On the Foundation of the Geometry in Microscopic and Macroscopic Space.

By

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(Received April 20, 1935.)

We know that the Quantum Theory, which has made remarkable progress in recent years, has a viewpoint definitely opposed to that of all recently unified field theories of gravitation, electromagnetism and matter, which explain physical phenomena on the assumption that space is metric geometrical. In the present paper the writer intends to construct a new geometry, which will have its foundation on the proper qualities of geometry, and which, based on common and actual physical demonstrations and observations, will express pure-mathematically the relations between the physical realities; in other words, to establish a most intimate connection between physical generic and geometrical conceptions. Again the writer is inclined to assert that the geometrical conceptions are produced by our actual experience of physical phenomena, and that the conceptions of position, mass, etc., are to be defined according to them.

In short, as the new geometry in question is to be constructed resting on physical phenomena, the writer has decided to give up the conventional method of deriving microscopic space from macroscopic, and, reversing the process, to explain macroscopic from microscopic, the metrical conception of the latter being defined by the physical phenomena.

The extension of Dirac's equation to projective relativity has been developed by O. Voblen, J. A. Schouten, W. Pauli and others. The method used in this paper should serve to illuminate the relations between these theories.

Finally, we will develop the theory of kinematic connections in what seems to us a natural manner.

Consider the set of matrices E_λ $(\lambda = 1, 2, 3, 4, 5)$:

$$E_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$E_4 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad E_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Then we get

$$(1) \qquad E_{(\lambda}E_{\mu)} = \frac{1}{2}(E_\lambda E_\mu + E_\mu E_\lambda) = \delta^\lambda_\mu ,$$

where the δ^λ_μ denotes Kronecker's delta. Also suppose that we have twenty-five real functions $p_{\mu}^{\cdot\lambda}$ of the independent real variables X^1, X^2, X^3, X^4, X^5. Let $q^{\lambda}_{\cdot\mu}$ be defined by the equations

$$(2) \qquad p_{\nu}^{\cdot\lambda} q^{\mu}_{\cdot\lambda} = \delta^\mu_\nu .$$

We now define the matrices M_λ and M^μ, by the following equations:

$$(3) \qquad M_\lambda = p_{\lambda}^{\cdot\mu} E_\mu , \qquad (\lambda, \mu = 1, 2, 3, 4, 5)$$

and

$$(3') \qquad M^\mu = q^{\mu}_{\cdot\lambda} E^\lambda ,$$

where $E^\lambda = E_\lambda$. From (1), we have

$$(4) \qquad \frac{1}{2}(M_\lambda M_\mu + M_\mu M_\lambda) = \sum_{\omega=1}^{5} p_{\lambda}^{\cdot\omega} p_{\mu}^{\cdot\omega} ,$$

and

$$(4') \qquad \frac{1}{2}(M^\lambda M^\mu + M^\mu M^\lambda) = \sum_{\varkappa=1}^{5} q^{\lambda}_{\cdot\varkappa} q^{\mu}_{\cdot\varkappa} .$$

From (3), also we get

$$(5) \qquad \frac{1}{4}(M^\lambda M^\mu + M^\mu M^\lambda)(M_\lambda M_\nu + M_\nu M_\lambda) = \delta^\mu_\nu .$$

As a special case, put $p_{\mu}^{\cdot\lambda} = \delta^\lambda_\mu$, then we have

$$M_\lambda = M^\lambda = E_\lambda$$

In like manner we shall consider the sixteen real functions $p_j^{\cdot i}$ of the independent real variables x^1, x^2, x^3, x^4, and let $q^{i}_{\cdot j}$ be defined by the equations $p_j^{\cdot i} q^{h}_{\cdot i} = \delta_j^h$, where $i, j, k = 1, 2, 3, 4$.

Let us make the connection that Greek indices run over the range $1, 2, 3, 4, 5$, whereas the Latin indices take on the values $1, 2, 3, 4$ only.

