On Wave Geometry. (Continued).

By

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§ 1. Introduction.

In my previous paper,[1] from the equation:

$$(\overline{ds}\psi)_x = (I + \Lambda_m \delta x^m)(ds\psi)_{x+\delta x} \tag{1.1}$$

in which the parallel displacements makes $ds\psi = 0$ invariant, we obtained the fundamental equation for ψ:

$$\frac{\partial \psi}{\partial x^m} = (-\Lambda_m + 2T^{\lambda}_{m}\gamma_\lambda + R_m I)\psi\,. \tag{1.2}$$

We then proceeded with the calculation that the expression:

$$\frac{1}{g_{ll}}\left(\gamma_l \Gamma^i_{lm} - \gamma_l \Lambda_m \gamma_l - \gamma_l \frac{\partial \gamma_l}{\partial x^m}\right) \qquad \text{(not summing by } l\text{)}$$

which is the coefficient of ψ in the right hand side of the equation:

$$\frac{\partial \psi}{\partial x^m} = \frac{1}{g_{ll}}\left(\gamma_l \Gamma^i_{lm} - \gamma_l \Lambda_m \gamma_l - \gamma_l \frac{\partial \gamma_l}{\partial x^m}\right)\psi \qquad \text{(not summing by } l\text{)} \tag{1.3}$$

is independent of the suffix l. In a later section again, we arrived at the same result from the invariancy of Λ_m for constant gauge transformations.

In this paper we will begin with (1.1) generally without putting any assumption.

In order to classify quantities in space exactly, hereafter we say the quantities, which are invariant by G, S-transformations, the quantities on the Ma-side of space e. g. g_{ij}, dx^i, Γ^i_{jk} etc.; and the quantities

(1) This Journal 5 (1935), 151.

which undergo S or G-transformation, the quantities of the Mi-side of space e. g. γ_i, ψ, Λ_m etc.

§ 2. The general fundamental equation for ψ and its integrability.

In the expression $ds\psi \equiv dx^i\gamma_i\psi$, putting $\gamma_i\psi = \Psi_i$ we consider $dx^i\Psi_i$ as the microscopic metric.

From the original equation of (1.3):

$$\gamma_l \frac{\partial\psi}{\partial x^m} = \left(\gamma_i \Gamma^i_{lm} - \Lambda_m \gamma_l - \frac{\partial \gamma_l}{\partial x^m}\right)\psi$$

we have easily the following equations:

$$\frac{\partial \Psi_i}{\partial x^m} = (-\Lambda_m \delta^j_i + \Gamma^j_{im})\Psi_j \tag{2.1}$$

and

$$\Psi_i = \gamma_i \psi^{(1)}, \tag{2.2}$$

It is easily seen that the above equations are invariant by the C, G, S-transformations.[2]

Thus we shall take (2.1) and (2.2) together as the fundamental equations for ψ instead of (1.2).

But if we put

$$\mathring{\gamma}_\lambda \psi = \begin{pmatrix} \mathring{\psi}_{\lambda 1} \\ \cdot \\ \cdot \\ \mathring{\psi}_{\lambda 4} \end{pmatrix} \qquad (\lambda = 1, 2, \ldots, 5),$$

it is to be noticed that the rank of the matrix of $\| \mathring{\psi}_{\lambda i} \|$ is less than 4. Therefore we must select the solution Ψ_i $(i = 1, 2, \cdot, 4)$ so that the rank of $\| \psi_{ik} \|$ is less than 4; if such a solution Ψ_i is obtained, then from this Ψ_i we can determine the values of γ_i and ψ so that $\Psi_i = \gamma_i\psi$. So we have the result: Given the coefficients of connection in the

(1) $\gamma_i = U h^\lambda_i \mathring{\gamma}_\lambda U^{-1}$.

(2) This Journal **5** (1935), 161.

Ma-side of space Γ^i_{jk} and the multiplier of parallel displacement Λ_m, we can determine γ_i and ψ from the equations (2.1) and (2.2).[1]

Let us now find the condition for integrability of (2.1) and (2.2).

Calculating the condition $\frac{\partial^2 \Psi_i}{\partial x^{[l} \partial x^{m]}}$, we have

$$\left\{R_{lmi}^{\cdot\cdot\cdot j} - 2\left(\frac{\partial \Lambda_{[l}}{\partial x^{m]}} - \Lambda_{[l}\Lambda_{m]}\right)\delta^j_i\right\}\Psi_j = 0 \tag{2.3}$$

and differentiating the above equation and using it again we have

$$\left.\begin{aligned}
&\left\{R_{lmi,r}^{\cdot\cdot\cdot j} - 2\left(\frac{\partial \Lambda_{[l}}{\partial x^{m]}} - \Lambda_{[l}\Lambda_{m]}\right)_{,r}\delta^j_i\right\}\Psi_j = 0 \\
&\left\{R_{lmi,rq}^{\cdot\cdot\cdot j} - 2\left(\frac{\partial \Lambda_{[l}}{\partial x^{m]}} - \Lambda_{[l}\Lambda_{m]}\right)_{,rq}\delta^j_i\right\}\Psi_j = 0 \\
&\cdots\cdots
\end{aligned}\right\} \tag{2.4}$$

