g = \exp(X)$, A_φ, B_φ, C_φ, D_φ, E_φ and F_φ are as in formula (4) and $\Sigma_{1,2}$, $\Sigma_{1,3}$ and $\Sigma_{2,3}$ are as in (9). Let now

$$\mathrm{Hom}_1(\mathfrak{l}, \mathfrak{g}) := \{\varphi \in \mathrm{Hom}(\mathfrak{l}, \mathfrak{g}) \mid C_\varphi = 0\}. \tag{10}$$

By Lemma 1, the correspondence $\varphi \mapsto \varphi_1$ gives a map: $\mathrm{Hom}(\mathfrak{l}, \mathfrak{g}) \to \mathrm{Hom}_1(\mathfrak{l}, \mathfrak{g})$. Then G also acts on $\mathrm{Hom}_1(\mathfrak{l}, \mathfrak{g})$ as follows:

$$g * \varphi_1 = \begin{pmatrix} A_\varphi & B_\varphi + \Sigma_{1,2}(X)D_\varphi & 0 \\ 0 & D_\varphi & E_\varphi + \Sigma_{2,3}(X)F_\varphi \\ 0 & 0 & F_\varphi \end{pmatrix}. \tag{11}$$

In other words, $g * \varphi_1$ is defined by $(g \cdot \varphi_1)_1$ where $(g \cdot \varphi_1) \in \mathrm{Hom}(\mathfrak{l}, \mathfrak{g})$. One can easily check that $g * \varphi_1$ defines a group action of G on $\mathrm{Hom}_1(\mathfrak{l}, \mathfrak{g})$. Hence, G acts on $\mathrm{Hom}_1(\mathfrak{l}, \mathfrak{g}) \times \mathscr{L}(\mathfrak{l}_2, \mathfrak{g}_0)$ as:

$$g \cdot (\varphi_1, C_\varphi) = \left(g * \varphi_1, C_\varphi + \Sigma_{1,2}(X)E_\varphi + \Sigma_{1,3}(X)F_\varphi + \frac{1}{2}\Sigma_{1,2}(X)\Sigma_{2,3}(X)F_\varphi\right). \quad (12)$$

We first have the following:

Lemma 2. *The map*

$$\psi : Hom(\mathfrak{l}, \mathfrak{g}) \to Hom_1(\mathfrak{l}, \mathfrak{g}) \times \mathscr{L}(\mathfrak{l}_2, \mathfrak{g}_0)$$
$$\varphi \mapsto (\varphi_1, C_\varphi)$$

is a G-equivariant homeomorphism, where φ_1 *is as in* (5).

Proof. The fact that ψ is a well defined homeomorphism comes directly from Lemma 3.1. Let $g = \exp(X) \in G$ and $\varphi \in \mathrm{Hom}(\mathfrak{l}, \mathfrak{g})$, then

$$\begin{aligned}\psi(g \cdot \varphi) &= \psi(\mathrm{Ad}_g \circ \varphi) \\ &= \left(g * \varphi_1, C_\varphi + \Sigma_{1,2}(X)E_\varphi + \Sigma_{1,3}(X)F_\varphi + \frac{1}{2}\Sigma_{1,2}(X)\Sigma_{2,3}(X)F_\varphi\right) \\ &= g \cdot \psi(\varphi),\end{aligned}$$

which proves the lemma. □

3.2.1. *Decomposition of* $Hom_1(\mathfrak{l}, \mathfrak{g})$. Now we consider the linear subspace Δ of $\mathscr{L}(\mathfrak{l}, \mathfrak{g})$ defined by

$$\Delta = \left\{ \varphi \in \mathscr{L}(\mathfrak{l}, \mathfrak{g}) \;\middle|\; \begin{array}{l} A_\varphi = 0,\ I_\varphi = 0,\ J_\varphi = 0,\ D_\varphi = 0, \\ K_\varphi = 0,\ C_\varphi = 0 \text{ and } F_\varphi = 0 \end{array} \right\} \cong \mathscr{L}(\mathfrak{l}_1, \mathfrak{g}_0) \times \mathscr{L}(\mathfrak{l}_2, \mathfrak{g}_1).$$

For $\varphi_1 \in \mathrm{Hom}_1(\mathfrak{l}, \mathfrak{g})$, we consider the linear map

$$l_{\varphi_1} : \mathfrak{g} \to \Delta$$
$$X \mapsto \begin{pmatrix} 0 & \Sigma_{1,2}(X)D_{\varphi_1} & 0 \\ 0 & 0 & \Sigma_{2,3}(X)F_{\varphi_1} \\ 0 & 0 & 0 \end{pmatrix}.$$

Then from equation (11) of the definition of the action of G on $\mathrm{Hom}_1(\mathfrak{l}, \mathfrak{g})$, we obtain immediately the following description of the orbits in $\mathrm{Hom}_1(\mathfrak{l}, \mathfrak{g})$.

Lemma 3. *The orbit* $G * \varphi_1 = \varphi_0 + (N_{\varphi_1} + Im(l_{\varphi_1}))$, *where*

$$\varphi_0 = \begin{pmatrix} A_{\varphi_1} & 0 & 0 \\ 0 & D_{\varphi_1} & 0 \\ 0 & 0 & F_{\varphi_1} \end{pmatrix} \quad \textit{and} \quad N_{\varphi_1} = \begin{pmatrix} 0 & B_{\varphi_1} & 0 \\ 0 & 0 & E_{\varphi_1} \\ 0 & 0 & 0 \end{pmatrix}.$$

Let $m = \dim \varDelta$ and $q = \dim \mathfrak{g}$. For $t = 0, \dots, q$, we define the sets

$$\mathrm{Hom}_1^t(\mathfrak{l}, \mathfrak{g}) := \{\varphi_1 \in \mathrm{Hom}_1(\mathfrak{l}, \mathfrak{g}) \mid \mathrm{rk}(l_{\varphi_1}) = t\}.$$

Then clearly,

$$\mathrm{Hom}_1(\mathfrak{l}, \mathfrak{g}) = \bigcup_{t=0}^{q} \mathrm{Hom}_1^t(\mathfrak{l}, \mathfrak{g}). \tag{13}$$

We fix a basis $\{e_1, \dots, e_m\}$ of $\varDelta$ and let

$$I(m, m-t) = \{(i_1, \dots, i_{m-t}) \in \mathbb{N}^{m-t} \mid 1 \le i_1 < \cdots < i_{m-t} \le m\}.$$

For $\beta = (i_1, \dots, i_{m-t}) \in I(m, m-t)$, we consider the subspace $V_\beta := \bigoplus_{j=1}^{m-t} \mathbb{R} e_{i_j}$ and for any $\varphi_1 \in \mathrm{Hom}_1^t(\mathfrak{l}, \mathfrak{g})$, let $P_{\varphi_1} : \varDelta \to \varDelta/\mathrm{Im}(l_{\varphi_1})$ and

$$\mathrm{Hom}_{1,\beta}^t(\mathfrak{l}, \mathfrak{g}) := \{\varphi_1 \in \mathrm{Hom}_1^t(\mathfrak{l}, \mathfrak{g}) \mid \det(P_{\varphi_1}(e_{i_1}), \dots, P_{\varphi_1}(e_{i_{m-t}})) \neq 0\}.$$

Then we have the following:

Lemma 4. *For each $t = 0, \dots, q = \dim \mathfrak{g}$, the family $\{Hom_{1,\beta}^t(\mathfrak{l}, \mathfrak{g})\}_{\beta \in I(m, m-t)}$ of subsets in $Hom_1^t(\mathfrak{l}, \mathfrak{g})$ gives an open covering of $Hom_1^t(\mathfrak{l}, \mathfrak{g})$.*

Proof. We know that for all $\varphi_1 \in \mathrm{Hom}_1^t(\mathfrak{l}, \mathfrak{g})$, the set $\mathrm{Im}(l_{\varphi_1})$ is a linear subspace of $\varDelta$ of dimension t. There exists therefore $(i_1, \dots, i_{m-t}) \in I(m, m-t)$ such that the family $\{P_{\varphi_1}(e_{i_1}), \dots, P_{\varphi_1}(e_{i_{m-t}})\}$ forms a basis of $\varDelta/\mathrm{Im}(l_{\varphi_1})$ and consequently $\det(P_{\varphi_1}(e_{i_1}), \dots, P_{\varphi_1}(e_{i_{m-t}})) \neq 0$. This shows that $\mathrm{Hom}_1^t(\mathfrak{l}, \mathfrak{g}) = \bigcup_{\beta \in I(m, m-t)} \mathrm{Hom}_{1,\beta}^t(\mathfrak{l}, \mathfrak{g})$. Now $\det(P_{\varphi_1}(e_{i_1}), \dots, P_{\varphi_1}(e_{i_{m-t}})) \neq 0$ if and only if the family $\{P_{\varphi_1}(e_{i_1}), \dots, P_{\varphi_1}(e_{i_{m-t}})\}$ is a basis of $\varDelta/\mathrm{Im}\, l_{\varphi_1}$ which is equivalent to $\varDelta = \mathrm{Im}\, l_{\varphi_1} \oplus V_\beta$. As $\dim \mathrm{Im}\, l_{\varphi_1} = t$, we get by Fact 2.2 (1) that there exists $(j_1, \dots, j_t) \in I(q, t)$ such that the family $\{l_{\varphi_1}(Y_{j_1}), \dots, l_{\varphi_1}(Y_{j_t}), e_{i_1}, \dots, e_{i_{m-t}}\}$ forms a basis of $\varDelta$, or similarly

$$\sum_{(j_1, \dots, j_t) \in I(q,t)} [\det(l_{\varphi_1}(Y_{j_1}), \dots, l_{\varphi_1}(Y_{j_t}), e_{i_1}, \dots, e_{i_{m-t}})]^2 \neq 0.$$

Then

$$\mathrm{Hom}_{1,\beta}^t(\mathfrak{l}, \mathfrak{g}) = \left\{ \varphi_1 \in \mathrm{Hom}_1^t(\mathfrak{l}, \mathfrak{g}) \;\middle|\; \sum_{\substack{(j_1, \dots, j_t) \\ \in I(q,t)}} [\det(l_{\varphi_1}(Y_{j_1}), \dots, l_{\varphi_1}(Y_{j_t}), e_{i_1}, \dots, e_{i_{m-t}})]^2 \neq 0 \right\}$$

which is open by continuity of the determinant. □

Proposition 2. *We have:*

$$Hom_1(\mathfrak{l}, \mathfrak{g}) = \bigcup_{t=0}^{q} \bigcup_{\beta \in I(m, m-t)} Hom_{1,\beta}^t(\mathfrak{l}, \mathfrak{g})$$

as a union of G-invariant subsets, where G acts on $Hom_1(\mathfrak{l}, \mathfrak{g})$ *as in* (11).

Proof. The decomposition is given by Lemma 4 and equation (13). To see the G-invariance, observe that $D_{\varphi_1} = D_{g*\varphi_1}$ and $F_{\varphi_1} = F_{g*\varphi_1}$, which means that $l_{\varphi_1} = l_{g*\varphi_1}$ and $P_{\varphi_1} = P_{g*\varphi_1}$ for all $\varphi_1 \in \mathrm{Hom}_1(\mathfrak{l}, \mathfrak{g})$ and $g \in G$. Then for all $\beta \in I(m, m-t)$ and $0 \le t \le q$ the set $\mathrm{Hom}_{1,\beta}^t(\mathfrak{l}, \mathfrak{g})$ is G-invariant. □

Let us fix $t = 0, \dots, q$ and $\beta = (i_1, \dots, i_{m-t}) \in I(m, m-t)$. Recall that V_β is a subspace of Δ spanned by $\{e_{i_k}\}_{k=1,\dots,m-t}$. We define the subset $\mathscr{M}_\beta^t(\mathfrak{l}, \mathfrak{g})$ of $\mathrm{Hom}_{1,\beta}^t(\mathfrak{l}, \mathfrak{g})$ by

$$\mathscr{M}_\beta^t(\mathfrak{l}, \mathfrak{g}) = \{\varphi_1 \in \mathrm{Hom}_{1,\beta}^t(\mathfrak{l}, \mathfrak{g}) \mid N_{\varphi_1} \in V_\beta\}$$

and we consider the map

$$\pi_\beta^t : \mathrm{Hom}_{1,\beta}^t(\mathfrak{l}, \mathfrak{g})/G \to \mathscr{M}_\beta^t(\mathfrak{l}, \mathfrak{g})$$

$$G * \varphi_1 \mapsto \varphi_0 + P_{\varphi_1|V_\beta}^{-1}(N_{\varphi_1} + \mathrm{Im}(l_{\varphi_1})).$$

We next prove the following lemmas:

Lemma 5. *For each* $\varphi_1 \in Hom_{1,\beta}^t(\mathfrak{l}, \mathfrak{g})$, *the intersection of the G-orbit* $G * \varphi_1$ *and* $\mathscr{M}_\beta^t(\mathfrak{l}, \mathfrak{g})$ *in* $Hom_{1,\beta}^t(\mathfrak{l}, \mathfrak{g})$ *is the singleton* $\{\varphi_0 + P_{\varphi_1|V_\beta}^{-1}(N_{\varphi_1} + Im(l_{\varphi_1}))\}$. *In particular, the map*

$$\pi_\beta^t : Hom_{1,\beta}^t(\mathfrak{l}, \mathfrak{g})/G \to \mathscr{M}_\beta^t(\mathfrak{l}, \mathfrak{g}); \qquad G * \varphi_1 \mapsto \varphi_0 + P_{\varphi_1|V_\beta}^{-1}(N_{\varphi_1} + Im(l_{\varphi_1}))$$

is well defined.