Also putting,

$$e_i = p_i^{\cdot h} E_h \,, \tag{6}$$

and

$$e^i = q^{i}_{\cdot h} E^h \,. \tag{6'}$$

Then we obtain

$$\frac{1}{2}(e_i e_j + e_j e_i) = \sum_{h=1}^{4} p_i^{\cdot h} p_j^{\cdot h} \,, \quad \frac{1}{2}(e^i e^j + e^j e^i) = \sum_{h=1}^{4} q^{i}_{\cdot h} q^{j}_{\cdot h} \,, \tag{7}$$

and

$$\frac{1}{4}(e^i e^j + e^j e^i)\,(e_i e_k + e_k e_i) = \delta_k^j \,. \tag{8}$$

Let us now consider the five independent real variables X^λ of which at least one is not zero. If $F^\lambda(X^1, X^2, X^3, X^4, X^5)$ for $\lambda = 1, 2, 3, 4, 5$ are real functions, the equations

$$\bar{X}^\lambda = F^\lambda(X^1, X^2, X^3, X^4, X^5) \tag{9}$$

define a transformation of the variables, where F^λ are homogeneous analytical functions of degree one, and independent in the domain considered. Also we introduce four arbitrary mutually independent functions f^k of the X^ν, homogeneous of degree zero:

$$X^\mu B_\mu^{\cdot k} = 0 \,; \qquad B_\mu^{\cdot k} = \frac{\partial f^k}{\partial X^\mu} \tag{10}$$

analytic, regular and independent in the domain considered. This may be written as the equations

$$x^k = f^k(X^1, X^2, X^3, X^4, X^5) \,. \tag{11}$$

Let $B^{\nu}_{\cdot h}$ be defined by the following equation:

$$B_\nu^{\cdot k} B^{\nu}_{\cdot h} = \delta_h^k \tag{12}$$

We put

$$B_{\nu}^{\cdot k} B_{\cdot k}^{\mu} = D_{\cdot \nu}^{\mu}, \tag{13}$$

then from (10) we have

$$X^{\mu} D_{\cdot \mu}^{\nu} = 0. \tag{14}$$

From (12) and (13), we get

$$B_{\cdot j}^{\nu}(D_{\cdot \nu}^{\mu} - \delta_{\nu}^{\mu}) = 0, \quad B_{\mu}^{\cdot l}(\delta_{\nu}^{\mu} - D_{\cdot \nu}^{\mu}) = 0. \tag{15}$$

Putting also

$$D_{\cdot \nu}^{\mu} - \delta_{\nu}^{\mu} = \rho_{\nu} X^{\mu}, \tag{16}$$

where

$$B_{\cdot j}^{\nu} \rho_{\nu} = 0. \tag{17}$$

From (14) and (16), we have

$$X^{\nu} \rho_{\nu} = -1. \tag{18}$$

We will now define the matrices γ_{λ} and γ^{μ}, by the following expressions:

$$\gamma^{\mu} = B_{\cdot k}^{\mu} q_{\cdot i}^{k} E^{i} + X^{\mu} E^{5} \tag{19}$$

and

$$\gamma_{\lambda} = B_{\lambda}^{\cdot k} p_{k}^{\cdot i} E_{i} + \rho_{\lambda} E_{5}. \tag{19'}$$

By the method used in (4), we get

$$\frac{1}{2}(\gamma^{\lambda}\gamma^{\mu} + \gamma^{\mu}\gamma^{\lambda}) = \sum_{i=1}^{4} B_{\cdot k}^{\lambda} B_{\cdot h}^{\mu} q_{\cdot i}^{k} q_{\cdot i}^{h} + X^{\lambda} X^{\mu} \tag{20}$$

$$\frac{1}{2}(\gamma_{\lambda}\gamma_{\mu} + \gamma_{\mu}\gamma_{\lambda}) = \sum_{i=1}^{4} B_{\lambda}^{\cdot k} B_{\mu}^{\cdot h} p_{k}^{\cdot i} p_{h}^{\cdot i} + \rho_{\lambda}\rho_{\mu}, \tag{20'}$$

and

$$\frac{1}{4}(\gamma^{\lambda}\gamma^{\mu} + \gamma^{\mu}\gamma^{\lambda})(\gamma_{\lambda}\gamma_{\nu} + \gamma_{\nu}\gamma_{\lambda}) = \delta_{\nu}^{\mu}. \tag{21}$$