where R_{lmij} is the component of the curvature tensor with respect to Γ^i_{jk}, and the notation (,) denotes the Φ-spinor covariant derivative with respect to Γ^i_{jk} and Λ_m.[2] We can also show that

$$\frac{\partial \Lambda_{[m}}{\partial x^{l]}} - \Lambda_{[l}\Lambda_{m]}$$

is covariant with respect to the suffices l and m by the C-transformation and that it is a spinor by the S-transformation namely

$$\left(\frac{\partial \Lambda_{[m}}{\partial x^{l]}} - \Lambda_{[l}\Lambda_{m]}\right) \longrightarrow S\left(\frac{\partial \Lambda_{m]}}{\partial x^{l]}} - \Lambda_{[l}\Lambda_{m]}\right)S^{-1}.^{(3)}$$

So we have the following result: *The condition that* (2.1) *and* (2.2) *have solutions is that the resulting equations obtained by putting* $\Psi_i = \gamma_i \psi$ *in* (2.3) *and* (2.4) *are compatible.*

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(1) Especially, it is noted that in this case $\frac{1}{g_{ll}}\gamma\left(-\Lambda_m\gamma_l + \Gamma^i_{lm} - \frac{\partial \gamma_l}{\partial x^m}\right)$ is not necessarily independent of l.

(2) If we write $\frac{\partial \Lambda_m}{\partial x^l} - \frac{\partial \Lambda_l}{\partial x^m} - \Lambda_l\Lambda_m + \Lambda_m\Lambda_l = \Lambda_{ml}$, we have

$$\Lambda_{ml,r} \equiv \frac{\partial \Lambda_{ml}}{\partial x^r} - \Gamma^p_{mr}\Lambda_{pl} - \Gamma^p_{lr}\Lambda_{mp} + \Lambda_r\Lambda_{ml} - \Lambda_{ml}\Lambda_r$$

(3) This Journal **5** (1935), 161.

If we call v^i, expressed by $dx^l\Psi_l \equiv \begin{pmatrix} v^1 \\ \vdots \\ v^4 \end{pmatrix}$, a metric component of the *Mi*-side of space for dx^l, the equation (1.1) shows that when dx^l undergoes a parallel displacement with the coefficients of connection Γ^i_{jk}, the corresponding metric component v^i is displaced parallel with the coefficients of connection

$$\Lambda_m \equiv \begin{pmatrix} \Lambda^1_{1m} & \Lambda^2_{1m} & \cdot & \Lambda^4_{1m} \\ \cdot & \cdot & \cdot & \\ \Lambda^1_{4m} & \cdot & \cdot & \Lambda^4_{4m} \end{pmatrix}$$

This shows that Λ_{mn} is regarded as the curvature tensor[1] in metric space on the *Ma*-side of space. Hence (2.3) and (2.4) together are the equations which define *the relation between the curvatures of the Ma-side of space and that of the metric space of the Mi-side of space.*

For convenience sake in future applications, if we write (2.3) and (2.4) in another form, we have after some calculation,[2] the following equations:

$$\Big[R_{lmi}{}^{j} + \Big\{\Big(\overset{*}{R}_{lm}{}^{\lambda\mu} - \frac{2}{4}A^{\lambda}_{[l}A^{\mu}_{m]}\Big)\gamma_\lambda\gamma_\mu + \frac{2}{4}(\overset{\circ}{\nabla}_{[m}A^{\lambda}_{l]})\gamma_\lambda + \frac{2}{4}\overset{\circ}{\nabla}_{[m}A_{l]} + \frac{2}{4}A^{\lambda\mu}_{l}A^{\nu}_{m}\gamma_\lambda g_{\nu\mu}\Big\}\delta^j_i\Big]\gamma_j\Psi = 0, \tag{2.5}$$

$$\left.\begin{aligned} &\Big[R_{lmi}{}^{j}{}_{;r} + \Big\{\Big(\overset{*}{R}_{lm}{}^{\lambda\mu} - \frac{2}{4}A^{\lambda}_{l}A^{\mu}_{m}\Big)_{;r}\gamma_\lambda\gamma_\mu + \frac{2}{4}(\overset{\circ}{\nabla}_{[m}A^{\lambda}_{l]})_{;r}\gamma_\lambda \\ &\qquad + \frac{2}{4}(\overset{\circ}{\nabla}_{[m}A_{l]})_{;r}I + \frac{2}{4}(A^{\lambda\mu}_{[l}A^{\nu}_{m]}g_{\nu\mu})_{;r}\gamma_\lambda\Big\}\delta^j_i\Big]\gamma_j\Psi = 0 \\ &\cdots\cdots \\ &\cdots\cdots \end{aligned}\right\} \tag{2.6}$$

where $\overset{*}{R}_{lm}{}^{\lambda\mu}$ is the component of the curvature tensor with respect to

$$\overset{*}{\Gamma}{}^{\lambda}_{\mu m} = \left\{{\lambda \atop \mu m}\right\} + \frac{1}{4}A^{\omega\lambda}_{m}g_{\omega\lambda}$$

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(1) See foot-note (2) in p. 105.