Proof. Let $\varphi_1 = \varphi_0 + N_{\varphi_1} \in \mathrm{Hom}_{1,\beta}^t(\mathfrak{l}, \mathfrak{g})$. Then from Lemma 3 the orbit $G * \varphi_1 = \varphi_0 + (N_{\varphi_1} + \mathrm{Im}\, l_{\varphi_1})$ and

$$\varphi_0 + P_{\varphi_1|V_\beta}^{-1}(N_{\varphi_1} + \mathrm{Im}\, l_{\varphi_1}) = \varphi_0 + (N_{\varphi_1} + \mathrm{Im}\, l_{\varphi_1}) \cap V_\beta \in \mathscr{M}_\beta^t(\mathfrak{l}, \mathfrak{g}).$$

Thus π_β^t is well defined. Note now that the intersection of $G * \varphi_1$ and $\mathscr{M}_\beta^t(\mathfrak{l}, \mathfrak{g})$ in $\mathrm{Hom}_{1,\beta}^t(\mathfrak{l}, \mathfrak{g})$ is not empty as it contains $\pi_\beta^t(G * \varphi_1)$. Let φ_1, φ_1' be two elements in the intersection. Then there exist $v, w \in \mathrm{Im}(l_{\varphi_1})$ such that $\varphi_1 = \varphi_0 + N_{\varphi_1} + v$ and $\varphi_1' = \varphi_0 + N_{\varphi_1} + w$ with $N_{\varphi_1} + v, N_{\varphi_1} + w \in V_\beta$. In particular $v - w \in V_\beta \cap \mathrm{Im}(l_{\varphi_1}) = \{0\}$, which means that $\varphi_1 = \varphi_1'$. □

Lemma 6. *The map*

$$h : Hom^t_{1,\beta}(\mathfrak{l}, \mathfrak{g}) \to \mathscr{M}^t_\beta(\mathfrak{l}, \mathfrak{g})$$
$$\varphi_1 \mapsto \varphi_0 + P^{-1}_{\varphi_1|V_\beta}(N_{\varphi_1} + Im\ l_{\varphi_1})$$

is continuous.

Proof. To show this lemma, we prove the following fact:

Fact 3.1. *The map*

$$Hom_1(\mathfrak{l}, \mathfrak{g}) \to \mathscr{L}(\mathfrak{g}, \Delta)$$
$$\varphi_1 \mapsto l_{\varphi_1}$$

is continuous.

Proof. Let $(\varphi_1^{(n)})_n$ be a sequence which converges to some element φ_1. Then obviously $(D_{\varphi_1^{(n)}})_n$ converges to D_{φ_1} and $(F_{\varphi_1^{(n)}})_n$ converges to F_{φ_1}. Then $(l_{\varphi_1^{(n)}})_n$ converges to l_{φ_1}. □

Now to prove the lemma, let $(\varphi_{1,n})_n$ be a sequence in $\mathrm{Hom}^t_{1,\beta}(\mathfrak{l}, \mathfrak{g})$ which converges to an element $\varphi_1 \in \mathrm{Hom}^t_{1,\beta}(\mathfrak{l}, \mathfrak{g})$. We have to show that $(h(\varphi_{1,n}))_n$ converges to $h(\varphi_1)$. Note first that for all $\varphi_1 \in \mathrm{Hom}^t_{1,\beta}(\mathfrak{l}, \mathfrak{g})$, $h(\varphi_1) = \varphi_0 + (N_{\varphi_1} + \mathrm{Im}\ l_{\varphi_1}) \cap V_\beta$. As $\Delta = \mathrm{Im}\ l_{\varphi_1} \oplus V_\beta$, by Fact 2.1 we see that $h(\varphi_1) = \varphi_0 + q_{\varphi_1}(N_{\varphi_1})$, where q_{φ_1} is the projection on V_β parallel to $\mathrm{Im}\ l_{\varphi_1}$. Let $\{X_1, \dots, X_n\}$ be a basis of $\mathfrak{g}$, using Fact 2.2 (1), one can find $X_{j_1}, \dots, X_{j_t} \in \{X_1, \dots, X_n\}$ such that

$$\mathrm{Im}\ l_{\varphi_1} = \mathbb{R}\text{-span}\{l_{\varphi_1}(X_{j_1}), \dots, l_{\varphi_1}(X_{j_t})\}.$$

Now by Fact 3.1, we get the convergence of the sequence $(l_{\varphi_{1,n}})_n$ to l_{φ_1}. By Fact 2.2 (2), for $S = \{X_{j_1}, \dots, X_{j_t}\}$,

$$A(S) = \{l \in \mathscr{L}(\mathfrak{g}, \Delta) \mid \dim \mathbb{R}\text{-span}\{l(X_{j_1}), \dots, l(X_{j_t})\} = t\}$$

is open in $\mathscr{L}(\mathfrak{g}, \Delta)$. Then there exists $N > 0$ such that for all $n > N$,

$$\mathrm{Im}\ l_{\varphi_1} = \mathbb{R}\text{-span}\{l_{\varphi_{1,n}}(X_{j_1}), \dots, l_{\varphi_{1,n}}(X_{j_t})\}.$$

As $(l_{\varphi_{1,n}})_n$ converges to l_{φ_1}, the sequence $(l_{\varphi_{1,n}}(X_{j_k}))_n$ converges to $l_{\varphi_1}(X_{j_k})$ for all $1 \le k \le t$. By Fact 2.3, let q_n be the projection of Δ on V_β parallel to $\mathrm{Im}(l_{\varphi_{1,n}})$, the sequence $(q_n)_n$ converges to the projection q_{φ_1} of Δ on V_β parallel to $\mathrm{Im}\ l_{\varphi_1}$. Finally, as $(\varphi_{0,n})_n$ converges to φ_0 and $(N_{\varphi_{1,n}})_n$ converges to N_{φ_1}, we get $(h(\varphi_{1,n}))_n$ converges to $h(\varphi_1)$. □

Lemma 7. *The map* $\pi_\beta^t : Hom_{1,\beta}^t(\mathfrak{l},\mathfrak{g})/G \to \mathcal{M}_\beta^t(\mathfrak{l},\mathfrak{g})$ *defined above is a homeomorphism.*

Proof. To see that π_β^t is surjective, observe that $\mathcal{M}_\beta^t(\mathfrak{l},\mathfrak{g}) \subset \mathrm{Hom}_{1,\beta}^t(\mathfrak{l},\mathfrak{g})$ and for $\varphi_1 \in \mathcal{M}_\beta^t(\mathfrak{l},\mathfrak{g})$ we have $\pi_\beta^t(G * \varphi_1) = \varphi_1$. Let $\varphi_1, \xi_1 \in \mathrm{Hom}_{1,\beta}^t(\mathfrak{l},\mathfrak{g})$ such that $\pi_\beta^t(G * \varphi_1) = \pi_\beta^t(G * \xi_1)$. Then obviously $\varphi_0 = \xi_0$ and

$$P_{\varphi_1|V_\beta}^{-1}(N_{\varphi_1} + \mathrm{Im}\ l_{\varphi_1}) = P_{\xi_1|V_\beta}^{-1}(N_{\xi_1} + \mathrm{Im}\ l_{\xi_1}).$$

As l_{φ_1} depends only on φ_0, we deduce that $l_{\varphi_1} = l_{\xi_1}$, which implies that $P_{\varphi_1|V_\beta}^{-1} = P_{\xi_1|V_\beta}^{-1}$. Hence, $N_{\varphi_1} + \mathrm{Im}\ l_{\varphi_1} = N_{\xi_1} + \mathrm{Im}\ l_{\xi_1}$ and in particular $G * \varphi_1 = G * \xi_1$. Thus π_β^t is injective. Now the following diagram commutes

$$\begin{array}{ccc} \mathrm{Hom}_{1,\beta}^t(\mathfrak{l},\mathfrak{g}) & & \\ \downarrow \pi & \searrow h & \\ \mathrm{Hom}_{1,\beta}^t(\mathfrak{l},\mathfrak{g})/G & \xrightarrow{\pi_\beta^t} & \mathcal{M}_\beta^t(\mathfrak{l},\mathfrak{g}) \end{array}$$

where $h(\varphi_1) = \varphi_0 + P_{\varphi_1|V_\beta}^{-1}(N_{\varphi_1} + \mathrm{Im}\ l_{\varphi_1})$. Since by Lemma 6, h is continuous and since π is open then π_β^t is continuous. The quotient canonical map $(\pi_\beta^t)^{-1} = \pi_{|\mathcal{M}_\beta^t(\mathfrak{l},\mathfrak{g})}$ is continuous and then the map π_β^t is a homeomorphism. □

Corollary 1. *For all* $t = \{0, \dots, q\}$, *the collection*

$$\mathrm{S}_\beta^t = (\pi_\beta^t, Hom_{1,\beta}^t(\mathfrak{l},\mathfrak{g})/G)_{\beta \in I(m,m-t)}$$

forms a family of local sections of the canonical surjection

$$\pi^t : Hom_1^t(\mathfrak{l},\mathfrak{g}) \to Hom_1^t(\mathfrak{l},\mathfrak{g})/G.$$

In particular,

$$\pi_\beta^t(Hom_{1,\beta}^t(\mathfrak{l},\mathfrak{g})/G) = \left\{ \varphi_1 \in Hom_1(\mathfrak{l},\mathfrak{g}) \;\middle|\; \begin{array}{l} \varphi_1 \in Hom_{1,\beta}^t(\mathfrak{l},\mathfrak{g}) \\ N_{\varphi_1} \in V_\beta \end{array} \right\}.$$

3.2.2. *Decomposition of* $Hom(\mathfrak{l},\mathfrak{g})$. Let $\varphi_1 \in \mathrm{Hom}_{1,\beta}^t(\mathfrak{l},\mathfrak{g})$ and

$$G_{\varphi_1} = \{g \in G \mid g * \varphi_1 = \varphi_1\}$$

be the isotropy group of φ_1. The group G_{φ_1} acts on $\{\varphi_1\} \times \mathcal{L}(\mathfrak{l}_2, \mathfrak{g}_0)$ through the following law

$$\exp(X) \cdot (\varphi_1, C) = (\varphi_1, C + \Sigma_{1,2}(X)E_{\varphi_1} + \Sigma_{1,3}(X)F_{\varphi_1}),$$

where $\Sigma_{1,2}$ and $\Sigma_{1,3}$ are as in (9). Indeed, for any $g = \exp(X) \in G_{\varphi_1}$, we have $g * \varphi_1 = \varphi_1$, then $\Sigma_{1,2}(X)D_{\varphi_1} = 0$ and $\Sigma_{2,3}(X)F_{\varphi_1} = 0$. We get therefore from (12):

$$g \cdot (\varphi_1, C) = (\varphi_1, C + \Sigma_{1,2}(X)E_{\varphi_1} + \Sigma_{1,3}(X)F_{\varphi_1}).$$

Let $\mathfrak{g}_{\pi^t_\beta(G*\varphi_1)} = \log(G_{\pi^t_\beta(G*\varphi_1)})$ and f_{φ_1} be the linear map defined by

$$f_{\varphi_1} : \mathfrak{g}_{\pi^t_\beta(G*\varphi_1)} \to \mathscr{L}(\mathfrak{l}_2, \mathfrak{g}_0)$$

$$X \mapsto \Sigma_{1,2}(X)E_{\pi^t_\beta(G*\varphi_1)} + \Sigma_{1,3}(X)F_{\pi^t_\beta(G*\varphi_1)}.$$

Then the range of f_{φ_1} is a linear subspace of $\mathscr{L}(\mathfrak{l}_2, \mathfrak{g}_0)$ and we can see immediately that:

Lemma 8. *For any $g \in G$ we have $f_{\varphi_1} = f_{g*\varphi_1}$. In addition,*

$$G_{\pi^t_\beta(G*\varphi_1)} \cdot (\pi^t_\beta(G * \varphi_1), C) = (\pi^t_\beta(G * \varphi_1), C + Im(f_{\varphi_1})).$$

Let $m' = \dim(\mathscr{L}(\mathfrak{l}_2, \mathfrak{g}_0))$ and $q' = \dim(\mathfrak{g}_{\pi^t_\beta(G*\varphi_1)})$. For $t' = 0, \dots, q'$, we define the sets

$$\mathrm{Hom}^{t,t'}_{1,\beta}(\mathfrak{l}, \mathfrak{g}) = \{\varphi_1 \in \mathrm{Hom}^t_{1,\beta}(\mathfrak{l}, \mathfrak{g}) \mid \mathrm{rk}(f_{\varphi_1}) = t'\}.$$

Then clearly

$$\mathrm{Hom}^t_{1,\beta}(\mathfrak{l}, \mathfrak{g}) = \bigcup_{t'=0}^{q'} \mathrm{Hom}^{t,t'}_{1,\beta}(\mathfrak{l}, \mathfrak{g}). \tag{14}$$

Let us fix a basis $\{e'_1, \dots, e'_{m'}\}$ of $\mathscr{L}(\mathfrak{l}_2, \mathfrak{g}_0)$. For $\beta' = (i'_1, \dots, i'_{m'-t'}) \in I(m', m'-t')$ and $\varphi_1 \in \mathrm{Hom}^{t,t'}_{1,\beta}(\mathfrak{l}, \mathfrak{g})$, we consider the subspace $V_{\beta'} := \bigoplus_{j=1}^{m'-t'} \mathbb{R}e'_{i'_j}$, the quotient map

$$P'_{\varphi_1} : \mathscr{L}(\mathfrak{l}_2, \mathfrak{g}_0) \to \mathscr{L}(\mathfrak{l}_2, \mathfrak{g}_0)/\mathrm{Im}(f_{\varphi_1})$$

and the set

$$\mathrm{Hom}^{t,t'}_{1,\beta,\beta'}(\mathfrak{l}, \mathfrak{g}) = \{\varphi_1 \in \mathrm{Hom}^{t,t'}_{1,\beta}(\mathfrak{l}, \mathfrak{g}) \mid \det(P'_{\varphi_1}(e'_{i'_1}), \dots, P'_{\varphi_1}(e'_{i'_{m'-t'}})) \neq 0\}.$$

Then we get the following:

Lemma 9. *For each $t = 0, \dots, q = \dim \mathfrak{g}$, $t' = 0, \dots, q' = \dim(\mathfrak{g}_{\pi^t_\beta(G*\varphi_1)})$ and $\beta \in I(m, m-t)$, the family $\{Hom^{t,t'}_{1,\beta,\beta'}(\mathfrak{l}, \mathfrak{g})\}_{\beta' \in I(m', m'-t')}$ of subsets in $Hom^{t,t'}_{1,\beta}(\mathfrak{l}, \mathfrak{g})$ gives an open covering of $Hom^{t,t'}_{1,\beta}(\mathfrak{l}, \mathfrak{g})$.*

Proof. For all $\varphi_1 \in \mathrm{Hom}^{t,t'}_{1,\beta}(\mathfrak{l}, \mathfrak{g})$, the set $\mathrm{Im}(f_{\varphi_1})$ is a linear subspace of $\mathscr{L}(\mathfrak{l}_2, \mathfrak{g}_0)$ of dimension t'. There exists therefore $(i'_1, \dots, i'_{m'-t'}) \in I(m', m'-t')$ such that the family $\{P'_{\varphi_1}(e'_{i'_1}), \dots, P'_{\varphi_1}(e'_{i'_{m'-t'}})\}$ forms a basis of $\mathscr{L}(\mathfrak{l}_2, \mathfrak{g}_0)/\mathrm{Im}(f_{\varphi_1})$ and consequently

$$\det(P'_{\varphi_1}(e'_{i'_1}), \dots, P'_{\varphi_1}(e'_{i'_{m'-t'}})) \neq 0.$$

