Unitary Equivalence of Self-Adjoint Operators and Constants of Motion.

By

Fumitomo MAEDA.

(Received April 20, 1936.)

In wave mechanics, the state of a dynamical system is represented at each instant of time by a wave function $\psi(q, t)$ which is the solution of

$$\frac{h}{2\pi i}\frac{\partial\psi}{\partial t} = -\boldsymbol{H}\psi\,. \tag{1}$$

When the Hamiltonian $\boldsymbol{H}$ does not contain t explicitly, we can write down the solution of (1), namely

$$\psi(q, t) = e^{-\frac{2\pi i}{h}\boldsymbol{H}t}\psi(q, 0)\,.$$

Since $e^{-\frac{2\pi i}{h}\boldsymbol{H}t}$ is a unitary operator, the function space representing the states at time t is obtained by operating the unitary operator $\boldsymbol{U}_t = e^{-\frac{2\pi i}{h}\boldsymbol{H}t}$ on the function space representing the states at time 0. Hence we may say that the motion of the dynamical system is represented by the unitary movement of the function space.

When the Hamiltonian $\boldsymbol{H}$ contains t explicitly, it is already proved that if $\{\psi_i(q, 0)\}$ is a complete normalised orthogonal system in the function space, then $\{\psi_i(q, t)\}$ is also a complete normalized orthogonal system in the function space, where $\psi_i(q, t)$ is the solution of (1) which is $\psi_i(q, 0)$ when $t = 0$.[(1)] Hence in this case also, we may consider the unitary movement of the function space.

Now the constants of motion are introduced as follows: An observable $\boldsymbol{A}$ is said to be a constant of motion when the elements a_{ij} of the matrix representing $\boldsymbol{A}$, is constant. When $\boldsymbol{H}$ does not contain t explicitly, $\boldsymbol{A}$ being independent of t, a_{ij} is defined by

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(1) V. Fock, Zeitschrift für Physik, **49** (1928), 323.

$$a_{ij} = \left(A\psi_j(q, t), \psi_i(q, t)\right),$$

where $\{\psi_i(q, t)\}$ is the complete system of eigenfunctions of $\boldsymbol{H}$, which satisfy (1). And it is proved that $\boldsymbol{A}$ is a constant of motion when and only when

$$\boldsymbol{AH} = \boldsymbol{HA}\,.\text{[1]} \tag{2}$$

And when $\boldsymbol{H}$ contains t explicitly, a_{ij} is defined by

$$a_{ij} = \left(A\psi_j(q, t), \psi_i(q, t)\right),$$

where $\{\psi_i(q, t)\}$ is a complete normalized orthogonal system of the solutions of (1), which are eigenfunctions $\psi_i(q, 0)$ of $\boldsymbol{H}$ when $t = 0$. And it is proved that $\boldsymbol{A}$ is a constant of motion when and only when

$$\frac{\partial \boldsymbol{A}}{\partial t} = \frac{2\pi i}{h}(\boldsymbol{AH}-\boldsymbol{HA})\,.\text{[2]} \tag{3}$$

From (2) and (3), the physical significance of the constant of motion is obtained as follows:

(i) The possible values of $\boldsymbol{A}$ are independent of time.

(ii) The probability of this possible value is also independent of time for each state.[3]

From a mathematical point of view, the above treatments are incomplete. For, first, only the cases where $\boldsymbol{H}$ and $\boldsymbol{A}$ have discrete spectra are considered, and secondly, the domains of $\boldsymbol{H}$ and $\boldsymbol{A}$ are not considered. But to treat these problems in the above way rigorously according to the standards of pure mathematics is very difficult.

Hence in this paper, considering the motion of the dynamical system as the unitary movement $\boldsymbol{U}_t$ of the Hilbert space, I define the constant of motion from its physical significance. And I prove that an observable $\boldsymbol{A}_t$, which generally depends on t, is a constant of motion when and only when

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(1) Cf. L. de Broglie, *Théorie de la Quantification dans la nouvelle Mécanique*, (1932), 204–206.

(2) Cf. ibid., 227–228.

(3) Cf. ibid., 207–209; 229–235.

$$A_t = U_t A_0 U_t^{-1}.^{(1)} \tag{4}$$

And next I show that when $U_t = e^{-\frac{2\pi i}{h}Ht}$, an observable A, which is independent of t, is a constant of motion when and only when A and H are permutable in the strict mathematical sense.(2)

In order to find these results, I first consider the unitary equivalence of two self-adjoint operators A_1 and A_2. I prove that the relation

$$A_2 = UA_1U^{-1}$$

where U is a unitary operator, holds when and only when

$$\| E_2(U)\mathfrak{f} \| = \| E_1(U)U^{-1}\mathfrak{f} \|$$

holds for all U and $\mathfrak{f}$, where $E_1(U)$ and $E_2(U)$ are resolutions of identity which correspond to A_1 and A_2 respectively.