(2) See **Note 1**, p. 109.

and $A^{\omega\lambda}_{m}$, A^{λ}_{m}, A_m are the functions defined by the relation

$$\Lambda_m = -\Gamma_m - \frac{1}{4}A_m{}^{(1)}$$

$$A_m = A^{\lambda\mu}_{m}\gamma_\lambda\gamma_\mu + A^{\lambda}_{m}\gamma_\lambda + A_m I,$$

and the notation (;) expresses the covariant derivative with respect to $\overset{*}{\Gamma}{}^{\lambda}_{\mu m}$

Especially, it is noted that all the results above obtained hold good in the case in which the dimension is 5 by putting the Greek suffices instead of the latin ones.

§ 3. Special cases.

1) When (2.1) and (2.2) are completely integrable, from (2.5) we have

$$\left.\begin{aligned} R_{lmij} &= -\frac{1}{4}\left(\frac{\partial A_l}{\partial x^m} - \frac{\partial A_m}{\partial x^l}\right)g_{ij} \\ \overset{*}{R}_{lm\nu\mu} &= \frac{1}{4}(A_{l\nu}A_{m\omega} - A_{l\omega}A_{m\nu}) \\ \overset{\circ}{\nabla}_{[m}A^{\lambda}_{l]} &= -A^{\lambda\mu}_{[l}A^{\nu}_{m]}g_{\nu\mu} \end{aligned}\right\} \qquad (3.1)$$

2) When in (2.5) and (2.6) all the coefficients of ψ are linear and homogeneous in γ_i, we have

$$\left.\begin{aligned} &P_{lmij}P_{l'm'i'}^{\cdot\;\cdot\;\cdot\;j} = 0\,, \quad P_{lmij;r..}P_{l'm'i';q..}^{\cdot\;\cdot\;\cdot\;j} = 0\,,{}^{(2)} \\ &\overset{*}{R}_{lm\nu\omega} = \frac{1}{4}(A_{l\nu}A_{m\omega} - A_{l\omega}A_{m\nu})\,, \\ &\overset{\circ}{\nabla}_{[m}A^{\lambda}_{l]} = -A^{\lambda\mu}_{[l}A^{\nu}_{m]}g_{\nu\mu} \end{aligned}\right\} \qquad (3.2)$$

where

$$P_{lmij} = R_{lmij} + \frac{1}{4}\left(\frac{\partial A_l}{\partial x^m} - \frac{\partial A_m}{\partial x^l}\right).$$

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(1) Γ_m is the quantity defined by the identity

$$\frac{\partial\gamma_\lambda}{\partial x^l} = \left\{{}^{\mu}_{\lambda l}\right\}\gamma_\mu + \Gamma_l\gamma_\lambda - \gamma_\lambda\Gamma_l\,.$$

(2) See **Note 2,** p. 110.

Specially, when the *Ma*-side of space is 5 dimensional and comforms (though not necessarily symmetric) namely when $\nabla_\lambda g_{\mu\nu} = \theta_\lambda g_{\mu\nu}$, (3.2) is reduced to the following form[1]:

$$\left.\begin{aligned} &P_{\lambda\mu\nu\omega} = 0 \qquad \text{or} \qquad R_{\lambda\mu\nu\omega} = -\frac{1}{4}\left(\frac{\partial A_\lambda}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\lambda}\right), \\ &\overset{*}{R}_{\lambda\mu\nu\omega} = \frac{1}{4}(A_{\lambda\nu}A_{\mu\omega} - A_{\lambda\omega}A_{\mu\nu}), \\ &\overset{\circ}{\nabla}_{[\mu} A^\nu_{\lambda]} = -A^{\nu\omega}_{[\lambda} A^\epsilon_{\mu]} g_{\omega\epsilon} \end{aligned}\right\} \quad (3.3)$$

So the case becomes to the case (1).

Special case i). In the special case of the above when $A_\lambda = A_\lambda I$, (3.3) becomes

$$R_{\lambda\mu\nu\omega} = \left(\frac{\partial \theta_\mu}{\partial x^\lambda} - \frac{\partial \theta_\lambda}{\partial x^\mu}\right) g_{\nu\omega}\,. \text{[2]} \qquad (3.4)$$

Special case ii). If θ_λ is a gradient vector in case i), we have

$$R_{\lambda\mu\nu\omega} = 0\,. \text{[3]}$$

Special case iii). When the *Ma*-side of space is real, i.e. $\Gamma^\lambda_{\mu\nu}$ are all real, we also have

$$R_{\lambda\mu\nu\omega} = 0\,.$$

3) When in (2.5) the coefficients of ψ are reduced to quadratic form in γ_i independent of suffix i, we have after some calculation the following relations:

$$R_{lm[ij]} = \overset{*}{R}_{lm[ij]} - \frac{2}{4} A_{[l[i} A_{m]j]}$$

$$\overset{*}{R}_{lm[5j]} - \frac{2}{4} A_{[l[5} A_{m]j]} = 0$$

$$\overset{*}{\nabla}_{[l} A^i_{m]} = 0$$

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(1) See **Note 3,** p. 11 .