This shows that $\mathrm{Hom}_{1,\beta}^{t,t'}(\mathfrak{l},\mathfrak{g}) = \bigcup_{\beta' \in I(m',m'-t')} \mathrm{Hom}_{1,\beta,\beta'}^{t,t'}(\mathfrak{l},\mathfrak{g})$. Now to prove that $\mathrm{Hom}_{1,\beta,\beta'}^{t,t'}(\mathfrak{l},\mathfrak{g})$ is open in $\mathrm{Hom}_{1,\beta}^{t,t'}(\mathfrak{l},\mathfrak{g})$, we need the following facts:

FACT 3.2. *Let $\varphi_1 \in Hom_{1,\beta}^{t}(\mathfrak{l},\mathfrak{g})$ and assume that $\pi_\beta^t(G * \varphi_1) = \psi_1$. Then $\mathfrak{g}_{\psi_1} = \ker l_{\psi_1}$.*

PROOF.

$$\begin{aligned}
\mathfrak{g}_{\psi_1} &= \{X \in \mathfrak{g} \mid \exp(X) * \psi_1 = \psi_1\} \\
&= \{X \in \mathfrak{g} \mid \psi_1 + l_{\psi_1}(X) = \psi_1\} \\
&= \{X \in \mathfrak{g} \mid l_{\psi_1}(X) = 0\} \\
&= \ker l_{\psi_1}.
\end{aligned}$$

□

Let $\mathscr{B} = \{X_1, \ldots, X_q\}$ be a basis of $\mathfrak{g}$, $\varphi_1 \in \mathrm{Hom}_{1,\beta}^{t}(\mathfrak{l},\mathfrak{g})$ and $\psi_1 = \pi_\beta^t(G * \varphi_1)$. For $\gamma = (j_1, \ldots, j_t) \in I(q,t)$ such that $\mathrm{Im}\, l_{\psi_1} = \mathbb{R}\text{-span}\{l_{\psi_1}(X_{j_1}), \ldots, l_{\psi_1}(X_{j_t})\}$, we define a linear map $l_{\varphi_1,\gamma} : \mathbb{R}^{q-t} \to \mathfrak{g}$ given by

$$l_{\varphi_1,\gamma}(u_i) = X_{s_i} - \sum_{r=1}^{t} \alpha_{r,s_i} X_{j_r},$$

where $\{u_1, \ldots, u_{q-t}\}$ is the canonical basis of $\mathbb{R}^{q-t}$, $\{s_1 < \cdots < s_{q-t}\} = \{1, \ldots, q\} \setminus \{j_1, \ldots, j_t\}$ and

$$l_{\psi_1}(X_{s_i}) = \sum_{r=1}^{t} \alpha_{r,s_i} l_{\psi_1}(X_{j_r}) \qquad \forall 1 \le i \le q - r.$$

FACT 3.3. *We have: $Im\, l_{\varphi_1,\gamma} = \mathfrak{g}_{\psi_1}$.*

PROOF. By Fact 2.4 and Fact 3.2, we have $\mathrm{Im}(l_{\varphi_1,\gamma}) = \ker l_{\psi_1} = \mathfrak{g}_{\psi_1}$. □

Consider now the map $\bar{l}_{\varphi_1,\gamma} : \mathbb{R}^{q-t} \to \mathscr{L}(\mathfrak{l}_2, \mathfrak{g}_0)$ defined by $\bar{l}_{\varphi_1,\gamma} = f_{\varphi_1} \circ l_{\varphi_1,\gamma}$. Then we have the following result

FACT 3.4. *We have:*
(1) *$\bar{l}_{\varphi_1,\gamma}$ is a linear map.*
(2) *$Im(\bar{l}_{\varphi_1,\gamma}) = Im\, f_{\varphi_1}$.*

PROOF. As $l_{\varphi_1,\gamma}$ and f_{φ_1} are linear maps, the map $\bar{l}_{\varphi_1,\gamma}$ is also linear. Now by Fact 3.3,

$$\mathrm{Im}(\bar{l}_{\varphi_1,\gamma}) = f_{\varphi_1}(\mathrm{Im}\, l_{\varphi_1,\gamma}) = f_{\varphi_1}(\mathfrak{g}_{\psi_1}) = \mathrm{Im}(f_{\varphi_1}).$$

□

Now for $\gamma = (j_1, \dots, j_t) \in I(q,t)$, let

$$\mathrm{Hom}^{t,t'}_{1,\beta,\beta'}(\gamma) = \{\varphi_1 \in \mathrm{Hom}^{t,t'}_{1,\beta,\beta'}(\mathfrak{l},\mathfrak{g}) \mid \mathrm{rk}(l_{\varphi_1}(X_{j_1}), \dots, l_{\varphi_1}(X_{j_t})) = t\}.$$

Then obviously,

$$\mathrm{Hom}^{t,t'}_{1,\beta,\beta'}(\mathfrak{l},\mathfrak{g}) = \bigcup_{\gamma \in I(q,t)} \mathrm{Hom}^{t,t'}_{1,\beta,\beta'}(\gamma).$$

To conclude that $\mathrm{Hom}^{t,t'}_{1,\beta,\beta'}(\mathfrak{l},\mathfrak{g})$ is open in $\mathrm{Hom}^{t,t'}_{1,\beta}(\mathfrak{l},\mathfrak{g})$, we have to show that for all $\gamma \in I(q,t)$, the set $\mathrm{Hom}^{t,t'}_{1,\beta,\beta'}(\gamma)$ is open in $\mathrm{Hom}^{t,t'}_{1,\beta}(\mathfrak{l},\mathfrak{g})$. First note that

$$\mathrm{Hom}^{t,t'}_{1,\beta,\beta'}(\gamma) = \left\{ \varphi_1 \in \mathrm{Hom}^{t,t'}_{1,\beta}(\mathfrak{l},\mathfrak{g}) \;\middle|\; \begin{array}{l} \mathrm{rk}(l_{\varphi_1}(X_{j_1}), \dots, l_{\varphi_1}(X_{j_t})) = t \\ \det(P'_{\varphi_1}(e'_{i'_1}), \dots, P'_{\varphi_1}(e'_{i'_{m'-t'}})) \neq 0 \end{array} \right\}.$$

The set

$$A^{t,t'}_{\beta}(\gamma) = \{\varphi_1 \in \mathrm{Hom}^{t,t'}_{1,\beta}(\mathfrak{l},\mathfrak{g}) \mid \mathrm{rk}(l_{\varphi_1}(X_{j_1}), \dots, l_{\varphi_1}(X_{j_t})) = t\}$$

is open in $\mathrm{Hom}^{t,t'}_{1,\beta}(\mathfrak{l},\mathfrak{g})$ and

$$\mathrm{Hom}^{t,t'}_{1,\beta,\beta'}(\gamma) = \{\varphi_1 \in A^{t,t'}_{\beta}(\gamma) \mid \det(P'_{\varphi_1}(e'_{i'_1}), \dots, P'_{\varphi_1}(e'_{i'_{m'-t'}})) \neq 0\}.$$

Then to obtain our result, it is sufficient to prove that $\mathrm{Hom}^{t,t'}_{1,\beta,\beta'}(\gamma)$ is open in $A^{t,t'}_{\beta}(\gamma)$. Indeed, the condition $\det(P'_{\varphi_1}(e'_{i'_1}), \dots, P'_{\varphi_1}(e'_{i'_{m'-t'}})) \neq 0$ is equivalent to $\mathscr{L}(\mathfrak{l}_2, \mathfrak{g}_0) = \mathrm{Im}\, f_{\varphi_1} \oplus V_{\beta'}$. By Fact 3.4, we get:

$$\begin{aligned}
&\det(P'_{\varphi_1}(e'_{i'_1}), \dots, P'_{\varphi_1}(e'_{i'_{m'-t'}})) \neq 0 \\
&\quad \Leftrightarrow \mathrm{Im}(\bar{l}_{\varphi_1,\gamma}) \oplus V_{\beta'} = \mathscr{L}(\mathfrak{l}_2, \mathfrak{g}_0) \\
&\quad \Leftrightarrow \exists \theta \in I(q-t, t'),\ \theta = (s_1, \dots, s_{t'})/ \\
&\qquad \det(\bar{l}_{\varphi_1,\gamma}(u_{s_1}), \dots, \bar{l}_{\varphi_1,\gamma}(u_{s_{t'}}), e'_{i'_1}, \dots, e'_{i'_{m'-t'}}) \neq 0 \\
&\quad \Leftrightarrow \sum_{\theta \in I(q-t,t')} [\det(\bar{l}_{\varphi_1,\gamma}(u_{s_1}), \dots, \bar{l}_{\varphi_1,\gamma}(u_{s_{t'}}), e'_{i'_1}, \dots, e'_{i'_{m'-t'}})]^2 \neq 0.
\end{aligned}$$

Then

$$\mathrm{Hom}^{t,t'}_{1,\beta,\beta'}(\gamma) = \left\{ \varphi_1 \in A^{t,t'}_{\beta}(\gamma) \;\middle|\; \sum_{\theta \in I(q-t,t')} [\det(\bar{l}_{\varphi_1,\gamma}(u_{s_1}), \dots, \bar{l}_{\varphi_1,\gamma}(u_{s_{t'}}), e'_{i'_1}, \dots, e'_{i'_{m'-t'}})]^2 \neq 0 \right\}$$

which is clearly an open subset of $A^{t,t'}_{\beta}(\gamma)$. □

As a consequence, we get the following:

Proposition 3. *We have the following decomposition:*

$$Hom_1(\mathfrak{l},\mathfrak{g}) = \bigcup_{t=0}^{q}\bigcup_{t'=0}^{q'}\bigcup_{\beta\in I(m,m-t)}\bigcup_{\beta'\in I(m',m'-t')} Hom_{1,\beta,\beta'}^{t,t'}(\mathfrak{l},\mathfrak{g})$$

as a union of G-invariant subsets, where G acts on $Hom_1(\mathfrak{l},\mathfrak{g})$ *as in* (11).

Proof. The decomposition is given by Proposition 2, Lemma 9 and equation (14). We showed already that for $\beta\in I(m,m-t)$ and $0\le t\le q$, the set $\mathrm{Hom}_{1,\beta}^{t}(\mathfrak{l},\mathfrak{g})$ is G-invariant. Likewise, by Lemma 8 we have $f_{\varphi_1}=f_{g*\varphi_1}$ and then $P'_{\varphi_1}=P'_{g*\varphi_1}$ for all $g\in G$ and $\varphi_1\in\mathrm{Hom}_1(\mathfrak{l},\mathfrak{g})$. As a consequence, for all $\beta\in I(m,m-t)$, $\beta'\in I(m',m'-t')$, $0\le t\le q$ and $0\le t'\le q'$ the set $\mathrm{Hom}_{1,\beta,\beta'}^{t,t'}(\mathfrak{l},\mathfrak{g})$ is G-invariant. □

Using the map ψ defined as in Lemma 2, we will identify in the rest of this section the set $\mathrm{Hom}(\mathfrak{l},\mathfrak{g})$ to $\mathrm{Hom}_1(\mathfrak{l},\mathfrak{g})\times\mathscr{L}(\mathfrak{l}_2,\mathfrak{g}_0)$. Let first for $\beta\in I(m,m-t)$, $\beta'\in I(m',m'-t')$, $0\le t\le q$ and $0\le t'\le q'$,

$$\mathrm{Hom}_{\beta,\beta'}^{t,t'}(\mathfrak{l},\mathfrak{g}) = \mathrm{Hom}_{1,\beta,\beta'}^{t,t'}(\mathfrak{l},\mathfrak{g})\times\mathscr{L}(\mathfrak{l}_2,\mathfrak{g}_0). \tag{15}$$

Then we have the following:

Proposition 4. *We have:*

$$Hom(\mathfrak{l},\mathfrak{g}) = \bigcup_{t=0}^{q}\bigcup_{t'=0}^{q'}\bigcup_{\beta\in I(m,m-t)}\bigcup_{\beta'\in I(m',m'-t')} Hom_{\beta,\beta'}^{t,t'}(\mathfrak{l},\mathfrak{g})$$

as a union of G-invariant subsets.

