Resolution of Identity.

By

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(Received April 20, 1935.)

Introduction.

Abstract Hilbert space $\mathfrak{H}$ is characterised by the following five properties:

(a) $\mathfrak{H}$ is a linear space.

(b) for any two elements $\mathfrak{f}$, $\mathfrak{g}$ of $\mathfrak{H}$ an inner product $(\mathfrak{f}, \mathfrak{g})$ is defined.

(c) $\mathfrak{H}$ is complete.

(d) $\mathfrak{H}$ is separable.

(e) for any positive integer n there exist n linearly independent elements.

As is well known, to every self-adjoint transformation H a resolution of identity $E(\lambda)$ corresponds, which is a projective transformation defined for $-\infty < \lambda < +\infty$ such that[1]

$$(1)\quad E(\lambda)E(\mu) = E(\mu)E(\lambda) = E(\lambda) \qquad \text{for} \qquad \lambda < \mu,$$

$$(2)\quad \lim_{\lambda \to \lambda_0 - 0} E(\lambda) = E(\lambda_0)$$

$$(3)\quad \lim_{\lambda \to -\infty} E(\lambda) = O, \qquad \text{and} \qquad \lim_{\lambda \to +\infty} E(\lambda) = 1.$$

And there exists a one-to-one correspondence between H and $E(\lambda)$ by the expression $(H\mathfrak{f}, \mathfrak{g}) = \int_{-\infty}^{+\infty} \lambda d\big(E(\lambda)\mathfrak{f}, \mathfrak{g}\big)$.

Using set U as parameter instead of λ, F. Maeda has generalised the concept of the resolution of identity as follows[2]: If $\boldsymbol{E}(U)$ is a self-adjoint transformation which depends on Borel subset U of a Borel set V in a metric space and satisfies the following conditions, then $\boldsymbol{E}(U)$ is said to be a resolution of identity.

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(1) F. Riesz, Acta Szeged, **5** (1930), 23–54.

(2) F. Maeda, This journal, **4** (1934), 57. K. Friedrichs has considered $E(\varDelta)$ which depends on an interval $\varDelta$ instead of λ. See, Math. Ann., **110** (1934), 54–62.

(α) $\boldsymbol{E}(U)\boldsymbol{E}(U') = \boldsymbol{E}(UU')$.

(β) $\boldsymbol{E}(U) = \boldsymbol{E}(U_1)+\boldsymbol{E}(U_2)+\ldots,$ when $U = U_1+U_2+\ldots$

(γ) $\boldsymbol{E}(V) = 1$.

Recently F. Rellich,(1) H. Löwig(2) and F. Riesz(3) have shown that the theory of the resolution of identity holds also in the space which is characterised by the conditions (a) (b) and (c) mentioned above. We shall call this space a complete complex Euclidean space and denote it by $\mathfrak{R}$.

In this paper I will extend $E(\lambda)$ to $\boldsymbol{E}(U)$ which has set U as parameter. The process to be used is analogous to that by which the theory of measure is established from the elementary figure. Then I shall simplify J. v. Neumann's calculation(4) for determining $\boldsymbol{E}(U)$ corresponding to every self-adjoint transformation H.

Extension $E(\lambda)$ to $\boldsymbol{E}(U)$

§ 1 We consider the one dimentional Euclidean space R_1. In this paper unless otherwise stated, "interval" means the semi-closed interval $[a, b)$, i. e. closed on the left and open on the right. And we shall denote it by $\varDelta$ generally. If $\varDelta$ and $\varDelta'$ are two intervals, then the common set $\varDelta\varDelta'$ is also an interval in the above sense. Next I define "elementary set" as the sum of the finite or infinite number of intervals of which any two have no point in common, and denote it by J generally, stated in symbols

$$J = \varDelta_1+\varDelta_2+\ldots, \qquad \text{or} \qquad J = \sum_n \varDelta_n .$$

If J and J' are elementary sets, common set JJ' is also an elementary set; that is, let be $J = \sum\limits_n \varDelta_n$ and $J' = \sum\limits_m \varDelta'_m$, then $JJ' = \sum\limits_{m,n} \varDelta_n\varDelta'_m$