(2) In this case we can easily show that

$$\frac{1}{4}\frac{\partial A_{\lambda]}}{\partial x^{[\mu}} = \frac{\partial \theta_{\lambda]}}{\partial x^{[\mu}} \qquad \text{and} \qquad S_{\omega\mu}^{\cdot\cdot\lambda}\theta_\lambda = 0\,.$$

(3) Independent of us H. T. Flint has treated this special case in which the parallel displacements which make $ds\Psi = 0$ invariant Proc. R. O. Y. Soc., 1935–432.

and
$$\left(R_{lm}^{\cdot\cdot[ij]}\gamma_i\gamma_j+\frac{2}{4}\overset{*}{\nabla}_{[l}A^5_{m]}\gamma_5+\frac{1}{2}\overset{\circ}{\nabla}_{[m}A_{l]}I\right)\psi = 0\,.$$

Therefore the case is just coincides to that which we have treated in my previous papers.[1]

4) When the *Ma*-side of space is euclidean, the condition for integrability of the equations (2.1) and (2.2) are reduced to the following form[2]:

$$(i\overset{*}{R}_{lm14}-i\overset{*}{R}_{lm23}+f_{lm}+t_{lm}):(i\overset{*}{R}_{lm12}+\overset{*}{R}_{lm13}+\overset{*}{R}_{lm24}-i\overset{*}{R}_{lm34}):$$
$$(\overset{*}{\nabla}_{[m}A^1_{l]}+i\overset{*}{\nabla}_{[m}A^4_{l]}-\overset{*}{R}_{lm51}-i\overset{*}{R}_{lm54}):(i\overset{*}{\nabla}_{[m}A^2_{l]}+\overset{*}{\nabla}_{[m}A^3_{l]}-i\overset{*}{R}_{lm52}-\overset{*}{R}_{lm53})$$
$$=(i\overset{*}{R}_{lm12}-\overset{*}{R}_{lm13}-\overset{*}{R}_{lm24}-i\overset{*}{R}_{lm34}):(-i\overset{*}{R}_{lm14}+i\overset{*}{R}_{lm23}+f_{lm}+t_{lm}):$$
$$(i\overset{*}{\nabla}_{[m}A^2_{l]}-\overset{*}{\nabla}_{[m}A^3_{l]}-i\overset{*}{R}_{lm52}+\overset{*}{R}_{lm53}):(\overset{*}{\nabla}_{[m}A^1_{l]}-i\overset{*}{\nabla}_{[m}A^4_{l]}-\overset{*}{R}_{lm51}+i\overset{*}{R}_{lm54})$$
$$=(\overset{*}{\nabla}_{[m}A^1_{l]}-i\overset{*}{\nabla}_{[m}A^4_{l]}+\overset{*}{R}_{ml51}-i\overset{*}{R}_{lm54}):(-i\overset{*}{\nabla}_{[m}A^2_{l]}-\overset{*}{\nabla}_{[m}A^3_{l]}-i\overset{*}{R}_{ml52}+\overset{*}{R}_{lm53}):$$
$$(-i\overset{*}{R}_{lm14}-i\overset{*}{R}_{lm23}+f_{lm}-t_{lm}):(-i\overset{*}{R}_{lm12}-\overset{*}{R}_{lm13}+\overset{*}{R}_{lm24}-i\overset{*}{R}_{lm34})$$
$$=(\overset{*}{\nabla}_{[m}A^2_{l]}-i\overset{*}{\nabla}_{[m}A^3_{l]}+\overset{*}{R}_{lm52}-\overset{*}{R}_{lm53}):(\overset{*}{\nabla}_{[m}A^1_{l]}+i\overset{*}{\nabla}_{[m}A^4_{l]}+\overset{*}{R}_{lm51}+i\overset{*}{R}_{lm54}):$$
$$(-i\overset{*}{R}_{lm12}+\overset{*}{R}_{lm13}-\overset{*}{R}_{lm24}-i\overset{*}{R}_{lm34}):(i\overset{*}{R}_{lm14}+i\overset{*}{R}_{lm23}+f_{lm}-t_{lm})\,,$$

where
$$t_{lm}=\frac{2}{4}\overset{*}{\nabla}_{[m}A^5_{l]}\quad\text{and}\quad f_{lm}=\frac{2}{4}\overset{\circ}{\nabla}_{[m}A_{l]}\,.$$

5) When the *Ma*-side of space is real and the metric space of *Mi*-side of space has null curvature, i. e. $\Lambda_{mn}=0$, the *Ma*-side of the space also has null curvature. (Ref. Case 2).

Note 1.