Proof. As in Proposition 3, for all $\beta\in I(m,m-t)$, $\beta'\in I(m',m'-t')$, $0\le t\le q$ and $0\le t'\le q'$, the set $\mathrm{Hom}_{1,\beta,\beta'}^{t,t'}(\mathfrak{l},\mathfrak{g})$ is G-invariant where G acts on $\mathrm{Hom}_1(\mathfrak{l},\mathfrak{g})$ as in (11) and then the set $\mathrm{Hom}_{\beta,\beta'}^{t,t'}(\mathfrak{l},\mathfrak{g})$ is also G-invariant. □

Let now

$$\begin{aligned}\mathscr{M}_{\beta}^{t,t'}(\mathfrak{l},\mathfrak{g}) &= \{\varphi_1\in\mathscr{M}_{\beta}^{t}(\mathfrak{l},\mathfrak{g})\mid \mathrm{rk}(f_{\varphi_1})=t'\}\\ &= \{\varphi_1\in\mathrm{Hom}_{1,\beta}^{t}(\mathfrak{l},\mathfrak{g})\mid N_{\varphi_1}\in V_\beta;\ \mathrm{rk}(f_{\varphi_1})=t'\}\\ &= \{\varphi_1\in\mathrm{Hom}_{1,\beta}^{t,t'}(\mathfrak{l},\mathfrak{g})\mid N_{\varphi_1}\in V_\beta\}.\end{aligned} \tag{16}$$

and

$$\begin{aligned}\mathscr{M}^{t,t'}_{\beta,\beta'}(\mathfrak{l},\mathfrak{g}) &= \{\varphi_1 \in \mathscr{M}^{t,t'}_{\beta}(\mathfrak{l},\mathfrak{g}) \mid \det(P'_{\varphi_1}(e'_{i'_1}),\dots,P'_{\varphi_1}(e'_{i'_{m'-t'}})) \neq 0\}\\ &= \{\varphi_1 \in \mathrm{Hom}^{t,t'}_{1,\beta}(\mathfrak{l},\mathfrak{g}) \mid N_{\varphi_1} \in V_\beta,\\ &\qquad \det(P'_{\varphi_1}(e'_{i'_1}),\dots,P'_{\varphi_1}(e'_{i'_{m'-t'}})) \neq 0\} \qquad \text{(by 16)}\\ &= \{\varphi_1 \in \mathrm{Hom}^{t,t'}_{1,\beta,\beta'}(\mathfrak{l},\mathfrak{g}) \mid N_{\varphi_1} \in V_\beta\}\\ &= \mathrm{Hom}^{t,t'}_{1,\beta,\beta'}(\mathfrak{l},\mathfrak{g}) \cap \mathscr{M}^{t}_{\beta}(\mathfrak{l},\mathfrak{g}). \end{aligned} \tag{17}$$

We show next the following lemmas:

Lemma 10. *Each G-orbit in* $Hom^{t,t'}_{\beta,\beta'}(\mathfrak{l},\mathfrak{g})$ *intersects* $\mathscr{M}^{t,t'}_{\beta,\beta'}(\mathfrak{l},\mathfrak{g}) \times V_{\beta'}$ *at exactly one element in* $Hom^{t,t'}_{\beta,\beta'}(\mathfrak{l},\mathfrak{g})$, *if we take an element* $g_0 \in G$ *such that* $g_0 * \varphi_1 \in \mathscr{M}^t_\beta(\mathfrak{l},\mathfrak{g})$, *then the element in the intersection of* $(G\cdot(\varphi_1, C))$ *and* $\mathscr{M}^{t,t'}_{\beta,\beta'}(\mathfrak{l},\mathfrak{g}) \times V_{\beta'}$ *can be written as*

$$(\pi^t_\beta(G*\varphi_1), (P'_{\varphi_1|V_{\beta'}})^{-1}(C(g_0) + Im\, f_{\varphi_1}))$$

where

$$C(g_0) = C + \Sigma_{1,2}(X_0)E_{\varphi_1} + \Sigma_{1,3}(X_0)F_{\varphi_1} + \frac{1}{2}\Sigma_{1,2}(X_0).\Sigma_{2,3}(X_0)F_{\varphi_1}.$$

Here $\Sigma_{1,2}$, $\Sigma_{1,3}$ *and* $\Sigma_{2,3}$ *are as in formula* (9) *in Section* 3.2.

Proof. From Lemma 5, the intersection of the G-orbit $G*\varphi_1$ and $\mathscr{M}^t_\beta(\mathfrak{l},\mathfrak{g})$ in $\mathrm{Hom}^t_{1,\beta}(\mathfrak{l},\mathfrak{g})$ is the singleton $\{\varphi_0 + P^{-1}_{\varphi_1|V_\beta}(N_{\varphi_1} + \mathrm{Im}(l_{\varphi_1}))\}$, then there exists $g_0 = \exp(X_0) \in G$ such that

$$g_0 * \varphi_1 = \varphi_0 + P^{-1}_{\varphi_1|V_\beta}(N_{\varphi_1} + \mathrm{Im}(l_{\varphi_1})) = \pi^t_\beta(G*\varphi_1).$$

Now $g_0\cdot(\varphi_1, C) = (g_0*\varphi_1, C(g_0))$ and

$$G_{g_0*\varphi_1}\cdot(g_0*\varphi_1, C(g_0)) = (g_0*\varphi_1, C(g_0) + \mathrm{Im}\, f_{\varphi_1}).$$

Thus

$$\begin{aligned}(G\cdot(\varphi_1, C)) \cap (\mathscr{M}^{t,t'}_{\beta,\beta'}(\mathfrak{l},\mathfrak{g}) \times V_{\beta'}) &= (G*\varphi_1 \cap \mathscr{M}^{t,t'}_{\beta,\beta'}(\mathfrak{l},\mathfrak{g}), (C(g_0) + \mathrm{Im}\, f_{\varphi_1}) \cap V_{\beta'})\\ &= (g_0*\varphi_1, P'^{-1}_{\varphi_1|V_{\beta'}}(C(g_0) + \mathrm{Im}\, f_{\varphi_1})).\end{aligned}$$

Let (φ_1, A), (φ'_1, A') be in the intersection. Then from Lemma 5, we have $\varphi_1 = \varphi'_1$. Let now $g = \exp(X)$ and $g' = \exp(X')$ in G such that $g*\varphi_1 = g'*\varphi_1 = \pi^t_\beta(G*\varphi_1)$. Then $g*\varphi_1 = g'*\varphi_1$ is equivalent to $g^{-1}g' \in G_{\varphi_1}$ and there exists $g_1 = \exp(X_1) \in G_{\varphi_1}$ such that $g' = gg_1$. Let $g_2 = \exp(X_2) = gg_1g^{-1}$, then $g_2 \in G_{g*\varphi_1}$ and

$$\begin{aligned} g' \cdot (\varphi_1, C) &= gg_1 \cdot (\varphi_1, C) = gg_1 g^{-1} g \cdot (\varphi_1, C) \\ &= g_2 g \cdot (\varphi_1, C) = g_2(g * \varphi_1, C(g)) \\ &= (g * \varphi_1, C(g) + f_{\varphi_1}(X_2)). \end{aligned}$$

Thus $C(g') - C(g) = f_{\varphi_1}(X_2) \in \mathrm{Im}\, f_{\varphi_1}$ which is equivalent to

$$C(g) + \mathrm{Im}\, f_{\varphi_1} = C(g') + \mathrm{Im}\, f_{\varphi_1}$$

which means that $A = A'$. □

Lemma 11. *The map*

$$h' : Hom^{t,t'}_{\beta,\beta'}(\mathfrak{l}, \mathfrak{g}) \to \mathscr{M}^{t,t'}_{\beta,\beta'}(\mathfrak{l}, \mathfrak{g}) \times V_{\beta'}$$

$$(\varphi_1, C) \mapsto (\pi^t_\beta(G * \varphi_1), P'^{-1}_{\varphi_1|V_{\beta'}}(C(g_0) + Im\, f_{\varphi_1}))$$

is continuous.

Proof. To show this lemma we prove the following facts

Fact 3.5. *Let $(\varphi_{1,n})_n$ be a sequence of $Hom^t_{1,\beta}(\mathfrak{l}, \mathfrak{g})$ which converges to φ_1, $\pi^t_\beta(G * \varphi_{1,n}) = \psi_{1,n}$ and $\pi^t_\beta(G * \varphi_1) = \psi_1$. Then there exists a convergent sequence $(X_n)_n$ in $\mathfrak{g}$ which converges to some element X such that*

$$\psi_{1,n} = \exp(X_n) * \varphi_{1,n} \qquad \text{and} \qquad \psi_1 = \exp(X) * \varphi_1.$$

Proof. Let $\{X_1, \dots, X_i\}$ be a basis of $\mathfrak{g}$ and $\alpha = (i_1, \dots, i_t) \in I(q, t)$ such that $\mathrm{Im}\, l_{\varphi_1} = l_{\varphi_1}(U_\alpha)$ where $U_\alpha = \mathbb{R}\text{-span}\{l_{\varphi_1}(X_{i_1}), \dots, l_{\varphi_1}(X_{i_t})\}$. As $G * \varphi_1 = \varphi_1 + \mathrm{Im}\, l_{\varphi_1}$ and $\pi^t_\beta(G * \varphi_1) \in G * \varphi_1$, there exists $X \in U_\alpha$ such that $\psi_1 = \varphi_1 + l_{\varphi_1}(X)$. By Fact 2.2 there exists $N > 0$ such that for all $n > N$, $\mathrm{Im}\, l_{\varphi_{1,n}} = l_{\varphi_{1,n}}(U_\alpha)$. Thus there exists $(X_n)_n$ in U_α such that $\psi_{1,n} = \varphi_{1,n} + l_{\varphi_{1,n}}(X_n)$ for all $n > N$. Now from the continuity of the map $\varphi_1 \mapsto \pi^t_\beta(G * \varphi_1)$ and by the convergence of $(\varphi_{1,n})_n$ to φ_1, we deduce that $(\varphi_{1,n} + l_{\varphi_{1,n}}(X_n))_n$ converges to $\{\varphi_1 + l_{\varphi_1}(X)\}$ and then $(l_{\varphi_{1,n}}(X_n))_n$ converges to $l_{\varphi_1}(X)$ (because $(\varphi_{1,n})_n$ converges to φ_1). Let $l'_{\varphi_{1,n}}$ be the restriction of $l_{\varphi_{1,n}}$ to U_α and l'_{φ_1} the restriction of l_{φ_1} to U_α. Then all the maps $l'_{\varphi_{1,n}}$, $n > N$ are injective and l'_{φ_1} is also injective. The sequence $(l'_{\varphi_{1,n}}(X_n))_n$ converges to $l'_{\varphi_1}(X)$ and obviously $(l'_{\varphi_{1,n}})_n$ converges to l'_{φ_1}. Using Fact 2.7 we conclude that $(X_n)_n$ converges to X. □

Fact 3.6. *Let $(\varphi_{1,n})_n$ be a sequence in $Hom^{t,t'}_{1,\beta,\beta'}(\mathfrak{l}, \mathfrak{g})$ which converges to $\varphi_1 \in Hom^{t,t'}_{1,\beta,\beta'}(\mathfrak{l}, \mathfrak{g})$. Then the following facts hold:*

(1) *There exists $\gamma = (j_1, \dots, j_t) \in I(q, t)$ such that*

$$Im\, l_{\psi_1} = \mathbb{R}\text{-}span\{l_{\psi_1}(X_{j_1}), \dots, l_{\psi_1}(X_{j_t})\}, \qquad \psi_1 = \pi^t_\beta(G * \varphi_1)$$

and

$$Im\ l_{\psi_{1,n}} = \mathbb{R}\text{-}span\{l_{\psi_{1,n}}(X_{j_1}), \dots, l_{\psi_{1,n}}(X_{j_t})\}, \qquad \psi_{1,n} = \pi_\beta^t(G * \varphi_{1,n}).$$

(2) *The sequence* $(\bar{l}_{\varphi_{1,n},\gamma})_n$ *converges to* $\bar{l}_{\varphi_1,\gamma}$.

Proof. As the quotient map $\varphi_1 \mapsto G * \varphi_1$ and π_β^t are continuous, the sequence $\{\psi_{1,n}\}$ converges to ψ_1. Then $(l_{\psi_{1,n}})_n$ converges to l_{ψ_1}. Let $\gamma = (j_1, \dots, j_t) \in I(q,t)$ be such that $\mathrm{Im}\, l_{\psi_1} = \mathbb{R}\text{-span}\{l_{\psi_1}(X_{j_1}), \dots, l_{\psi_1}(X_{j_t})\}$. By Fact 2.2, we can assume that

$$\mathrm{Im}\ l_{\psi_{1,n}} = \mathbb{R}\text{-span}\{l_{\psi_{1,n}}(X_{j_1}), \dots, l_{\psi_{1,n}}(X_{j_t})\}.$$

To prove the second result, let $\gamma = (j_1, \dots, j_t) \in I(q,t)$ satisfying (1) and for $\{s_1, \dots, s_{q-t}\} = \{1, \dots, q\} \backslash \{j_1, \dots, j_t\}$, $l_{\psi_{1,n}}(X_{s_i}) = \sum_{r=1}^t \alpha_{r,s_i}^n l_{\psi_{1,n}}(X_{j_r})$ and $l_{\psi_1}(X_{s_i}) = \sum_{r=1}^t \alpha_{r,s_i} l_{\psi_1}(X_{j_r})$. Let $v_{n,s_i} = X_{s_i} - \sum_{r=1}^t \alpha_{r,s_i}^n X_{j_r}$ and $v_{s_i} = X_{s_i} - \sum_{r=1}^t \alpha_{r,s_i} X_{j_r}$. Then by Fact 2.5, $(v_{n,s_i})_n$ converges to v_{s_i}. This shows that $(l_{\varphi_{1,n},\gamma})_n$ converges to $l_{\varphi_1,\gamma}$. Now, for all $1 \le i \le q-t$ we have

$$\bar{l}_{\varphi_{1,n},\gamma}(u_i) = \Sigma_{1,2}(l_{\varphi_{1,n},\gamma}(u_i))E_{\psi_{1,n}} + \Sigma_{1,3}(l_{\varphi_{1,n},\gamma}(u_i))F_{\psi_{1,n}}.$$

As the matrix multiplication is continuous and the maps $\varphi_1 \mapsto E_{\psi_1}$ and $\varphi_1 \mapsto F_{\psi_1}$ are continuous, we deduce that $(\bar{l}_{\varphi_{1,n},\gamma}(u_i))_n$ converges to $\bar{l}_{\varphi_1,\gamma}(u_i)$. Thus $(\bar{l}_{\varphi_{1,n},\gamma})_n$ converges to $\bar{l}_{\varphi_1,\gamma}$. □

Let $q_{\varphi_1,\gamma}$ be the projection of $\mathscr{L}(\mathfrak{l}_2, \mathfrak{g}_0)$ on V_β parallel to $\mathrm{Im}(\bar{l}_{\varphi_1,\gamma})$. Then

Fact 3.7. *We have:*

$$h'(\varphi_1, C) = (\pi_\beta^t(G * \varphi_1), q_{\varphi_1,\gamma}(C(g_0))),$$

where $C(g_0)$ *is given in Lemma* 10.