Unitary Equivalence of Self-Adjoint Operators.

1. Let A_1 and A_2 be two self-adjoint operators in the abstract Hilbert space $\mathfrak{H}$, and let $E_1(U)$ and $E_2(U)$ be the corresponding resolu-

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(1) This expression means that when $\mathfrak{f}$ is in the domain of A_t, $U_t^{-1}\mathfrak{f}$ is in the domain of A_0 and $A_t\mathfrak{f} = U_t A_0 U_t^{-1}\mathfrak{f}$, and vice versa.

If we permit the formal calculation, we can show that (4) is equivalent to (3), as follows: Differentiate (4) by t,

$$\frac{\partial A_t}{\partial t} = \frac{\partial U_t}{\partial t} A_0 U_t^{-1} + U_t A_0 \frac{\partial U_t^{-1}}{\partial t},$$

by (4)
$$= \frac{\partial U}{\partial t} U_t^{-1} A_t + A_t U_t \frac{\partial U_t^{-1}}{\partial t}.$$

Since $U_t U_t^{-1} = \mathbf{1}$ we have $\frac{\partial U_t}{\partial t} U_t^{-1} + U_t \frac{\partial U_t^{-1}}{\partial t} = 0$. Hence

$$\frac{\partial A_t}{\partial t} = \frac{\partial U_t}{\partial t} U_t^{-1} A_t - A_t \frac{\partial U_t}{\partial t} U_t^{-1}.$$

Consequently, from the equation of motion which may be written

$$\frac{\partial U_t}{\partial t} = -\frac{2\pi i}{h} H_t U_t,$$

we have (3). Reversing this calculation, we have (4) from (3).

(2) That is, $E(U)F(U') = F(U')E(U)$ for all U and U', $E(U)$ and $F(U)$ being the resolutions of identity which correspond to A and H respectively. (Cf. M. H. Stone, *Linear Transformations in Hilbert Space*, (1932), 301.)

tions of identity which are defined for all open intervals and points in the space of real numbers R_1.[1] Then it is already known that the relation

$$A_2 = UA_1U^{-1},$$

where U is a unitary operator, holds when and only when the relation

$$E_2(U) = UE_1(U)U^{-1}$$

holds for all U.[2]

Now I prove the following lemma:

Let E_1 and E_2 be two projections. In order that

$$E_2 = UE_1U^{-1} \tag{1}$$

holds for a unitary operator U, it is necessary and sufficient that

$$\| E_2\mathfrak{f} \| = \| E_1U^{-1}\mathfrak{f} \| \tag{2}$$

holds for every $\mathfrak{f}$ in $\mathfrak{H}$.

The necessity of (2) is evident.

To prove the sufficiency of (2), let E_2 be the projection on the closed linear manifold $\mathfrak{M}$. Take any vector $\mathfrak{f}$ in $\mathfrak{M}$, then since $E_2\mathfrak{f} = \mathfrak{f}$, we have by (2) $\| E_1U^{-1}\mathfrak{f} \| = \| \mathfrak{f} \|$. Hence

$$\| (1-UE_1U^{-1})\mathfrak{f} \|^2 = \| (1-E_1)U^{-1}\mathfrak{f} \|^2 = \left((1-E_1)U^{-1}\mathfrak{f}, U^{-1}\mathfrak{f}\right)$$
$$= \| U^{-1}\mathfrak{f} \|^2 - \| E_1U^{-1}\mathfrak{f} \|^2 = 0.$$

That is
$$\mathfrak{f} = UE_1U^{-1}\mathfrak{f}. \tag{3}$$

Next take any vector $\mathfrak{f}$ in $\mathfrak{H} \ominus \mathfrak{M}$, then $E_2\mathfrak{f} = 0$. Hence by (2)

$$E_1U^{-1}\mathfrak{f} = 0 \quad \text{and} \quad UE_1U^{-1}\mathfrak{f} = 0. \tag{4}$$

Since every vector $\mathfrak{f}$ in $\mathfrak{H}$ can be written as a sum

$$\mathfrak{f} = \mathfrak{f}_1 + \mathfrak{f}_2 \qquad \text{where } \mathfrak{f}_1 \in \mathfrak{M},\ \mathfrak{f}_2 \in \mathfrak{H} \ominus \mathfrak{M},$$

from (3) and (4) we have

$$E_2\mathfrak{f} = UE_1U^{-1}\mathfrak{f}.$$

(1) Cf. F. Maeda, this volume, 39.