Λ_{lm} is written as follows

(1) We can easily see from equation (6.10) or (6.11) in author's previous papers (loc. cit., 168) that the condition
$$\frac{\sqrt{\Delta}}{2}\epsilon_{stpq}R_{lm}^{\cdot\cdot pq}=\pm R_{lm[st]}$$
is a sufficient condition for integrability of fundamental equation in all cases which are not necessarily positive definite.

(2) These equations are obtained by the same method by which we have (3.2).

$$-\left\{\frac{\partial \Gamma_l}{\partial x^m}-\frac{\partial \Gamma_m}{\partial x^l}-\Gamma_l\Gamma_m+\Gamma_m\Gamma_l\right\}-\frac{2}{4}\left\{\frac{\partial A_{[l}}{\partial x^{m]}}-\Gamma_{[l}A_{m]}+A_{[l}\Gamma_{m]}\right\}+\frac{2}{16}A_{[l}A_{m]}$$

where $$\Lambda_m=-\Gamma_m-\frac{1}{4}A_m.$$

The first term in the above expression is equal to

$$-\frac{1}{4}K_{lm}^{\cdot\cdot\lambda\mu}\gamma_\lambda\gamma_\mu;$$

by using the identity

$$\frac{\partial\gamma_\lambda}{\partial x^l}=\{^{\mu}_{\lambda l}\}\gamma_\mu+\Gamma_l\gamma_\lambda-\gamma_\lambda\Gamma_l$$

the second term is reduced to the following form

$$-\frac{2}{4}\{(\mathring{\nabla}_{[m}A_{l]}^{\lambda\mu})\gamma_\lambda\gamma_\mu+(\mathring{\nabla}_{[m}A_{l]}^{\lambda})\gamma_\lambda+\mathring{\nabla}_{[m}A_{l]}\}$$

where $$A=A_m^{\lambda\mu}\gamma_\lambda\gamma_\mu+A_m^{\lambda}\gamma_\lambda+A_mI$$

and $\mathring{\nabla}_m A{:}{:}$ expresses the covariant derivative of $A{:}{:}$ with respect to $\{^{\lambda}_{\mu m}\}$. The third term is reduced to the form:

$$-\frac{2}{4}A_{[l}^{\nu\omega}A_{m]}^{\lambda\mu}\gamma_\nu\gamma_\lambda g_{\omega\mu}-\frac{2}{4}A_{[l}^{\lambda\mu}A_{m]}^{\nu}\gamma_\lambda g_{\mu\nu}+\frac{2}{16}A_{[l}^{\lambda}A_{m]}^{\mu}\gamma_\lambda\gamma_\mu$$

So we have

$$\Lambda_{lm}=-\frac{1}{4}(\overset{*}{R}{}_{lm}^{\cdot\cdot\lambda\mu}-\frac{2}{4}A_{[l}^{\lambda}A_{m]}^{\mu})\gamma_\lambda\gamma_\mu-\frac{2}{4}(\mathring{\nabla}_{[m}A_{l]}^{\lambda})\gamma_\lambda$$
$$-\frac{2}{4}\mathring{\nabla}_{[m}A_{l]}-\frac{2}{4}A_{[l}^{\lambda\mu}A_{m]}^{\nu}\gamma_\lambda g_{\mu\nu}$$

where $\overset{*}{R}{}_{lm}^{\cdot\cdot\lambda\mu}$ is the curvature tensor made from $\{^{\lambda}_{\mu m}\}+\frac{1}{4}A_m^{\omega\lambda}g_{\mu\omega}\equiv\overset{*}{\Gamma}{}^{\lambda}_{\mu m}$.

Note 2.

In order that in (2.6) all the coefficients of ψ should be linear and homogeneous in γ_i, the following relation must hold

$$\left\{\left(\overset{*}{R}{}_{lm}^{\cdot\cdot\lambda\mu}-\frac{2}{4}A_{[l}^{\lambda}A_{m]}^{\mu}\right)\gamma_\lambda\gamma_\mu+\frac{2}{4}(\mathring{\nabla}_{[m}A_{l]}^{\lambda}+A_{[m}^{\lambda\mu}A_{l]}^{\nu}g_{\mu\nu})\gamma_\lambda=a_{lmi}^{\cdot\cdot\cdot\lambda}\gamma_\lambda\gamma_i\right.$$

(not summing by i)

where $a_{lmi}^{\cdot\cdot\cdot\lambda}$ is any function.

So we have

$$\left.\begin{aligned}\overset{*}{R}_{lm}{}^{\cdot\cdot\lambda\mu}-\frac{2}{4}A^{\lambda}_{[l}A^{\mu}_{m]}=0\\ \overset{\circ}{\nabla}_{[m}A^{\lambda}_{l]}+A^{\lambda\mu}_{[m}A^{\nu}_{l]}g_{\mu\nu}=0\end{aligned}\right\}. \tag{N2.1}$$

And by this equation (2.6) is reduced to the form

$$\left\{R_{lmi}^{\cdot\cdot\cdot j}+\frac{1}{4}\left(\frac{\partial A_l}{\partial x^m}-\frac{\partial A_m}{\partial x^l}\right)\delta^j_i\right\}\gamma_j\psi=0\,.$$