Proof. By Fact 3.4 (2), we have $\mathrm{Im}(\bar{l}_{\varphi_1,\gamma}) = \mathrm{Im}(f_{\varphi_1})$. Then

$$\begin{aligned} P'^{-1}_{\varphi_1|V_{\beta'}}(C(g_0) + \mathrm{Im}\ f_{\varphi_1}) &= (C(g_0) + \mathrm{Im}(f_{\varphi_1})) \cap V_{\beta'} \\ &= (C(g_0) + \mathrm{Im}(\bar{l}_{\varphi_1,\gamma})) \cap V_{\beta'} \\ &= q_{\varphi_1,\gamma}(C(g_0)) \qquad \text{by Fact 2.1.} \end{aligned}$$

□

We now prove the lemma. Let $(\varphi_{1,n}, C_n)_n$ be a sequence in $\mathrm{Hom}_{\beta,\beta'}^{t,t'}(\mathfrak{l}, \mathfrak{g})$ which converges to an element $(\varphi_1, C) \in \mathrm{Hom}_{\beta,\beta'}^{t,t'}(\mathfrak{l}, \mathfrak{g})$. To see that h' is continuous, we have to show that $(h'(\varphi_{1,n}, C_n))_n$ converges to $h'(\varphi_1, C)$. As $(\varphi_{1,n})_n$ converges to φ_1, by Fact 3.6 there exists $\gamma \in I(q,t)$ such that the sequence $(\bar{l}_{\varphi_{1,n},\gamma})_n$ converges to $\bar{l}_{\varphi_1,\gamma}$. As $\mathscr{L}(\mathfrak{l}_2, \mathfrak{g}_0) = \mathrm{Im}(l_{\varphi_{1,n},\gamma}) \oplus V_{\beta'}$, $\mathscr{L}(\mathfrak{l}_2, \mathfrak{g}_0) =$

$\mathrm{Im}(l_{\varphi_1,\gamma}) \oplus V_{\beta'}$,

$$\mathrm{Im}(\bar{l}_{\varphi_1,\gamma}) = \mathbb{R}\text{-span}\{\bar{l}_{\varphi_1,\gamma}(X_{i_1}), \dots, \bar{l}_{\varphi_1,\gamma}(X_{i_t})\}$$

and

$$\mathrm{Im}(\bar{l}_{\varphi_{1,n},\gamma}) = \mathbb{R}\text{-span}\{\bar{l}_{\varphi_{1,n},\gamma}(X_{i_1}), \dots, \bar{l}_{\varphi_{1,n},\gamma}(X_{i_t})\},$$

where $\gamma = (i_1, \dots, i_t)$, then by Fact 2.3 the sequence $(q_{\varphi_{1,n},\gamma})_n$ converges to $q_{\varphi_1,\gamma}$. Now $(C_n)_n$ converges to C and by Fact 3.5 there exists a sequence $(X_n)_n$ in $\mathfrak{g}$ such that $(\exp(X_n))_n$ converges to $\exp(X)$. Now by Fact 3.7,

$$h'(\varphi_{1,n}, C_n) = (\pi_\beta^t(G * \varphi_{1,n}), q_{\varphi_{1,n},\gamma}(C_n(\exp(X_n))))$$

and it is clear that $(C_n(\exp(X_n)))_n$ converges to $C(\exp(X))$. Then $(h'(\varphi_{1,n}, C_n))_n$ converges to $h'(\varphi_1, C)$. □

By Lemma 10 above, for each G-orbit $\mathcal{O}$ in $\mathrm{Hom}_{\beta,\beta'}^{t,t'}(\mathfrak{l}, \mathfrak{g})$, there uniquely exists an element $A_{\mathcal{O}}$ in $\mathcal{O} \cap (\mathcal{M}_{\beta,\beta'}^{t,t'}(\mathfrak{l}, \mathfrak{g}) \times V_{\beta'})$. We define the map

$$\varepsilon_{\beta,\beta'}^{t,t'} : \mathrm{Hom}_{\beta,\beta'}^{t,t'}(\mathfrak{l}, \mathfrak{g})/G \to \mathcal{M}_{\beta,\beta'}^{t,t'}(\mathfrak{l}, \mathfrak{g}) \times V_{\beta'}$$

$$\mathcal{O} \mapsto A_{\mathcal{O}}.$$

Then $\mathcal{M}_{\beta,\beta'}^{t,t'}(\mathfrak{l}, \mathfrak{g}) \times V_{\beta'}$ is a fundamental domain of the G-action on $\mathrm{Hom}_{\beta,\beta'}^{t,t'}(\mathfrak{l}, \mathfrak{g})$ in the sense below:

Lemma 12. *The map* $\varepsilon_{\beta,\beta'}^{t,t'} : Hom_{\beta,\beta'}^{t,t'}(\mathfrak{l}, \mathfrak{g})/G \to \mathcal{M}_{\beta,\beta'}^{t,t'}(\mathfrak{l}, \mathfrak{g}) \times V_{\beta'}$ *defined above is a homeomorphism.*

Proof. Let $(\varphi_1, C) \neq (\varphi_1', C') \in \mathrm{Hom}_{\beta,\beta'}^{t,t'}(\mathfrak{l}, \mathfrak{g})$ such that $G \cdot (\varphi_1, C) = G \cdot (\varphi_1', C')$. Then there exist g_0 and g_0' in G such that

$$\varepsilon_{\beta,\beta'}^{t,t'}(G \cdot (\varphi_1, C)) = (g_0 * \varphi_1, P'^{-1}_{\varphi_1|V_{\beta'}}(C(g_0) + \mathrm{Im}(f_{\varphi_1})))$$

and

$$\varepsilon_{\beta,\beta'}^{t,t'}(G \cdot (\varphi_1', C')) = (g_0' * \varphi_1', P'^{-1}_{\varphi_1'|V_{\beta'}}(C'(g_0') + \mathrm{Im}(f_{\varphi_1'}))).$$

Since $G \cdot (\varphi_1, C) = G \cdot (\varphi_1', C')$, there exists $g \in G$ such that $(\varphi_1, C) = g \cdot (\varphi_1', C') = (g * \varphi_1', C'(g))$. We get thus the following:

$$G * \varphi_1 = G * (g * \varphi_1') = G * \varphi_1' \Leftrightarrow \pi_\beta^t(G * \varphi_1) = \pi_\beta^t(G * \varphi_1')$$

$$\Leftrightarrow g_0 * \varphi_1 = g_0' * \varphi_1'.$$

This means in particular that $f_{\varphi_1} = f_{\varphi_1'}$. Besides, there exists $g_1' = \exp(X_1') \in G_{g_0' * \varphi_1'}$ such that $g_1' g_0' \cdot (\varphi_1', C') = g_0 \cdot (\varphi_1, C)$. This entails that

$$(g_0 * \varphi_1, C(g_0)) = (g_0' * \varphi_1', C'(g_0') + \Sigma_{1,2}(X_1') E_{g_0' * \varphi_1'} + \Sigma_{1,3}(X_1') F_{g_0' * \varphi_1'})$$

which is equivalent to

$$C(g_0) + \mathrm{Im}\, f_{\varphi_1} = C'(g_0') + f_{\varphi_1'}(X_1') + \mathrm{Im}\, f_{\varphi_1} = C'(g_0') + \mathrm{Im}\, f_{\varphi_1'}.$$

Thus $\varepsilon^{t,t'}_{\beta,\beta'}(G \cdot (\varphi_1, C)) = \varepsilon^{t,t'}_{\beta,\beta'}(G \cdot (\varphi_1', C'))$ and $\varepsilon^{t,t'}_{\beta,\beta'}$ is a well defined map. We now prove that the map $\varepsilon^{t,t'}_{\beta,\beta'}$ is a homeomorphism. In fact, we first show that $\varepsilon^{t,t'}_{\beta,\beta'}$ is a bijection. Let (φ_1, C) and (φ_1', C') be in $\mathrm{Hom}^{t,t'}_{\beta,\beta'}(\mathfrak{l}, \mathfrak{g})$ such that

$$\varepsilon^{t,t'}_{\beta,\beta'}(G \cdot (\varphi_1, C)) = \varepsilon^{t,t'}_{\beta,\beta'}(G \cdot (\varphi_1', C')).$$

Then there exist $g_0, g_0' \in G$ such that

$$(g_0 * \varphi_1, P'^{-1}_{\varphi_1|V_{\beta'}}(C(g_0) + \mathrm{Im}\, f_{\varphi_1})) = (g_0' * \varphi_1', P'^{-1}_{\varphi_1'|V_{\beta'}}(C'(g_0') + \mathrm{Im}\, f_{\varphi_1'})).$$

This implies that $g_0 * \varphi_1 = g_0' * \varphi_1'$, which means that $f_{\varphi_1} = f_{\varphi_1'}$ and $P'_{\varphi_1} = P'_{\varphi_1'}$. Then from the equality

$$P'^{-1}_{\varphi_1|V_{\beta'}}(C(g_0) + \mathrm{Im}\, f_{\varphi_1}) = P'^{-1}_{\varphi_1'|V_{\beta'}}(C'(g_0') + \mathrm{Im}\, f_{\varphi_1'}),$$

we get $C(g_0) + \mathrm{Im}\, f_{\varphi_1} = C'(g_0') + \mathrm{Im}\, f_{\varphi_1'}$. Then $G * \varphi_1 = G * \varphi_1'$ and

$$G_{g_0 * \varphi_1} \cdot (g_0 * \varphi_1, C(g_0)) = G_{g_0' * \varphi_1'} \cdot (g_0' * \varphi_1', C'(g_0')),$$

which is equivalent to $G \cdot (\varphi_1, C) = G \cdot (\varphi_1', C')$ and the map $\varepsilon^{t,t'}_{\beta,\beta'}$ is injective. Let now $(\psi_1, C_1) \in \mathscr{M}^{t,t'}_{\beta,\beta'}(\mathfrak{l}, \mathfrak{g}) \times V_{\beta'}$. Since the map π^t_β is a homeomorphism, there exist $\varphi_1 \in \mathrm{Hom}^{t,t'}_{1,\beta,\beta'}(\mathfrak{l}, \mathfrak{g})$ and $g' \in G$ such that

$$(G * \varphi_1) \cap \mathscr{M}^{t,t'}_{\beta,\beta'}(\mathfrak{l}, \mathfrak{g}) = \psi_1 = g' * \varphi_1.$$

Hence, there exists $C \in \mathscr{L}(\mathfrak{l}_2, \mathfrak{g}_0)$, such that $g' \cdot (\varphi_1, C) = (g' * \varphi_1, C(g'))$,

$$C_1 = P'^{-1}_{\psi_1|V_{\beta'}}(C(g') + \mathrm{Im}\, f_{\psi_1})$$

and $\varepsilon^{t,t'}_{\beta,\beta'}(G \cdot (\varphi_1, C)) = (\psi_1, C_1)$ and then the map $\varepsilon^{t,t'}_{\beta,\beta'}$ is surjective. Now the diagram below commutes

$$\begin{array}{ccc} \mathrm{Hom}^{t,t'}_{\beta,\beta'}(\mathfrak{l}, \mathfrak{g}) & & \\ \downarrow \pi' & \searrow h' & \\ \mathrm{Hom}^{t,t'}_{\beta,\beta'}(\mathfrak{l}, \mathfrak{g})/G & \xrightarrow{\varepsilon^{t,t'}_{\beta,\beta'}} & \mathscr{M}^{t,t'}_{\beta,\beta'}(\mathfrak{l}, \mathfrak{g}) \times V_{\beta'} \end{array}$$

where $h'((\varphi_1, C)) = (\pi^t_\beta(G * \varphi_1), P'^{-1}_{\varphi_1|V_{\beta'}}(C(g_0) + \mathrm{Im}(f_{\varphi_1}))$. Since by Lemma 11, h' is continuous and since π' is open then $\varepsilon^{t,t'}_{\beta,\beta'}$ is continuous. Now the quotient canonical map

$$(\varepsilon^{t,t'}_{\beta,\beta'})^{-1} = \pi'_{|\mathscr{M}^{t,t'}_{\beta,\beta'}(\mathfrak{l}, \mathfrak{g}) \times V_{\beta'}}$$

is continuous and then the map $\varepsilon^{t,t'}_{\beta,\beta'}$ is a homeomorphism. $\square$

As an immediate consequence, we get:

Proposition 5. *We have the following:*

$$Hom(\mathfrak{l},\mathfrak{g})/G = \bigcup_{\substack{0\le t\le q\\ 0\le t'\le q'}} \bigcup_{\substack{\beta\in I(m,m-t)\\ \beta'\in I(m',m'-t')}} Hom^{t,t'}_{\beta,\beta'}(\mathfrak{l},\mathfrak{g})/G,$$

where for all $t\in\{0,\dots,q\}$, $t'\in\{0,\dots,q'\}$, $\beta\in I(m,m-t)$ *and* $\beta'\in I(m', m'-t')$, $Hom^{t,t'}_{\beta,\beta'}(\mathfrak{l},\mathfrak{g})$ *is defined as in formula* (15) *and the set* $Hom^{t,t'}_{\beta,\beta'}(\mathfrak{l},\mathfrak{g})/G$ *is homeomorphic to* $\mathscr{M}^{t,t'}_{\beta,\beta'}(\mathfrak{l},\mathfrak{g})\times V_{\beta'}$.

Let now

$$\mathrm{Hom}^{t,t'}_1(\mathfrak{l},\mathfrak{g}) = \bigcup_{\substack{\beta\in I(m,m-t)\\ \beta'\in I(m',m'-t')}} \mathrm{Hom}^{t,t'}_{1,\beta,\beta'}(\mathfrak{l},\mathfrak{g}). \tag{18}$$

and

$$\mathrm{Hom}^{t,t'}(\mathfrak{l},\mathfrak{g}) = \mathrm{Hom}^{t,t'}_1(\mathfrak{l},\mathfrak{g})\times\mathscr{L}(\mathfrak{l}_2,\mathfrak{g}_2). \tag{19}$$

We then show the following:

Lemma 13. *Retain definitions* (18) *and* (19). *Then the collection*

$$S^{t,t'}_{\beta,\beta'} = (\varepsilon^{t,t'}_{\beta,\beta'}, Hom^{t,t'}_{\beta,\beta'}(\mathfrak{l},\mathfrak{g})/G)_{\substack{\beta\in I(m,m-t)\\ \beta'\in I(m',m'-t')}} \qquad (t=0,\dots,q \text{ and } t'=0,\dots,q')$$

constitutes a family of local sections of the canonical surjection

$$\pi^{t,t'} : Hom^{t,t'}(\mathfrak{l},\mathfrak{g}) \to Hom^{t,t'}(\mathfrak{l},\mathfrak{g})/G.$$

Proof. We have to show that $\pi^{t,t'}\circ\varepsilon^{t,t'}_{\beta,\beta'} = \mathrm{Id}_{\mathrm{Hom}^{t,t'}_{\beta,\beta'}(\mathfrak{l},\mathfrak{g})/G}$ for all t, t', β, β'. Let $\varphi=(\varphi_1,C_\varphi)\in\mathrm{Hom}(\mathfrak{l},\mathfrak{g})$ be such that the orbit $G\cdot(\varphi_1,C_\varphi)\in \mathrm{Hom}^{t,t'}_{\beta,\beta'}(\mathfrak{l},\mathfrak{g})/G$. Then

$$\pi^{t,t'}\circ\varepsilon^{t,t'}_{\beta,\beta'}(G\cdot(\varphi_1,C_\varphi)) = \pi^{t,t'}(g_0*\varphi_1, P'^{-1}_{\varphi_1|V_{\beta'}}(C_\varphi(g_0)+\mathrm{Im}(f_{\varphi_1}))).$$