§ 2 Let $E(\lambda)$ be a resolution of identity as mentioned in the introduction. If we put $\alpha_{\mathfrak{f}}(\lambda) = \|E(\lambda)\mathfrak{f}\|^2$, $\mathfrak{f}$ being an element of $\mathfrak{R}$, then $\alpha_{\mathfrak{f}}(\lambda)$ is non negative and for $\lambda' < \lambda$

(1) F. Rellich, Math. Ann., **110** (1934), 342–356.
(2) H. Löwig, Acta Szeged, **7** (1934), 1–33.
(3) F. Riesz, acta Szeged, **7** (1934), 34–38.
(4) J. v. Neumann, Math. Ann., **102** (1929), 91–96.

$$\alpha_{\mathfrak{f}}(\lambda)-\alpha_{\mathfrak{f}}(\lambda') = \| E(\lambda)\mathfrak{f}\|^2 - \| E(\lambda)\mathfrak{f}\|^2$$
$$= \| E(\lambda)\mathfrak{f} - E(\lambda')\mathfrak{f}\|^2 .$$

Then we have

$$\alpha_{\mathfrak{f}}(\lambda) \geqq \alpha_{\mathfrak{f}}(\lambda') \qquad \text{for} \qquad \lambda > \lambda'.$$

$$\lim_{\lambda'\to\lambda-0} \alpha_{\mathfrak{f}}(\lambda') = \alpha_{\mathfrak{f}}(\lambda),$$

$$\lim_{\lambda\to-\infty} \alpha_{\mathfrak{f}}(\lambda) = 0, \qquad \text{and} \qquad \lim_{\lambda\to+\infty} \alpha_{\mathfrak{f}}(\lambda) = \|\mathfrak{f}\|^2 .$$

Therefore $\alpha_{\mathfrak{f}}(\lambda)$ is a non negative monotone increasing and left hand continuous function of λ for fixed $\mathfrak{f}$. Hence, if we put

$$\sigma_{\mathfrak{f}}(\varDelta) = \alpha_{\mathfrak{f}}(b) - \alpha_{\mathfrak{f}}(a) \qquad \text{for} \qquad \varDelta = [a, b),$$

$$\alpha_{\mathfrak{f}}(-\infty) = 0, \qquad \text{and} \qquad \alpha_{\mathfrak{f}}(+\infty) = \|\mathfrak{f}\|^2,$$

then we can determine the complete additive set function $\sigma_{\mathfrak{f}}(U)$[1]; it is convenient to follow the Saks method[2] for my discussion:

Let U be any set in R_1, and $J=\sum_n \varDelta_n$ an elementary set covering U, then $\sum_n \sigma_{\mathfrak{f}}(\varDelta_n)$ has a finite value, and we denote it by $\sigma_{\mathfrak{f}}(J)$. We call the greatest lower bound of such $\sigma_{\mathfrak{f}}(J)$ the outer $\sigma_{\mathfrak{f}}$-measure of U, and denote it by $\overline{\sigma_{\mathfrak{f}}(U)}$. Obviously we have

Lemma 1. If $U \geqq U'$, then $\overline{\sigma_{\mathfrak{f}}(U)} \geqq \overline{\sigma_{\mathfrak{f}}(U')}$.

Lemma 2. *Let $\mathfrak{f}$, $\mathfrak{g}$ be two elements of $\mathfrak{R}$. If $\sigma_{\mathfrak{f}}(\varDelta) \geqq \sigma_{\mathfrak{g}}(\varDelta)$ for every interval $\varDelta$, then for any set U $\overline{\sigma_{\mathfrak{f}}(U)} \geqq \overline{\sigma_{\mathfrak{g}}(U)}$.*

If, as stated in lemma 2, $\sigma_{\mathfrak{f}}(\varDelta) \geqq \sigma_{\mathfrak{g}}(\varDelta)$ for every interval $\varDelta$, then I say that $\mathfrak{f}$ is a majorant of $\mathfrak{g}$ and $\mathfrak{g}$ is a minorant of $\mathfrak{f}$.