Since we have $\gamma_i=Uh^{\lambda}_i\overset{\circ}{\gamma}_\lambda U^{-1}$, the above equation can be rewritten as follows:

$$P_{lmi}^{\cdot\cdot\cdot j}h^{\lambda}_j\overline{\psi}=0 \tag{N2.2}$$

where

$$\overline{\psi}=U\psi \qquad \text{and} \qquad P_{lmi}^{\cdot\cdot\cdot j}=R_{lmi}^{\cdot\cdot\cdot j}+\frac{1}{4}\left(\frac{\partial A_l}{\partial x^m}-\frac{\partial A_m}{\partial x^l}\right)\delta^j_i\,.$$

If we rewrite (N2.2) in actual form from the matrix form we have

$$\left.\begin{aligned}&p_{lmi}^{\cdot\cdot\cdot 5}\overline{\psi}_1+(p_{lmi}^{\cdot\cdot\cdot 1}-ip_{lmi}^{\cdot\cdot\cdot 4})\overline{\psi}_3+(-ip_{lmi}^{\cdot\cdot\cdot 2}+p_{lmi}^{\cdot\cdot\cdot 3})\overline{\psi}_4=0\\ &p_{lmi}^{\cdot\cdot\cdot 5}\overline{\psi}_2+(-ip_{lmi}^{\cdot\cdot\cdot 2}-p_{lmi}^{\cdot\cdot\cdot 3})\overline{\psi}_3+(p_{lmi}^{\cdot\cdot\cdot 1}+ip_{lmi}^{\cdot\cdot\cdot 4})\overline{\psi}_4=0\\ &(p_{lmi}^{\cdot\cdot\cdot 1}+ip_{lmi}^{\cdot\cdot\cdot 4})\overline{\psi}_1+(ip_{lmi}^{\cdot\cdot\cdot 2}-p_{lmi}^{\cdot\cdot\cdot 3})\overline{\psi}_2-p_{lmi}^{\cdot\cdot\cdot 5}\overline{\psi}_3=0\\ &(ip_{lmi}^{\cdot\cdot\cdot 2}+p_{lmi}^{\cdot\cdot\cdot 3})\overline{\psi}_1+(p_{lmi}^{\cdot\cdot\cdot 1}-ip_{lmi}^{\cdot\cdot\cdot 4})\overline{\psi}_2-p_{lmi}^{\cdot\cdot\cdot 5}\overline{\psi}_4=0\end{aligned}\right\} \tag{N2.3}$$

where

$$p_{lmi}^{\cdot\cdot\cdot\lambda}=P_{lmi}^{\cdot\cdot\cdot j}h^{5}_j\,.$$

Therefore as a necessary condition for which (N.23) may have a non-vanishing solution $\overline{\psi}$ we have the following equations:

$$\left.\begin{aligned}&\sum_{\lambda}p_{lmi}^{\cdot\cdot\cdot\lambda}p_{lmi}^{\cdot\cdot\cdot\lambda}=0\\ \text{or}\quad &\sum_{j}P_{lmij}P_{lmi}^{\cdot\cdot\cdot j}=0 \qquad \text{(not summing by } l, m, i)\end{aligned}\right\}. \tag{N2.4}$$

And we also have from (N2.2) the following equations:

$$\left.\begin{aligned}&p_{lmi}^{\cdot\cdot\cdot 5}\overline{\psi}_1+(p_{lmi}^{\cdot\cdot\cdot 1}-ip_{lmi}^{\cdot\cdot\cdot 4})\overline{\psi}_3+(-ip_{lmi}^{\cdot\cdot\cdot 2}+p_{lmi}^{\cdot\cdot\cdot 3})\overline{\psi}_4=0\\ &p_{lmi}^{\cdot\cdot\cdot 5}\overline{\psi}_2+(-ip_{lmi}^{\cdot\cdot\cdot 2}-p_{lmi}^{\cdot\cdot\cdot 3})\overline{\psi}_3+(p_{lmi}^{\cdot\cdot\cdot 1}+ip_{lmi}^{\cdot\cdot\cdot 4})\overline{\psi}_4=0\\ &(p_{l'm'i'}^{\cdot\cdot\cdot 1}+ip_{l'm'i'}^{\cdot\cdot\cdot 4})\overline{\psi}_1+(ip_{l'm'i'}^{\cdot\cdot\cdot 2}-p_{l'm'i'}^{\cdot\cdot\cdot 3})\overline{\psi}_2-p_{l'm'i'}^{\cdot\cdot\cdot 5}\overline{\psi}_3=0\\ &(ip_{l'm'i'}^{\cdot\cdot\cdot 2}+p_{l'm'i'}^{\cdot\cdot\cdot 3})\overline{\psi}_1+(p_{l'm'i'}^{\cdot\cdot\cdot 1}-ip_{l'm'i'}^{\cdot\cdot\cdot 4})\overline{\psi}_2-p_{l'm'i'}^{\cdot\cdot\cdot 5}\overline{\psi}_4=0\end{aligned}\right\}. \tag{N2.5}$$