There exists $g_1\in G_{g_0*\varphi_1}$ such that

$$g_1g_0\cdot(\varphi_1,C_\varphi) = (g_0*\varphi_1, (C_\varphi(g_0)+\mathrm{Im}\, f_{\varphi_1})\cap V_{\beta'}).$$

Thus

$$\begin{aligned}\pi^{t,t'}(g_0*\varphi_1, P'^{-1}_{\varphi_1|V_{\beta'}}(C_\varphi(g_0)+\mathrm{Im}(f_{\varphi_1}))) &= \pi^{t,t'}(g_1g_0\cdot(\varphi_1,C_\varphi))\\ &= G\cdot(g_1g_0\cdot(\varphi_1,C_\varphi))\\ &= G\cdot(\varphi_1,C_\varphi).\end{aligned}$$

In particular,

$$\varepsilon_{\beta,\beta'}^{t,t'}(\mathrm{Hom}_{\beta,\beta'}^{t,t'}(\mathfrak{l},\mathfrak{g})/G) = \left\{ \varphi \in \mathrm{Hom}(\mathfrak{l},\mathfrak{g}) \;\middle|\; \begin{array}{l} \varphi_1 \in \mathscr{M}_{\beta,\beta'}^{t,t'}(\mathfrak{l},\mathfrak{g}) \\ C_\varphi \in V_{\beta'} \end{array} \right\}. \qquad \square$$

Write

$$\mathrm{Hom}^{t,t'}(\mathfrak{l},\mathfrak{g}) = \bigcup_{\substack{\beta \in I(m,m-t) \\ \beta' \in I(m',m'-t')}} \mathrm{Hom}_{\beta,\beta'}^{t,t'}(\mathfrak{l},\mathfrak{g})$$

Then from Proposition 4, the set $\mathrm{Hom}^{t,t'}(\mathfrak{l},\mathfrak{g})$ is a G-invariant subset. More precisely all the subsets of the union are G-invariant and open in $\mathrm{Hom}^{t,t'}(\mathfrak{l},\mathfrak{g})$. Our main result is the following:

Theorem 2. *The writing*

$$Hom(\mathfrak{l},\mathfrak{g})/G = \bigcup_{\substack{0 \le t \le q \\ 0 \le t' \le q'}} Hom^{t,t'}(\mathfrak{l},\mathfrak{g})/G$$

is a decomposition of $Hom(\mathfrak{l},\mathfrak{g})/G$ as a union of Hausdorff subspaces. The sets $Hom^{t,t'}(\mathfrak{l},\mathfrak{g})/G$ may fail to be open in $Hom(\mathfrak{l},\mathfrak{g})/G$.

To prove this result, we need the following Lemma:

Lemma 14. *Let $\varphi = (\varphi_1, C)$ and $\xi = (\xi_1, C')$ be two elements in $Hom^{t,t'}(\mathfrak{l},\mathfrak{g})$. If $[\varphi_1]$ and $[\xi_1]$ are separated in $Hom_1^{t,t'}(\mathfrak{l},\mathfrak{g})/G$ then so are $[\varphi]$ and $[\xi]$ in*

$$Hom^{t,t'}(\mathfrak{l},\mathfrak{g})/G.$$

Proof. We consider the following diagram

$$\begin{array}{ccc} \mathrm{Hom}^{t,t'}(\mathfrak{l},\mathfrak{g}) & \xrightarrow{P_1} & \mathrm{Hom}_1^{t,t'}(\mathfrak{l},\mathfrak{g}) \\ {\scriptstyle \pi_1}\big\downarrow & & \big\downarrow{\scriptstyle \pi_2} \\ \mathrm{Hom}^{t,t'}(\mathfrak{l},\mathfrak{g})/G & \xrightarrow[\tilde{P}_1]{} & \mathrm{Hom}_1^{t,t'}(\mathfrak{l},\mathfrak{g})/G. \end{array}$$

where π_1, π_2 are the quotient maps, $P_1(\varphi_1, C) = \varphi_1$ and $\widetilde{P_1}(G \cdot (\varphi_1, C)) = G * \varphi_1$. Then obviously this diagram commutes and the maps P_1 and $\widetilde{P_1}$ are continuous. Assume that $[P_1(\varphi)]$ and $[P_1(\xi)]$ are separated, then there exist neighborhoods U_1 of $\pi_2 \circ P_1(\varphi)$ and U_2 of $\pi_2 \circ P_1(\xi)$ such that $U_1 \cap U_2 = \varnothing$. Now $\pi_2 \circ P_1(\varphi) = \tilde{P}_1 \circ \pi_1(\varphi) \in U_1$ and $\pi_2 \circ P_1(\xi) = \tilde{P}_1 \circ \pi_1(\xi) \in U_2$. Thus $\pi_1(\varphi) \in \tilde{P}_1^{-1}(U_1)$, $\pi_1(\xi) \in \tilde{P}_1^{-1}(U_2)$ and we have $\tilde{P}_1^{-1}(U_1) \cap \tilde{P}_1^{-1}(U_2) = \varnothing$. $\square$

Proof of Theorem 2. Let $\varphi = (\varphi_1, C)$ and $\xi = (\xi_1, C')$ be two elements in $\mathrm{Hom}^{t,t'}(\mathfrak{l}, \mathfrak{g})$ and assume that $[\varphi]$ and $[\xi]$ are not separated. From Lemma 14, $[\varphi_1]$ and $[\xi_1]$ are not separated in $\mathrm{Hom}_1^{t,t'}(\mathfrak{l}, \mathfrak{g})/G$, then there exist $(\varphi_{1,n})_n \subset \mathrm{Hom}_1^t(\mathfrak{l}, \mathfrak{g})$ and $g_n = \exp(X_n) \in G$ such that $\varphi_{1,n}$ converges to φ_1 and $g_n * \varphi_{1,n}$ converges to ξ_1 in $\mathrm{Hom}_1^t(\mathfrak{l}, \mathfrak{g})$. This means $A_{\varphi_1} = A_{\xi_1}$, $D_{\varphi_1} = D_{\xi_1}$ and $F_{\varphi_1} = F_{\xi_1}$. In particular $l_{\varphi_1} = l_{\xi_1}$ and $P_{\varphi_1} = P_{\xi_1}$. Thus $[\varphi_1]$ and $[\xi_1]$ belong to the open set $\mathrm{Hom}_{1,\beta}^{t,t'}(\mathfrak{l}, \mathfrak{g})/G$ for some $\beta \in I(m, m-t)$. Now, $\mathrm{Hom}_{1,\beta}^{t,t'}(\mathfrak{l}, \mathfrak{g})/G$ is a Hausdorff space as it is included in $\mathrm{Hom}_{1,\beta}^{t}(\mathfrak{l}, \mathfrak{g})/G$, which is a Hausdorff space as being homeomorphic to $\mathscr{M}_\beta^t(\mathfrak{l}, \mathfrak{g})$ by Lemma 7. Then $[\varphi_1] = [\xi_1]$ and this implies that $f_{\varphi_1} = f_{\xi_1}$ and $P'_{\varphi_1} = P'_{\xi_1}$. Finally $[\varphi]$ and $[\xi]$ belong to the open set $\mathrm{Hom}_{\beta,\beta'}^{t,t'}(\mathfrak{l}, \mathfrak{g})/G$ for some $\beta \in I(m, m-t)$ and $\beta' \in I(m', m'-t')$. Now from Proposition 5, $\mathrm{Hom}_{\beta,\beta'}^{t,t'}(\mathfrak{l}, \mathfrak{g})/G$ is homeomorphic to the Hausdorff space $\mathscr{M}_{\beta,\beta'}^{t,t'}(\mathfrak{l}, \mathfrak{g}) \times V_{\beta'}$, then $\mathrm{Hom}_{\beta,\beta'}^{t,t'}(\mathfrak{l}, \mathfrak{g})/G$ is a Hausdorff space and $[\varphi] = [\xi]$. □

4. Description of the parameter and the deformation spaces

We use the same setting and notation in Section 3. Let us take a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and consider the decompositions

$$\mathfrak{g} = (\mathfrak{g}_0 \cap \mathfrak{h}) \oplus \mathfrak{g}_0' \oplus \mathfrak{h}' \oplus V \oplus \mathfrak{h}'' \oplus W \qquad \text{and} \qquad \mathfrak{l} = \mathfrak{l}_0 \oplus \mathfrak{l}_1 \oplus \mathfrak{l}_2,$$

where $\mathfrak{g}_0'$, $\mathfrak{h}'$ and $\mathfrak{h}''$ designate some subspaces of $\mathfrak{g}$ such that $\mathfrak{g}_0 = (\mathfrak{g}_0 \cap \mathfrak{h}) \oplus \mathfrak{g}_0'$, $[\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{h} = (\mathfrak{g}_0 \cap \mathfrak{h}) \oplus \mathfrak{h}'$, $\mathfrak{h} = [\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{h} \oplus \mathfrak{h}''$, V is a linear subspace supplementary to $(\mathfrak{g}_0 \cap \mathfrak{h}) \oplus \mathfrak{g}_0' \oplus \mathfrak{h}'$ in $[\mathfrak{g}, \mathfrak{g}]$ and W a linear supplementary subspace to $(\mathfrak{g}_0 \cap \mathfrak{h}) \oplus \mathfrak{g}_0' \oplus \mathfrak{h}' \oplus V \oplus \mathfrak{h}''$ in $\mathfrak{g}$. Then with respect to these decompositions, the adjoint representation Ad_g, $g = \exp(X) \in G$ can once again be written down as

$$\mathrm{Ad}_g = \begin{pmatrix} I_1 & 0 & \sigma_{1,3}(X) & \delta_{1,4}(X) & \gamma_{1,5}(X) & \omega_{1,6}(X) \\ 0 & I_2 & \sigma_{2,3}(X) & \delta_{2,4}(X) & \gamma_{2,5}(X) & \omega_{2,6}(X) \\ 0 & 0 & I_3 & 0 & \gamma_{3,5}(X) & \omega_{3,6}(X) \\ 0 & 0 & 0 & I_4 & \gamma_{4,5}(X) & \omega_{4,6}(X) \\ 0 & 0 & 0 & 0 & I_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_6 \end{pmatrix},$$

where

$$\Sigma_{1,2}(X) = \begin{pmatrix} \sigma_{1,3}(X) & \delta_{1,4}(X) \\ \sigma_{2,3}(X) & \delta_{2,4}(X) \end{pmatrix},$$

$$\Sigma_{1,3}(X) + \frac{1}{2}\Sigma_{1,2}(X)\Sigma_{2,3}(X) = \begin{pmatrix} \gamma_{1,5}(X) & \omega_{1,6}(X) \\ \gamma_{2,5}(X) & \omega_{2,6}(X) \end{pmatrix}$$

and

$$\Sigma_{2,3}(X) = \begin{pmatrix} \gamma_{3,5}(X) & \omega_{3,6}(X) \\ \gamma_{4,5}(X) & \omega_{4,6}(X) \end{pmatrix}$$

with: $\sigma_{1,3}(X) \in \mathscr{L}(\mathfrak{h}', \mathfrak{g}_0 \cap \mathfrak{h})$, $\delta_{1,4}(X) \in \mathscr{L}(V, \mathfrak{g}_0 \cap \mathfrak{h})$, $\sigma_{2,3}(X) \in \mathscr{L}(\mathfrak{h}', \mathfrak{g}_0')$, $\delta_{2,4}(X) \in \mathscr{L}(V, \mathfrak{g}_0')$, $\gamma_{1,5}(X) \in \mathscr{L}(\mathfrak{h}'', \mathfrak{g}_0 \cap \mathfrak{h})$, $\omega_{1,6}(X) \in \mathscr{L}(W, \mathfrak{g}_0 \cap \mathfrak{h})$, $\gamma_{2,5}(X) \in \mathscr{L}(\mathfrak{h}'', \mathfrak{g}_0')$, $\omega_{2,6}(X) \in \mathscr{L}(W, \mathfrak{g}_0')$, $\gamma_{3,5}(X) \in \mathscr{L}(\mathfrak{h}'', \mathfrak{h}')$, $\omega_{3,6}(X) \in \mathscr{L}(W, \mathfrak{h}')$, $\gamma_{4,5}(X) \in \mathscr{L}(\mathfrak{h}'', V)$ and $\omega_{4,6}(X) \in \mathscr{L}(W, V)$. Here I_1, I_2, I_3, I_4, I_5 and I_6 designate the identity maps of $\mathfrak{g}_0 \cap \mathfrak{h}$, $\mathfrak{g}_0'$, $\mathfrak{h}'$, V, $\mathfrak{h}''$ and W respectively. This leads to the fact that any element of $\mathrm{Hom}(\mathfrak{l}, \mathfrak{g})$ can be written accordingly, as a matrix

$$\varphi := \varphi(A, B, C) = \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ 0 & B_3 & C_3 \\ 0 & B_4 & C_4 \\ 0 & 0 & C_5 \\ 0 & 0 & C_6 \end{pmatrix},$$

where

$$A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}, \qquad B = \begin{pmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{pmatrix} \qquad \text{and} \qquad C = \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \end{pmatrix}.$$

Here $A_1 \in \mathscr{L}(\mathfrak{l}_0, \mathfrak{g}_0 \cap \mathfrak{h})$, $A_2 \in \mathscr{L}(\mathfrak{l}_0, \mathfrak{g}_0')$, $B_1 \in \mathscr{L}(\mathfrak{l}_1, \mathfrak{g}_0 \cap \mathfrak{h})$, $B_2 \in \mathscr{L}(\mathfrak{l}_1, \mathfrak{g}_0')$, $B_3 \in \mathscr{L}(\mathfrak{l}_1, \mathfrak{h}')$, $B_4 \in \mathscr{L}(\mathfrak{l}_1, V)$, $C_1 \in \mathscr{L}(\mathfrak{l}_2, \mathfrak{g}_0 \cap \mathfrak{h})$, $C_2 \in \mathscr{L}(\mathfrak{l}_2, \mathfrak{g}_0')$, $C_3 \in \mathscr{L}(\mathfrak{l}_2, \mathfrak{h}')$, $C_4 \in \mathscr{L}(\mathfrak{l}_2, V)$, $C_5 \in \mathscr{L}(\mathfrak{l}_2, \mathfrak{h}'')$ and $C_6 \in \mathscr{L}(\mathfrak{l}_2, W)$. We can now state our first result.