Putting

$$\gamma_{k} = B_{\cdot k}^{\lambda} \gamma_{\lambda}, \quad \gamma^{k} = B_{\lambda}^{\cdot k} \gamma^{\lambda}, \quad \gamma^{5} = X^{\lambda} \gamma_{\lambda}, \quad \gamma_{5} = \rho_{\lambda} \gamma^{\lambda},$$

then from (10) and (19), we obtain

$$\gamma_k = p_{k}^{\cdot i} E_i \tag{22}$$

$$\gamma^k = q^{k}_{\cdot j} E^j \tag{22'}$$

and

$$\gamma^5 = \gamma_5 = -E_5 . \tag{22''}$$

Hence we have

$$\left\{\begin{aligned} &\frac{1}{2}(\gamma_i\gamma_j+\gamma_j\gamma_i) = \sum_{k=1}^{4} p_i^{\cdot k} p_j^{\cdot k}\,, \quad \frac{1}{2}(\gamma^i\gamma^j+\gamma^j\gamma^i) = \sum_{k=1}^{4} q^{i}_{\cdot k} q^{j}_{\cdot k}\,, \\ &\frac{1}{2}(\gamma^5\gamma^i+\gamma^i\gamma^5) = 0\,, \qquad (\gamma^5)^2 = 1\,. \end{aligned}\right. \tag{23}$$

Consequently we see that the case is the special one defined by (3) but more general than those defined by (6).

If the matrices of $p^{\lambda}_{\cdot\mu}$ and $q^{\lambda}_{\cdot\mu}$ are

$$\begin{pmatrix} p_1^{\cdot 1} & p_2^{\cdot 1} & p_3^{\cdot 1} & p_4^{\cdot 1} & 0 \\ p_1^{\cdot 2} & p_2^{\cdot 2} & p_3^{\cdot 2} & p_4^{\cdot 2} & 0 \\ p_1^{\cdot 3} & p_2^{\cdot 3} & p_3^{\cdot 3} & p_4^{\cdot 3} & 0 \\ p_1^{\cdot 4} & p_2^{\cdot 4} & p_3^{\cdot 4} & p_4^{\cdot 4} & 0 \\ 0 & 0 & 0 & 0 & p_5^{\cdot 5} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} q^1_{\cdot 1} & q^1_{\cdot 2} & q^1_{\cdot 3} & q^1_{\cdot 4} & 0 \\ q^2_{\cdot 1} & q^2_{\cdot 2} & q^2_{\cdot 3} & q^2_{\cdot 4} & 0 \\ q^3_{\cdot 1} & q^3_{\cdot 2} & q^3_{\cdot 3} & q^3_{\cdot 4} & 0 \\ q^4_{\cdot 1} & q^4_{\cdot 2} & q^4_{\cdot 3} & q^4_{\cdot 4} & 0 \\ 0 & 0 & 0 & 0 & q^5_{\cdot 5} \end{pmatrix}$$

where $p^{\cdot 5}_{5} = q^5_{\cdot 5} = 1$, then (23) may substituted in (3) and (3′). Accordingly, we obtain the matrices which have been treated by W. Pauli.[1]

Also, as the matrices of $p_{\mu}^{\cdot\lambda}$ and $q^{\lambda}_{\cdot\mu}$; we put

$$\begin{pmatrix} p_1^{\cdot 1} & p_2^{\cdot 1} & p_3^{\cdot 1} & p_4^{\cdot 1} & 0 \\ p_1^{\cdot 2} & p_2^{\cdot 2} & p_3^{\cdot 2} & p_4^{\cdot 2} & 0 \\ p_1^{\cdot 3} & p_2^{\cdot 3} & p_3^{\cdot 3} & p_4^{\cdot 3} & 0 \\ p_1^{\cdot 4} & p_2^{\cdot 4} & p_3^{\cdot 4} & p_4^{\cdot 4} & 0 \\ \varphi_1 & \varphi_2 & \varphi_3 & \varphi_4 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} q^1_{\cdot 1} & q^1_{\cdot 2} & q^1_{\cdot 3} & q^1_{\cdot 4} & 0 \\ q^2_{\cdot 1} & q^2_{\cdot 2} & q^2_{\cdot 3} & q^2_{\cdot 4} & 0 \\ q^3_{\cdot 1} & q^3_{\cdot 2} & q^3_{\cdot 3} & q^3_{\cdot 4} & 0 \\ q^4_{\cdot 1} & q^4_{\cdot 2} & q^4_{\cdot 4} & q^4_{\cdot 4} & 0 \\ \phi_1 & \phi_2 & \phi_3 & \phi_4 & 1 \end{pmatrix}$$

where
$$\phi_i = -\sum q^{k}_{\cdot i}\varphi_k .$$

Moreover we put

(1) W. Pauli: Ann. der Phys. **18** (1933), 337–372.