Therefore as a necessary condition for which (N2.3) may have a non-vanishing solution ψ, we have by using (N2.4) if $p_{lmi}^{\cdot\cdot\cdot 5}, p_{l'm'i'}^{\cdot\cdot\cdot 5} \neq 0$ the following equations

$$\sum_{\lambda} p_{lmi}^{\cdot\cdot\cdot\lambda} p_{l'm'i'}^{\cdot\cdot\cdot\lambda} = 0 . \qquad \text{(N2.6)}$$

The equation (N2.6) is also the sufficient condition for which (N2.2) may have a non-vanishing solution if $p_{lmi}^{\cdot\cdot\cdot 5} \neq 0$: for, in the case where l, m, i are fixed we have only two independent equations which are taken from (N2.3) arbitrarily.

When $p_{lmi}^{\cdot\cdot\cdot 5} = 0$, we have from (N2.3) the equations

$$\frac{-ip_{lmi}^{\cdot\cdot\cdot 2} - p_{lmi}^{\cdot\cdot\cdot 3}}{p_{lmi}^{\cdot\cdot\cdot 1} + ip_{lmi}^{\cdot\cdot\cdot 4}} = \frac{p_{l'm'i'}^{\cdot\cdot\cdot 1} - ip_{l'm'i'}^{\cdot\cdot\cdot 4}}{-ip_{lmi}^{\cdot\cdot\cdot 2} + p_{l'm'i'}^{\cdot\cdot\cdot 3}}$$

or

$$\frac{p_{lmi}^{\cdot\cdot\cdot 1} - ip_{lmi}^{\cdot\cdot\cdot 4}}{ip_{lmi}^{\cdot\cdot\cdot 2} + p_{lmi}^{\cdot\cdot\cdot 3}} = \frac{ip_{l'm'i'}^{\cdot\cdot\cdot 2} - p_{l'm'i'}^{\cdot\cdot\cdot 3}}{p_{l'm'i'}^{\cdot\cdot\cdot 1} - ip_{l'm'i'}^{\cdot\cdot\cdot 4}}$$

Then these equations become (N2.6).

So (N2.6) expresses the necessary and sufficient condition in all cases for which (N2.6) has the non-vanishing solution ψ.

Note 3.

Specially, when the *Ma*-side of space is comformal namely $\nabla_{\mu} g_{\lambda\omega} = \theta_{\mu} g_{\lambda\omega}$ we have the following relations between the components $R_{\lambda\mu\nu p}$ with respect to an orthogonal ennuple

$$\left.\begin{aligned} R_{\omega\mu(\lambda\nu)} &= \left(\frac{\partial\theta_{\mu}}{\partial x^{\omega}} - \frac{\partial\theta_{\omega}}{\partial x^{\mu}}\right)\delta_{\lambda\nu}\epsilon_{\lambda} \\ R_{\omega\mu\lambda\nu} &= -R_{\omega\mu\nu\lambda} \qquad \text{for} \qquad \lambda \neq \nu \end{aligned}\right\}^{(1)} . \qquad \text{(N3.1)}$$

(1) In this case we have

$$2\nabla_{[\omega}\nabla_{\mu]} w_{\lambda} = R_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu} w_{\nu} + 2S_{\omega\mu}^{\cdot\cdot\alpha}\nabla_{\alpha} w_{\lambda}$$

and

$$2\nabla_{[\omega}(\theta_{\mu]} g_{\lambda\nu}) = 2\nabla_{[\omega}\theta_{\mu]} g_{\lambda\nu} .$$

So we have

$$g_{\lambda\nu}\left(\frac{\partial\theta_{\mu}}{\partial x^{\omega}} - \frac{\partial\theta_{\omega}}{\partial x^{\mu}}\right) = R_{\omega\mu\lambda}^{\cdot\cdot\cdot\alpha} g_{\alpha\nu} + R_{\omega\mu\nu}^{\cdot\cdot\cdot\alpha} g_{\alpha\lambda} .$$

(Ref. J. A. Schouten. *Der Ricci Kalkül*, 1924, 93).

(2) $\epsilon_{\lambda} = \pm 1$.

We will now show that in this case $P_{\lambda\mu\nu\omega}$, which satisfies (3.2), vanishes. We fix the point x and the suffices $\lambda\mu$ in $P_{\lambda\mu\nu\omega}$ and consider the quantities $q_{ab\nu\omega} \equiv P^{\epsilon}_{\nu} P^{\eta}_{\omega} P_{ab\epsilon\eta}$. If we select suitable values of P^{ϵ}_{ν} we have the following relations(3)

i) $q_{ab12}, q_{ab34} \neq 0$ and other $q_{ab[\lambda\mu]} = 0$ when the rank $\| P_{ab[\nu\omega]} \|$ is 4, and

ii) $q_{ab12} \neq 0$ and other $q_{ab[\lambda\mu]} = 0$ when the rank $\| P_{ab[\nu\omega]} \|$ is 2.