Theorem 3. *Let G be a 3-step nilpotent Lie group, H a connected subgroup of G and Γ a discontinuous group for G/H. The syndetic hull of Γ in G and its Lie algebra are denoted by L and $\mathfrak{l}$ respectively. Then the parameter space $\mathscr{R}(\mathfrak{l}, \mathfrak{g}, \mathfrak{h})$ writes as a disjoint union $\mathscr{R}_1 \sqcup \mathscr{R}_2$, where $\mathscr{R}_1$ is the open set defined by*

$$\mathscr{R}_1 := \left\{ \varphi(A, B, C) \in Hom(\mathfrak{l}, \mathfrak{g}) \;\middle|\; \begin{array}{l} rk(C_6) = \dim(\mathfrak{l}_2) \\ rk(B_4) = \dim(\mathfrak{l}_1) \\ and\;\; rk(A_2) = \dim(\mathfrak{l}_0) \end{array} \right\},$$

$$\mathscr{R}_2 := \left\{ \varphi(A, B, C) \in Hom(\mathfrak{l}, \mathfrak{g}) \;\middle|\; \begin{array}{l} rk(B_4) + rk(C_6) < \dim(\mathfrak{l}_1 \oplus \mathfrak{l}_2) \;\; and \\ rk(M_{\varphi, X}) = \dim(\mathfrak{l}) \;\; for\ all\;\; X \in \mathfrak{g} \end{array} \right\}$$

which may fail to be open. Here,

$$M_{\varphi,X} = \begin{pmatrix} A_2 & B_2+\sigma_{2,3}(X)B_3+\delta_{2,4}(X)B_4 & C_2+\sigma_{2,3}(X)C_3+\delta_{2,4}(X)C_4 \\ & & +\gamma_{2,5}(X)C_5+\omega_{2,6}(X)C_6 \\ 0 & B_4 & C_4+\gamma_{4,5}(X)C_5+\omega_{4,6}(X)C_6 \\ 0 & 0 & C_6 \end{pmatrix}.$$

PROOF. As the pair (G,H) has the Lipsman property ([1] and [16]), Theorem 1 enables us to state that

$$\mathscr{R}(\mathfrak{l},\mathfrak{g},\mathfrak{h}) = \left\{ \varphi \in \mathrm{Hom}(\mathfrak{l},\mathfrak{g}) \;\middle|\; \begin{array}{l} \dim \varphi(\mathfrak{l}) = \dim(\mathfrak{l}) \\ \mathrm{Ad}_g \circ \varphi(\mathfrak{l}) \cap \mathfrak{h} = \{0\} \ \text{ for all } \ g = \exp(X) \in G \end{array} \right\}.$$

Now,

$$\mathrm{Ad}_g \circ \varphi = \begin{pmatrix} A_1 & B_1+\sigma_{1,3}(X)B_3+\delta_{1,4}(X)B_4 & C_1+\sigma_{1,3}(X)C_3+\delta_{1,4}(X)C_4 \\ & & +\gamma_{1,5}(X)C_5+\omega_{1,6}(X)C_6 \\ A_2 & B_2+\sigma_{2,3}(X)B_3+\delta_{2,4}(X)B_4 & C_2+\sigma_{2,3}(x)C_3+\delta_{2,4}(X)C_4 \\ & & +\gamma_{2,5}(X)C_5+\omega_{2,6}(X)C_6 \\ 0 & B_3 & C_3+\gamma_{3,5}(X)C_5+\omega_{3,6}(X)C_6 \\ 0 & B_4 & C_4+\gamma_{4,5}(X)C_5+\omega_{4,6}(X)C_6 \\ 0 & 0 & C_5 \\ 0 & 0 & C_6 \end{pmatrix},$$

which means that the condition $\mathrm{Ad}_g \circ \varphi(\mathfrak{l}) \cap \mathfrak{h} = \{0\}$ is equivalent to the fact that $\mathrm{rk}(M_{\varphi,X}) = \dim(\mathfrak{l})$, which is in turn equivalent to

$$\mathrm{rk}(C_6) = \dim(\mathfrak{l}_2), \qquad \mathrm{rk}(B_4) = \dim(\mathfrak{l}_1) \qquad \text{and} \qquad \mathrm{rk}(A_2) = \dim(\mathfrak{l}_0),$$

or

$$\begin{cases} \mathrm{rk}(C_6) < \dim(\mathfrak{l}_2), \ \mathrm{rk}(B_4) = \dim(\mathfrak{l}_1) \ \text{ and } \ \mathrm{rk}(M_{\varphi,X}) = \dim(\mathfrak{l}) & \text{or} \\ \mathrm{rk}(B_4) < \dim(\mathfrak{l}_1), \ \mathrm{rk}(C_6) = \dim(\mathfrak{l}_2) \ \text{ and } \ \mathrm{rk}(M_{\varphi,X}) = \dim(\mathfrak{l}) & \text{or} \\ \mathrm{rk}(C_6) < \dim(\mathfrak{l}_2), \ \mathrm{rk}(B_4) < \dim(\mathfrak{l}_1) \ \text{ and } \ \mathrm{rk}(M_{\varphi,X}) = \dim(\mathfrak{l}). \end{cases}$$

The latter three cases are equivalent to say that

$$\mathrm{rk}(B_4) + \mathrm{rk}(C_6) < \dim(\mathfrak{l}_1 \oplus \mathfrak{l}_2). \qquad \square$$

We are now ready to present our main result in this section.

THEOREM 4. *Let* $\mathfrak{g}$, $\mathfrak{h}$ *and* $\mathfrak{l}$ *be as before. The deformation space reads*

$$\mathscr{T}(\mathfrak{l},\mathfrak{g},\mathfrak{h}) = \bigcup_{i=1}^{2} \bigcup_{\substack{0 \le t \le q \\ 0 \le t' \le q'}} \bigcup_{\substack{\beta \in I(m,m-t) \\ \beta' \in I(m',m'-t')}} \mathscr{T}_{t,t',\beta,\beta',i}(\mathfrak{l},\mathfrak{g},\mathfrak{h}),$$

where for $\beta = (i_1, \dots, i_{m-t})$ *and* $\beta' = (i'_1, \dots, i'_{m'-t'})$, *the set* $\mathscr{T}_{t,t',\beta,\beta',1}$ *is homeomorphic to the semi-algebraic subset in* $\mathscr{L}(\mathfrak{l}, \mathfrak{g})$

$$\mathscr{T}_{t,t',\beta,\beta',1} \simeq \left\{ \varphi(A,B,C) \in Hom(\mathfrak{l},\mathfrak{g}) \;\middle|\; \begin{array}{l} \varphi_1 \in \mathscr{M}^{t,t'}_{\beta,\beta'}(\mathfrak{l},\mathfrak{g}), \\ \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \in V_{\beta'} \ \ and \\ rk\begin{pmatrix} A_2 & 0 & 0 \\ 0 & B_4 & 0 \\ 0 & 0 & C_6 \end{pmatrix} = \dim(\mathfrak{l}) \end{array} \right\}$$

and $\mathscr{T}_{t,t',\beta,\beta',2}$ *is homeomorphic to*

$$\mathscr{T}_{t,t',\beta,\beta',2} \simeq \left\{ \varphi(A,B,C) \in Hom(\mathfrak{l},\mathfrak{g}) \;\middle|\; \begin{array}{l} \varphi_1 \in \mathscr{M}^{t,t'}_{\beta,\beta'}(\mathfrak{l},\mathfrak{g}), \\ \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \in V_{\beta'}, \\ rk(B_4) + rk(C_6) < \dim(\mathfrak{l}_1 \oplus \mathfrak{l}_2) \\ and\ \ rk(M_{\varphi,X}) = \dim(\mathfrak{l})\ \ for\ all\ \ X \in \mathfrak{g} \end{array} \right\}.$$

Proof. Recall first that

$$\mathscr{R}(\mathfrak{l},\mathfrak{g},\mathfrak{h}) = \bigcup_{i=1}^{2} \bigcup_{\substack{0 \le t \le q \\ 0 \le t' \le q'}} \bigcup_{\substack{\beta \in I(m,m-t) \\ \beta' \in I(m',m'-t')}} \mathrm{Hom}^{t,t'}_{\beta,\beta'}(\mathfrak{l},\mathfrak{g}) \cap \mathscr{R}_i.$$

On the other hand, $\mathrm{Hom}^{t,t'}_{\beta,\beta'}(\mathfrak{l},\mathfrak{g}) \cap \mathscr{R}_i$ is a G-invariant set as in formula (5). Hence,

$$\mathscr{T}(\mathfrak{l},\mathfrak{g},\mathfrak{h}) = \bigcup_{i=1}^{2} \bigcup_{\substack{0 \le t \le q \\ 0 \le t' \le q'}} \bigcup_{\substack{\beta \in I(m,m-t) \\ \beta' \in I(m',m'-t')}} (\mathrm{Hom}^{t,t'}_{\beta,\beta'}(\mathfrak{l},\mathfrak{g}) \cap \mathscr{R}_i)/G,$$

and the result follows from Theorem 3. Now to see the semi-algebraicness of $\mathscr{T}_{t,t',\beta,\beta',1}$, we state first the following claim which explains the semi-algebraicness of $\mathrm{Hom}(\mathfrak{l},\mathfrak{g})$ in $\mathscr{L}(\mathfrak{l},\mathfrak{g})$.

Claim 4.1. *The set* $Hom(\mathfrak{l},\mathfrak{g})$ *is algebraic in* $\mathscr{L}(\mathfrak{l},\mathfrak{g})$.

Proof. Let $\{Y_1, \dots, Y_k\}$ be a basis of $\mathfrak{l}$ and $\{X_1, \dots, X_q\}$ a basis of $\mathfrak{g}$. Assume that the Lie brackets of $\mathfrak{l}$ are given by $[Y_i, Y_j] = \sum_{u=1}^{k} c^u_{i,j} Y_u$ for all $1 \le i, j \le k$ and the Lie brackets of $\mathfrak{g}$ are given by $[X_s, X_{s'}] = \sum_{v=1}^{q} d^v_{s,s'} X_v$ for all $1 \le s, s' \le q$. Let now $\varphi \in \mathscr{L}(\mathfrak{l},\mathfrak{g})$ and assume that $\varphi(Y_i) = \sum_{s=1}^{q} a_{s,i} X_s$ for all $1 \le i \le k$. Now

$$\begin{aligned}\mathrm{Hom}(\mathfrak{l},\mathfrak{g}) &= \{\varphi\in\mathscr{L}(\mathfrak{l},\mathfrak{g})\mid\varphi([Y,T])=[\varphi(Y),\varphi(T)],\ \forall Y,T\in\mathfrak{l}\}\\ &= \{\varphi\in\mathscr{L}(\mathfrak{l},\mathfrak{g})\mid\varphi([Y_i,Y_j])=[\varphi(Y_i),\varphi(Y_j)],\ \forall 1\le i,j\le k\}.\end{aligned}$$

As

$$\varphi([Y_i,Y_j])=\sum_{u=1}^{k}c_{i,j}^{u}\varphi(Y_u)=\sum_{u=1}^{k}\sum_{v=1}^{q}c_{i,j}^{u}a_{v,u}X_v=\sum_{v=1}^{q}\left(\sum_{u=1}^{k}c_{i,j}^{u}a_{v,u}\right)X_v$$

and

$$\begin{aligned}[\varphi(Y_i),\varphi(Y_j)] &= \left[\sum_{s=1}^{q}a_{s,i}X_s,\sum_{s'=1}^{q}a_{s',j}X_{s'}\right]=\sum_{s,s'=1}^{q}a_{s,i}a_{s',j}[X_s,X_{s'}]\\ &= \sum_{s,s'=1}^{q}\sum_{v=1}^{q}a_{s,i}a_{s',j}d_{s,s'}^{v}X_v=\sum_{v=1}^{q}\left(\sum_{s,s'=1}^{q}a_{s,i}a_{s',j}d_{s,s'}^{v}\right)X_v,\end{aligned}$$

we get

$$[\varphi(Y_i),\varphi(Y_j)]=\varphi([Y_i,Y_j])\Leftrightarrow\sum_{u=1}^{k}c_{i,j}^{u}a_{v,u}=\sum_{s,s'=1}^{q}a_{s,i}a_{s',j}d_{s,s'}^{v}$$

for any $v=1,\dots,q$. Hence, if we identify $\mathscr{L}(\mathfrak{l},\mathfrak{g})$ to $M_{q,k}(\mathbb{R})$ via the map $\varphi\mapsto M_\varphi=(\varphi(Y_1)|\dots|\varphi(Y_k))$, then

$$\mathrm{Hom}(\mathfrak{l},\mathfrak{g})=\left\{(a_{s,i})_{\substack{1\le s\le q\\1\le i\le k}}\ \middle|\ \sum_{u=1}^{k}c_{i,j}^{u}a_{v,u}=\sum_{s,s'=1}^{q}a_{s,i}a_{s',j}d_{s,s'}^{v},\ 1\le i,j\le k\right\}$$

which is an algebraic set. □

Now we see that the conditions $\begin{pmatrix}C_1\\C_2\end{pmatrix}\in V_{\beta'}$ and $\mathrm{rk}\begin{pmatrix}A_2&0&0\\0&B_4&0\\0&0&C_6\end{pmatrix}=\dim(\mathfrak{l})$ are semi-algebraic conditions and as $\mathrm{Hom}_{1,\beta,\beta'}^{t,t'}(\mathfrak{l},\mathfrak{g})$ is semi-algebraic and the condition $N_{\varphi_1}\in V_\beta$ is an algebraic condition, then by (17) we conclude that $\mathscr{M}_{\beta,\beta'}^{t,t'}(\mathfrak{l},\mathfrak{g})$ is semi-algebraic. □

5. Hausdorfness of the deformation space

This section aims to study the Hausdorfness of the deformation space $\mathscr{T}(\mathfrak{l},\mathfrak{g},\mathfrak{h})$ in the setting where $\mathfrak{g}$ is 3-step nilpotent. Let

$$p : \mathscr{R}(\mathfrak{l}, \mathfrak{g}, \mathfrak{h}) \to \mathrm{Hom}_{\mathrm{I}}(\mathfrak{l}, \mathfrak{g})$$
$$(\varphi_1, C) \mapsto \varphi_1$$

where $\mathrm{Hom}_{\mathrm{I}}(\mathfrak{l}, \mathfrak{g})$ is as in (10) in Section 3.2, φ_1 is as in (5) in Section 3.1 and

$$(\varphi_1, C) \in \mathscr{R}(\mathfrak{l}, \mathfrak{g}, \mathfrak{h}) \subset \mathrm{Hom}_{\mathrm{I}}(\mathfrak{l}, \mathfrak{g}) \times \mathscr{L}(\mathfrak{l}_2, \mathfrak{g}_0).$$