$$a^i = q^i_{\cdot j} E^j\,, \qquad a^5 = \phi_i E^i + E^5 \tag{24}$$

and

$$a_i = p_i^{\cdot k} E_k + \varphi_i E_5\,,\ a_4 = E_4 \tag{25}$$

then we get

$$\frac{1}{2}(a^i a^j + a^j a^i) = \sum_{k=1}^{4} q^i_{\cdot k} q^j_{\cdot k}\,, \tag{26}$$

$$\frac{1}{2}(a^i a^5 + a^5 a^i) = -\varphi_j \sum_{k=1}^{4} q^i_{\cdot k} q^j_{\cdot k}\,,\ (a^5)^2 = 1 + \varphi_i \varphi_j \sum_{k=1}^{4} q^i_{\cdot k} q^j_{\cdot k}$$

and

$$\begin{cases} \frac{1}{2}(a_i a_j + a_j a_i) = \sum\limits_{k=1}^{4} p_i^{\cdot k} p_j^{\cdot k} + \varphi_i \varphi_j \\ \frac{1}{2}(a_i a_5 + a_5 a_i) = \varphi_i \\ (a_5)^2 = 1 \end{cases} \tag{27}$$

We see that this is nothing other than the case which has been treated by O. Veblen.[1]

In like manner, putting

$$\begin{pmatrix} p_1^{\cdot 1} & p_2^{\cdot 1} & p_3^{\cdot 1} & p_4^{\cdot 1} & \phi^1 \\ p_1^{\cdot 2} & p_2^{\cdot 2} & p_3^{\cdot 2} & p_4^{\cdot 2} & \phi^2 \\ p_1^{\cdot 3} & p_2^{\cdot 3} & p_3^{\cdot 3} & p_4^{\cdot 3} & \phi^3 \\ p_1^{\cdot 4} & p_2^{\cdot 4} & p_3^{\cdot 4} & p_4^{\cdot 4} & \phi^4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} q^1_{\cdot 1} & q^1_{\cdot 2} & q^1_{\cdot 3} & q^1_{\cdot 4} & \varphi^1 \\ q^2_{\cdot 1} & q^2_{\cdot 2} & q^2_{\cdot 3} & q^2_{\cdot 4} & \varphi^2 \\ q^3_{\cdot 1} & q^3_{\cdot 2} & q^3_{\cdot 3} & q^3_{\cdot 4} & \varphi^3 \\ q^4_{\cdot 1} & q^4_{\cdot 2} & q^4_{\cdot 3} & q^4_{\cdot 4} & \varphi^4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

where $\phi^i = -p_j^{\cdot i} \varphi^j$, then we have the case which has been given by the present writer.[2] Then we have

$$a_i = p_i^{\cdot j} E_j\,,\ a_5 = \phi^j E_j + E_5 \tag{28}$$

and

$$a^i = q^i_{\cdot j} E^j + \varphi^i E_5,\ a^5 = E^5\,. \tag{28$'$}$$

As the special one of the transformations (9), we consider the following equations:

(1) O. Veblen, Proc. Nat. Acad. Sci. **20** (1934), 383–388.
(2) T. Hosokawa, Proc. Imp. Acad. Tokyo, **10** (1934), 49–52.

$$\left\{\begin{array}{l}\bar{X}^5 = X^5 \\ \bar{X}^i = \bar{X}^i(X^1, X^2, X^3, X^4, X^5)\,.\end{array}\right. \tag{29}$$