In both cases i) and ii) we have from (3.2) and (N3.1)

$$\sum_{\lambda} (q_{ab1\lambda})^2 = (q_{ab11})^2 + (q_{ab12})^2 = 0\,,$$

$$\sum_{\lambda} (q_{ab5\lambda})^2 = (q_{ab55})^2 = 0\,,$$

and

$$q_{ab55} = q_{ab11}\,,$$

So we have

$$q_{ab12} = 0\,.$$

This contradicts the assumption. So it is also not the cases.

Therefore the rank of $\| P_{ab[\nu\omega]} \|$ is zero. This conclusion holds good for all values of $a, b = 1, 2, \ldots . 5$, so in this case we have the result: $P_{\lambda\mu\nu\omega} = 0$.

This problem was discussed at a special Seminar of geometry and Theoretical Physics at this University.

Specially the writer's best thanks for kind directions are offered to Prof. Iwatsuki.

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(3) J. A. Schouten. ibid., 49, 50.

Errata to the authors previous papers in these Journal vol. 5, No. 3.

p. 159, line 17: for $(\gamma_i dx^i \psi)_{x+dx}$ read $(\gamma_i dx^i \psi)_{x+\delta x}$.
p. 162, line 17: for $4(L_k^{il}+C_k^{il})g_{li}$ read $4(L_k^{il}+C_k^{il})g_{lj}$.
p. 164, line 14: for W_i read $W\gamma_i$.
p. 174, line 11: for $2R_{st\cdot[c}^{\cdot\cdot l}R_{|pql|p],r\dots}$ read $2R_{st\cdot[c}^{\cdot\cdot l}R_{|pql|d],r\dots}$.
p. 173, all (i)s in the equations (8.3) (8.6) will be read $(-)$.
p. 184, line 10: for (7.2) read (6.2).
p. 187, line 10: for $h_l^1 h_m^2 i - h_l^1 h_m^3 - h_l^2 h_m^4 - h_l^3 h_m^4$
read $(h_l^1 h_m^2 i - h_l^1 h_m^3 - h_l^2 h_m^4 - h_l^3 h_m^4)i$
p. 187, line 12: for $h_i^1 h_j^2 i - h_i^1 h_j^3 - h_i^2 h_j^4 - h_i^3 h_j^4$
read $(h_i^1 h_j^2 i - h_i^1 h_j^3 - h_i^2 h_j^4 - h_i^3 h_j^4)i$
p. 187, line 16: for $h_{[j}^4 h_{m]}^a i - h_{[j}^4 h_{m]}^3 - h_{[j}^1 h_{m]}^2 - h_{[j}^1 h_{m]}^3 i$
read $(h_{[j}^4 h_{m]}^2 i - h_{[j}^4 h_{m]}^3 - h_{[j}^1 h_{m]}^2 - h_{[j}^1 h_{m]}^3)i$
p. 188, for the sentences from lines 3 to 7 will be read as follows:

Applying the same method as above to the equations in (6.10)

$$\frac{-i\overset{3}{k}_{st}+i\overset{4}{k}_{st}+f_{st}+t_{st}}{i\overset{1}{k}_{st}+\overset{2}{k}_{st}+\overset{5}{k}_{st}-i\overset{6}{k}_{st}} = \frac{-i\overset{3}{k}_{pq}+i\overset{4}{k}_{pq}+f_{pq}+t_{pq}}{i\overset{1}{k}_{pq}-\overset{2}{k}_{pq}+\overset{5}{k}_{pq}-i\overset{6}{k}_{pq}},$$

$$\frac{i\overset{3}{k}_{st}-i\overset{4}{k}_{st}+f_{st}+t_{st}}{i\overset{1}{k}_{st}-\overset{2}{k}_{st}-\overset{5}{k}_{st}-i\overset{6}{k}_{st}} = \frac{i\overset{1}{k}_{pq}+\overset{2}{k}_{pq}+\overset{5}{k}_{pq}-i\overset{6}{k}_{pq}}{-i\overset{3}{k}_{pq}+i\overset{4}{k}_{pq}+f_{pq}+t_{pq}},$$

$$\frac{i\overset{3}{k}_{pq}-i\overset{4}{k}_{pq}+f_{pq}+t_{pq}}{i\overset{1}{k}_{pq}-\overset{2}{k}_{pq}-\overset{5}{k}_{pq}-i\overset{6}{k}_{pq}} = \frac{i\overset{1}{k}_{st}+\overset{2}{k}_{st}+\overset{5}{k}_{st}-i\overset{6}{k}_{st}}{-i\overset{3}{k}_{st}+i\overset{4}{k}_{st}+f_{st}+t_{st}},$$

we get the following equations

$$N^{jm}(h_j^4 h_m^2 i + h_j^4 h_m^3 + h_j^1 h_m^2 - h_j^1 h_m^3 i) = 0\,,$$

$$N^{jm}(h_j^2 h_m^3 + h_j^4 h_m^1) = 0\,.$$

From (N9.1) and above equations we get three equations....

p. 188, lines 15 and 16: for (i)s read $(-)$; and for f read $(f+t)$.
p. 188, foot-note is omitted.