Then p is a G-equivariant map and we can state that:

Theorem 5. *Let $G = \exp \mathfrak{g}$ be a 3-step nilpotent Lie group, $H = \exp \mathfrak{h}$ a closed connected subgroup of G, Γ a discontinuous group for the homogeneous space G/H and $L = \exp \mathfrak{l}$ its syndetic hull. If the dimensions of G-orbits in $\mathscr{R}(\mathfrak{l}, \mathfrak{g}, \mathfrak{h})$ and those in $p(\mathscr{R}(\mathfrak{l}, \mathfrak{g}, \mathfrak{h}))$ are constant respectively, then $\mathscr{T}(\mathfrak{l}, \mathfrak{g}, \mathfrak{h})$ is a Hausdorff space.*

Proof. In such a situation, there is $t \in \{0, \dots, q\}$ and $t' \in \{0, \dots, q'\}$ such that $\mathscr{R}(\mathfrak{l}, \mathfrak{g}, \mathfrak{h}) \subset \mathrm{Hom}^{t,t'}(\mathfrak{l}, \mathfrak{g})$. Indeed, let $\varphi = (\varphi_1, C_\varphi) \in \mathscr{R}(\mathfrak{l}, \mathfrak{g}, \mathfrak{h})$ and assume that $\mathrm{rk}\ l_{\varphi_1} = t$ and $\mathrm{rk}\ f_{\varphi_1} = t'$. As $G * \varphi_1 = \varphi_0 + \mathrm{Im}(l_{\varphi_1})$ we have $\dim G * \varphi_1 = t$ and

$$\begin{aligned}\dim G \cdot \varphi &= \dim G \cdot (\varphi_1, C_\varphi) \\ &= \dim G * \varphi_1 + \dim(\pi^t_\beta(G * \varphi_1), C_\varphi + \mathrm{Im}\ f_{\varphi_1}) \\ &= \dim(\varphi_0 + (N_{\varphi_1} + \mathrm{Im}\ l_{\varphi_1})) + \mathrm{rk}\ f_{\varphi_1} \\ &= \mathrm{rk}\ l_{\varphi_1} + \mathrm{rk}\ f_{\varphi_1} = t + t'.\end{aligned}$$

Since the dimensions of G-orbits of $\mathscr{R}(\mathfrak{l}, \mathfrak{g}, \mathfrak{h})$ and of $p(\mathscr{R}(\mathfrak{l}, \mathfrak{g}, \mathfrak{h}))$ are constant, then so are t and t'. The deformation space is therefore contained in $\mathrm{Hom}^{t,t'}(\mathfrak{l}, \mathfrak{g})/G$, which is a Hausdorff space by Theorem 2. □

6. Illustrating examples

For the convenience of the readers, we close the paper by giving the following series of examples for which the hypotheses of Theorem 5 are met and then the corresponding deformation space turns out to be a Hausdorff space. Let $\mathfrak{g} = \mathbb{R}\text{-span}\{X_0, X_1, X_2, X_3\}$ be the (3-step nilpotent) threadlike Lie algebra, whose pairwise brackets equal zero, except the followings:

$$[X_0, X_i] = X_{i+1}, \qquad i = 1, 2.$$

The center of $\mathfrak{g}$ is the space $\mathbb{R}\text{-span}\{X_3\}$, $\mathfrak{g}_1 = \mathbb{R}\text{-span}\{X_2\}$, $\mathfrak{g}_2 = \mathbb{R}\text{-span}\{X_0, X_1\}$ and for

$$X = x_0X_0 + x_1X_1 + x_2X_2 + x_3X_3 \in \mathfrak{g},$$

we have

$$\mathrm{ad}_X(X_0) = -x_1X_2 - x_2X_3, \qquad \mathrm{ad}_X(X_1) = x_0X_2 \qquad \text{and} \qquad \mathrm{ad}_X(X_2) = x_0X_3.$$

On the other hand, through the basis $\mathscr{B} = \{X_3, X_2, X_1, X_0\}$, the matrices of the endomorphisms ad_X and ad_X^2 are written as:

$$\mathrm{ad}_X = \begin{pmatrix} 0 & x_0 & 0 & -x_2 \\ 0 & 0 & x_0 & -x_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathrm{ad}_X^2 = \begin{pmatrix} 0 & 0 & x_0^2 & -x_0x_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence, the matrix of the adjoint representation $\mathrm{Ad}_{\exp(X)}$ can be expressed as:

$$\mathrm{Ad}_{\exp(X)} = \begin{pmatrix} 1 & x_0 & \frac{1}{2}x_0^2 & -x_2 - \frac{1}{2}x_0x_1 \\ 0 & 1 & x_0 & -x_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and finally

$$\mathrm{Ad}_{\exp(X)} \circ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a + bx_0 + \frac{1}{2}x_0^2c - d(x_2 + \frac{1}{2}x_0x_1) \\ b + cx_0 - dx_1 \\ c \\ d \end{pmatrix}, \tag{20}$$

where the vector ${}^t(a\ b\ c\ d)$ represents a vector of $\mathfrak{g}$ through the basis $\mathscr{B}$.

EXAMPLE 1. Let $\mathfrak{h} = \mathbb{R}\text{-span}\{X_1, X_2, X_3\}$ and $\mathfrak{l} = \mathbb{R}\text{-span}\{X_0\}$. Then if G, H designate the Lie groups associated to $\mathfrak{g}$ and $\mathfrak{h}$ respectively and $\Gamma = \exp(\mathbb{Z}X_0)$, then obviously the resulting Clifford-Klein form $\Gamma\backslash G/H$ turns out to be compact. Clearly $\Gamma \simeq \mathbb{Z}$, $G/H \simeq \mathbb{R}$ and $\Gamma\backslash G/H \simeq S^1$. As such, it is straightforward that through the basis $\mathscr{B}$, any $\varphi \in \mathrm{Hom}(\mathfrak{l}, \mathfrak{g})$ is given by

$$\varphi : \mathfrak{l} \to \mathfrak{g}; \qquad \lambda X_0 \mapsto \lambda(dX_0 + cX_1 + bX_2 + aX_3)$$

and

$$\varphi = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \in \mathscr{R}(\mathfrak{l}, \mathfrak{g}, \mathfrak{h}) \Leftrightarrow d \neq 0. \tag{21}$$

Indeed, from Theorem 3,

$$\mathscr{R}(\mathfrak{l},\mathfrak{g},\mathfrak{h}) = \left\{ \varphi = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \in \mathrm{Hom}(\mathfrak{l},\mathfrak{g}) \,\middle|\, \mathrm{Ad}_{\exp(X)} \circ \varphi(\mathfrak{l}) \oplus \mathfrak{h} = \mathfrak{g} \right\}$$

$$= \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \in \mathrm{Hom}(\mathfrak{l},\mathfrak{g}) \,\middle|\, d \neq 0 \right\}$$

On the other hand and according to our construction, for φ as in equation (21),

$$\varphi_1 = \begin{pmatrix} 0 \\ b \\ c \\ d \end{pmatrix}.$$

By equation (20), we get

$$\mathrm{Ad}_{\exp(X)} \circ \varphi - \varphi = \begin{pmatrix} bx_0 + \frac{1}{2}x_0^2 c - d\left(x_2 + \frac{1}{2}x_0 x_1\right) \\ cx_0 - dx_1 \\ 0 \\ 0 \end{pmatrix}$$

and

$$\mathrm{Ad}_{\exp(X)} * \varphi_1 - \varphi_1 = \begin{pmatrix} 0 \\ cx_0 - dx_1 \\ 0 \\ 0 \end{pmatrix},$$

which means that

$$G_{\varphi} = \left\{ \exp(x_0X_0 + x_1X_1 + x_2X_2 + x_3X_3) \in G \,\middle|\, x_1 = \frac{c}{d}x_0,\ x_2 = \frac{b}{d}x_0 \right\} \qquad \text{and}$$

$$G_{\varphi_1} = \left\{ \exp(x_0X_0 + x_1X_1 + x_2X_2 + x_3X_3) \in G \,\middle|\, x_1 = \frac{c}{d}x_0 \right\}.$$

Finally, $\dim G \cdot \varphi = 2$ and $\dim G * \varphi_1 = 1$ for any $\varphi \in \mathscr{R}(\mathfrak{l},\mathfrak{g},\mathfrak{h})$. Then by Theorem 5, the deformation space $\mathscr{T}(\mathfrak{l},\mathfrak{g},\mathfrak{h})$ is a Hausdorff space. Let us also mention that the explicit description of $\mathscr{T}(\mathfrak{l},\mathfrak{g},\mathfrak{h})$ is given in [2] and [6].

Example 2. Let now $\mathfrak{h} = \mathbb{R}\text{-span}\{X_0\}$ and $\mathfrak{l} = \mathbb{R}\text{-span}\{X_1, X_2, X_3\}$. Then again the resulting Clifford-Klein form $\Gamma\backslash G/H$ is compact. Clearly $\Gamma \simeq \mathbb{Z}^3$, $G/H \simeq \mathbb{R}^3$ and $\Gamma\backslash G/H$ is homeomorphic to the 3-dimensional torus. As $\varphi \in \mathscr{R}(\mathfrak{l},\mathfrak{g},\mathfrak{h})$ if and only if $\varphi(\mathfrak{l}) = \mathfrak{l}$, φ takes the following form:

$$\varphi = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence,

$$\begin{aligned}\mathrm{Ad}_{\exp(X)} \circ \varphi &= \begin{pmatrix} a_1 + b_1x_0 + \frac{1}{2}x_0^2c_1 & a_2 + b_2x_0 + \frac{1}{2}x_0^2c_2 & a_3 + b_3x_0 + \frac{1}{2}x_0^2c_3 \\ b_1 + x_0c_1 & b_2 + x_0c_2 & b_3 + x_0c_3 \\ c_1 & c_2 & c_3 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \varphi + x_0 \begin{pmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{2}x_0^2 \begin{pmatrix} c_1 & c_2 & c_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\end{aligned}$$

and then $\dim G \cdot \varphi = 1$. Besides,

$$\varphi_1 = \begin{pmatrix} 0 & 0 & 0 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\mathrm{Ad}_{\exp(X)} * \varphi_1 = \varphi_1 + x_0 \begin{pmatrix} 0 & 0 & 0 \\ c_1 & c_2 & c_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and likewise $\dim G * \varphi_1 = 1$ for any $\varphi \in \mathscr{R}(\mathfrak{l},\mathfrak{g},\mathfrak{h})$. Then by Theorem 5, the deformation space is a Hausdorff space.

Example 3. The following example treats a non-compact Clifford-Klein form case. Let $\mathfrak{h} = \mathbb{R}\text{-span}\{X_3\}$ and $\mathfrak{l} = \mathbb{R}\text{-span}\{X_1, X_2\}$. Clearly $\Gamma \simeq \mathbb{Z}^2$

and $G/H \simeq \mathbb{R}^3$. Therefore $\Gamma\backslash G/H$ is not compact. We first prove the following:

Claim 6.1. *For any* $\varphi \in \mathscr{R}(\mathfrak{l}, \mathfrak{g}, \mathfrak{h})$, *we have* $\varphi(\mathfrak{l}) \subset \mathbb{R}$-*span*$\{X_1, X_2, X_3\}$.

Proof. If not, there exists $v = X_0 + u \in \varphi(\mathfrak{l})$ for some $u \in \mathbb{R}\text{-span}\{X_1, X_2, X_3\}$. Since $\dim(\varphi(\mathfrak{l})) = 2$, there exists $w \neq 0$ such that $w \in \varphi(\mathfrak{l}) \cap \mathbb{R}\text{-span}\{X_1, X_2, X_3\}$. Thus $\mathbb{R}\text{-span}\{[v,[v,w]], [v,w], w\} \cap \mathfrak{h} \neq \{0\}$, which leads to a contradiction as $\varphi(\mathfrak{l}) \cap \mathfrak{h} = \{0\}$. □

Now, any $\varphi \in \mathscr{R}(\mathfrak{l}, \mathfrak{g}, \mathfrak{h})$ reads:

$$\varphi = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \\ 0 & 0 \end{pmatrix} \tag{22}$$

and we have the following:

Claim 6.2. *Let* $\varphi \in \mathscr{R}(\mathfrak{l}, \mathfrak{g}, \mathfrak{h})$ *be as in equation* (22), *then* $(c_1, c_2) \neq (0,0)$.

Proof. From Claim 6.1 any $\varphi \in \mathscr{R}(\mathfrak{l}, \mathfrak{g}, \mathfrak{h})$ reads $\varphi = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \\ 0 & 0 \end{pmatrix}$. Now from Theorem 3,

$$\varphi \in \mathscr{R}(\mathfrak{l}, \mathfrak{g}, \mathfrak{h}) \Leftrightarrow \dim \mathfrak{h} + \dim \varphi(\mathfrak{l}) = 3,$$

$$\varphi \in \mathscr{R}(\mathfrak{l}, \mathfrak{g}, \mathfrak{h}) \Leftrightarrow \det\begin{pmatrix} b_1 & b_2 \\ c_1 & c_2 \end{pmatrix} \neq 0.$$ □

Now for $\varphi \in \mathscr{R}(\mathfrak{l}, \mathfrak{g}, \mathfrak{h})$ be as in equation (22),

$$\mathrm{Ad}_{\exp(X)} \circ \varphi = \varphi + x_0 \begin{pmatrix} b_1 & b_2 \\ c_1 & c_2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{2} x_0^2 \begin{pmatrix} c_1 & c_2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and then $\dim G \cdot \varphi = 1$. It is also obviously the case for

$$\varphi_1 = \begin{pmatrix} 0 & 0 \\ b_1 & b_2 \\ c_1 & c_2 \\ 0 & 0 \end{pmatrix}$$

as

$$\mathrm{Ad}_{\exp(X)} * \varphi_1 = \varphi_1 + x_0 \begin{pmatrix} 0 & 0 \\ c_1 & c_2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then by Theorem 5, the deformation space is a Hausdorff space.

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