We can now consider the special transformation of (29)

$$d\bar{X}^\nu = A^\nu_\lambda dX^\lambda\,, \tag{30}$$

which satisfies the following condition: if for the transformation (30), we put

$$\bar{\alpha}^\mu = A^\mu_\lambda \alpha^\lambda \tag{31}$$

and

$$\alpha_\nu = A^\mu_\nu \bar{\alpha}_\mu\,, \tag{31'}$$

then there must exist a matrix S, which will satisfy the following equations:

$$S^{-1}\alpha^\mu S = A^\mu_\nu \alpha^\nu \tag{32}$$

and

$$A^\mu_\nu S^{-1}\alpha_\mu S = \alpha_\nu\,. \tag{32'}$$

An infinitesimal transformation of (30) is defined by the equations

$$\bar{X}^\mu = X^\mu + \eta^\mu \delta t\,. \tag{33}$$

Then we get

$$\bar{\alpha}_\mu = \alpha_\mu + \varepsilon^\rho \xi_{\mu\rho} \tag{34}$$

and

$$S = I + \varepsilon^\rho \Gamma_\rho\,. \tag{35}$$

Accordingly, we obtain

$$\xi_{\mu\rho} = -\Gamma_\rho \alpha_\mu + \alpha_\mu \Gamma_\rho \tag{36}$$

From (34) and (36), we have

$$\bar{\alpha}_\mu = \alpha_\mu + \varepsilon^\rho(-\Gamma_\rho \alpha_\mu + \alpha_\mu \Gamma_\rho) \tag{37}$$

and

$$\xi_{\mu\rho}\alpha_\nu + \alpha_\nu \xi_{\mu\rho} + \xi_{\nu\rho}\alpha_\mu + \alpha_\mu \xi_{\nu\rho} = 0\,. \tag{38}$$

We introduce the functions $\Gamma^\sigma_{\mu\rho}$ satisfying the following equations:

$$\frac{\partial a_\mu}{\partial X^\rho} - \Gamma^\sigma_{\mu\rho} a_\sigma + \Gamma_\rho a_\mu - a_\mu \Gamma_\rho = 0\,.$$

We put

$$a_{\mu;\rho} \equiv \frac{\partial a_\mu}{\partial X^\rho} - \Gamma^\sigma_{\mu\rho} a_\sigma + \Gamma_\rho a_\mu - a_\mu \Gamma_\rho \tag{39}$$

then we have

$$\xi_{\mu\rho} = \frac{\partial a_\mu}{\partial X^\rho} - \Gamma^\rho_{\mu\rho} a_\sigma\,. \tag{40}$$

We will call $\Gamma^\sigma_{\mu\rho}$ and Γ_σ *the parameters of the covariant differentiation* in macroscopic and microscopic space respectively.

Let us now consider the following differential equations

$$a^\mu \psi_{;\mu} + N\psi = 0 \tag{41}$$

where N is an arbitrary function. Using the solutions ψ satisfying the equations (41), we will introduce the equation:

$$a_i dx^i \psi = ds\psi\,, \tag{42}$$

which defines the *metrics $ds\psi$ in the microscopic space.*(1)

By repeating the operator of the left side of (42), we have

$$a_{(i}a_{j)} dx^i dx^j \psi = ds^2 \psi\,.$$

We will now assume that the equation will satisfy any ψ. Then we have

$$g_{ij} dx^i dx^j = ds^2 \tag{43}$$

where

$$g_{ij} = \frac{1}{2}(a_i a_j + a_j a_i)\,. \tag{44}$$

Let us now consider that g_{ij} define the *gravitational potentials* in macroscopic space.

Any ordered set of five independent real variables X^λ, which define the function ψ, may be thought of as homogeneous coordinates of point in four dimentional manifold. Accordingly, the transformation (29) can

(1) In making this proposal I have been greatly helped by questions by Prof. Y. Mimura; see Y. Mimura, this journal **5** (1935), 99.

be regarded as follows: Two points Y^ν and X^ν are called coincident if a factor exists, so that $Y^\nu = \sigma X^\nu$. Each totality of all points coincident with any point is called a spot. The totality of all ∞^4 spots is called the four dimensional projective manifold P_4. The set of all points of P_4, with the exception of those on a single three dimensional projective manifold P_3 contained in P_4, is called the affine manifold. By choosing P_3 as the hyperplane at infinity, the equation of P_3 may be written in the form $X^5 = 0$. Thus (29) are transformations of coordinates in the affine manifold, and by them P_3 is transformed into itself.

This problem was discussed at a special Seminar of Geometry and Theoretical Physics of this University.

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