(2) We can prove this as Stone did (Stone, loc. cit.. 242), with slight modifications. Cf. F. Maeda, this volume, 45.

From this lemma, the following theorem follows:

The relation $$A_2 = UA_1U^{-1}$$

holds when and only when

$$\| E_2(U)\mathfrak{f} \| = \| E_1(U)U^{-1}\mathfrak{f} \|$$

for all U *and* $\mathfrak{f}$ *in* $\mathfrak{H}$.

Constants of Motion in the Unitary Movement of Hilbert Space.

2. From the theory of quantum mechanics, the state of a dynamical system at each instant of time is represented by a vector $\mathfrak{f}$ in a Hilbert space $\mathfrak{H}$, and if the representative vector of a state is multiplied by any number, not zero, the resulting vector will represent the same state. And each observable is represented by a self-adjoint operator that can operate on the vectors of $\mathfrak{H}$.[1]

Now we consider the connexion between different instants of time.[2] For this purpose we give the following assumption for the law of variation of $\mathfrak{f}$ with the time t. Let the state at time t_1, represented by $\mathfrak{f}_{t_1}$, vary with the time, and it is represented by $\mathfrak{f}_{t_2}$, at time t_2, so that

$$\mathfrak{f}_{t_2} = U_{(t_1, t_2)}\mathfrak{f}_{t_1}, \tag{1}$$

where $U_{(t_1,t_2)}$ is a unitary operator which depends on the time interval (t_1, t_2). Formula (1) shows how all the states of our system vary with the time and it may be considered as the equation of motion in the general form. If we give a special condition to $U_{(t_1,t_2)}$ we have a special motion of the dynamical system.

When $U_{(t_1,t_2)}$ depends only on the length of the time interval and not on the position of the time interval, then we have

$$U_{s+t} = U_sU_t,$$

where U_t means $U_{(0,t)}$. Hence U_t forms a one-parameter group of unitary operators. If U_t is continuous with respect to t, that is,

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(1) P. A. M. Dirac, *The Principles of Quantum Mechanics*, sec. ed. (1935), 20–30. F. Maeda, this volume, 127.

(2) In this paper, the time t is considered as a parameter, and not an operator.

$$\lim_{s \to t} \| \boldsymbol{U}_s \mathfrak{f} - \boldsymbol{U}_t \mathfrak{f} \| = 0$$

for all $\mathfrak{f}$ in $\mathfrak{H}$, then we can prove, as J. v. Neumann did in the proof of Stone's theorem,[1] there exists a self-adjoint operator $\boldsymbol{R}$ so that

$$[\lim]_{\Delta t \to 0} \frac{\boldsymbol{U}_{t+\Delta t}\mathfrak{f} - \boldsymbol{U}_t \mathfrak{f}}{\Delta t} \text{ (2)}$$

exists when and only when $\boldsymbol{U}_t \mathfrak{f}$ is in the domain of $\boldsymbol{R}$ and is equal to $i\boldsymbol{R}\boldsymbol{U}_t\mathfrak{f}$. If we put $\boldsymbol{U}_t\mathfrak{f} = \mathfrak{f}_t$, then it becomes

$$[\lim]_{\Delta t \to 0} \frac{\mathfrak{f}_{t+\Delta t} - \mathfrak{f}_t}{\Delta t} = i\boldsymbol{R}\mathfrak{f}_t .$$

The left hand side may be written as $\frac{d\mathfrak{f}_t}{dt}$, and if we put $-\frac{2\pi}{h}\boldsymbol{H}$ instead of $\boldsymbol{R}$, then we have

$$\frac{h}{2\pi i} \frac{d\mathfrak{f}_t}{dt} = -\boldsymbol{H}\mathfrak{f}_t . \tag{2}$$

Thus we have Schrödinger's form of the equation of motion with constant Hamiltonian $\boldsymbol{H}$. And in this case, by Stone's theorem $\boldsymbol{U}_t$ is expressed by $e^{-\frac{2\pi i}{h}\boldsymbol{H}t}$.

Generally $\boldsymbol{U}_t$ is not a one-parameter group with continuity property, and we can not have the equation of motion (2) with constant $\boldsymbol{H}$. But in some cases, $\boldsymbol{U}_t$ varies in the neighbourhood of t as follows

$$[\lim]_{\Delta t \to 0} \frac{\boldsymbol{U}_{(t,t+\Delta t)}\mathfrak{f} - \boldsymbol{U}'_{(t,t+\Delta t)}t}{\Delta t} = 0$$

for all $\mathfrak{f}$ in $\mathfrak{H}$, where $\boldsymbol{U}'_{(t,t+\Delta t)} = e^{-\frac{2\pi i}{h}\boldsymbol{H}_t \Delta t}$, $\boldsymbol{H}_t$ being a self-adjoint operator which varies with t. Then, as above we have

$$\frac{h}{2\pi i} \frac{d\mathfrak{f}_t}{dt} = -\boldsymbol{H}_t\mathfrak{f}_t .$$

This is Schrödinger's form of the equation of motion where the Hamiltonian $\boldsymbol{H}_t$ varies with the time.

(1) Cf. J. v. Neumann, Annals of Math., (2) **33** (1932), 571.

(2) [lim] means the strong convergence of the limit.

3. Let A_t be an observable which generally depends on the time t, and $E_t(U)$ be the corresponding resolution of identity. Then the probability of A_t having a value lying within a specified range U for the state $U_t\mathfrak{f}$, is $\| E_t(U)U_t\mathfrak{f} \|^2$, provided $\mathfrak{f}$ is normalized.[1] Hence if this probability is independent of t, that is,

$$\| E_t(U)U_t\mathfrak{f} \| = \| E_0(U)\mathfrak{f} \|$$

for all U and $\mathfrak{f}$ in $\mathfrak{H}$, then we say that A_t is a *constant of motion* in U_t.

Consequently, from sec. 1, we have the following result:

In order that A_t may be a constant of motion in U_t, it is necessary and sufficient that

$$A_t = U_t A_0 U_t^{-1}.$$

In the special cases where $U_t = e^{-\frac{2\pi i}{h}Ht}$, $[\lim\limits_{t\to 0}] \dfrac{U_t\mathfrak{f}-\mathfrak{f}}{t}$ exists when and only when $\mathfrak{f}$ belongs to the domain of H and is equal to $-\dfrac{2\pi i}{h}H\mathfrak{f}$.[2] Let the observable A be independent of t, and let $E(U)$ be the resolution of identity which corresponds to A. Then by sec. 1, A is a constant of motion when and only when

$$E(U) = U_t E(U) U_t^{-1}. \qquad (1)$$

When $\mathfrak{f}$ is any vector in the domain of H, we have by (1)

$$-\frac{2\pi i}{h}E(U)H\mathfrak{f} = [\lim_{t\to 0}]\frac{E(U)U_t\mathfrak{f}-E(U)\mathfrak{f}}{t} = [\lim_{t\to 0}]\frac{U_tE(U)\mathfrak{f}-E(U)\mathfrak{f}}{t}.$$

Hence $E(U)\mathfrak{f}$ is in the domain of H, and

$$E(U)H\mathfrak{f} = HE(U)\mathfrak{f},$$

that is $E(U)$ and H are permutable.[3] Consequently, when we denote by $F(U)$ the resolution of identity which corresponds to H, then

$$E(U)F(U') = F(U')E(U) \qquad (2)$$

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(1) Cf. J. v. Neumann, loc. cit., 104.

(2) Cf. sec. 2.

(3) For the definition of permutability, cf M. H. Stone, loc. cit., 299. For this result, cf. M. H. Stone, Annals of Mathematics, (2) **33** (1932), 647.

for all U and U'.(1) In this case, we say, by the definition,(2) that $\boldsymbol{A}$ and $\boldsymbol{H}$ are permutable.

Conversely, when (2) holds, then since

$$\boldsymbol{U}_t\mathfrak{f} = \int_{R_1} e^{-\frac{2\pi i}{h}\lambda t} \boldsymbol{F}(dU)\mathfrak{f},$$

we have, by the definition of the integral,(3)

$$\boldsymbol{E}(U)\boldsymbol{U}_t\mathfrak{f} = \int_{R_1} e^{-\frac{2\pi i}{h}\lambda t} \boldsymbol{F}(dU)\boldsymbol{E}(U)\mathfrak{f} = \boldsymbol{U}_t\boldsymbol{E}(U)\mathfrak{f}.$$

That is, (1) holds.

Hence, *when* $\boldsymbol{U}_t = e^{-\frac{2\pi i}{h}\boldsymbol{H}t}$, *an observable* $\boldsymbol{A}$, *which is independent of* t, *is a constant of motion in* $\boldsymbol{U}_t$, *when and only when* $\boldsymbol{A}$ *and* $\boldsymbol{H}$ *are permutable.*

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(1) Cf. M. H. Stone, *Linear Transformations in Hilbert Space,* (1932), 300.

(2) Ibid., 301.

(3) F. Maeda, this volume, 34.