If for any positive number ε there exists an open set O covering U such that $\overline{\sigma_{\mathfrak{f}}(O-U)} < \varepsilon$, then we say that U is $\sigma_{\mathfrak{f}}$-measurable. We denote $\overline{\sigma_{\mathfrak{f}}(U)}$ by $\sigma_{\mathfrak{f}}(U)$, and call it $\sigma_{\mathfrak{f}}$-measure of U.

From this definition, we have

Lemma 3. *If $\mathfrak{f}$ is a majorant of $\mathfrak{g}$, and U is $\sigma_{\mathfrak{f}}$-measurable, then U is also $\sigma_{\mathfrak{g}}$-measurable and*

(1) J. Radon, Sitzbr. Wien., **72** Π_a (1913), 11.

(2) S. Saks, Théorie de L'intégrale, (1933), 22–33, and 250.

$$\sigma_{\mathfrak{f}}(U) \geqq \sigma_{\mathfrak{g}}(U).$$

A family of $\sigma_{\mathfrak{f}}$-measurable sets, for all $\mathfrak{f}$ of $\mathfrak{R}$, contains all Borel and analytic sets.

§ 3 Let U be a $\sigma_{\mathfrak{f}}$-measurable set. From the definition of $\sigma_{\mathfrak{f}}(U)$ we can take a sequence of elementary sets $\{J^n\}$ such that

$$\lim_{n\to\infty} \sigma_{\mathfrak{f}}(J^n) = \sigma_{\mathfrak{f}}(U)\,.$$

I shall call this sequence a $\sigma_{\mathfrak{f}}$-admissible sequence of U.

LEMMA 4. *If $\mathfrak{f}$ is a majorant of $\mathfrak{g}$, and $\{J^n\}$ is a $\sigma_{\mathfrak{f}}$-admissible sequence of U, then $\{J^n\}$ is also a $\sigma_{\mathfrak{g}}$-admissible sequence of U.*

Proof. We know from § 2 lemma 3 that U is $\sigma_{\mathfrak{g}}$-measurable, hence it is sufficient to show that

$$\lim_{n\to\infty} \sigma_{\mathfrak{g}}(J^n) = \sigma_{\mathfrak{g}}(U)\,. \tag{1}$$

But
$$\begin{aligned} |\,\sigma_{\mathfrak{g}}(J^n)-\sigma_{\mathfrak{g}}(U)\,| &= \sigma_{\mathfrak{g}}(J^n-U) \\ &\leqq \sigma_{\mathfrak{f}}(J^n-U) \qquad \text{by § 2 lemma 2.} \end{aligned}$$

From the definition of $\sigma_{\mathfrak{f}}$-measure, we have

$$\lim_{n\to\infty} \sigma_{\mathfrak{f}}(J'') = \sigma_{\mathfrak{f}}(U)\,.$$

Hence we have (1).

LEMMA 5. *Let $\mathfrak{f}$, $\mathfrak{g}$ be any two elements of $\mathfrak{R}$. If $\{J^n\}$ is a $\sigma_{\mathfrak{f}}$-admissible sequence of U and $\{J'^n\}$ a $\sigma_{\mathfrak{g}}$-admissible sequence of U, then $\{J^nJ'^n\}$ is a $\sigma_{\mathfrak{f}}$- and $\sigma_{\mathfrak{g}}$-admissible sequence of U.*

Proof. Obviously $J^nJ'^n$ covers U and $J^nJ'^n \subseteqq J^n$.

Then we have $$\sigma_{\mathfrak{f}}(U) \leqq \sigma_{\mathfrak{f}}(J^nJ'^n) \leqq \sigma_{\mathfrak{f}}(J^n)\,,$$

hence $$\lim_{n\to\infty} \sigma_{\mathfrak{f}}(J^nJ'^n) = \sigma_{\mathfrak{f}}(U)\,.$$

Similarly $$\lim_{n\to\infty} \sigma_{\mathfrak{g}}(J_nJ'_n) = \rho_{\mathfrak{g}}(U)\,.$$

LEMMA 6. *If $\{J^n\}$ is a $\sigma_{\mathfrak{f}}$-admissible sequence of U and $\{J'^n\}$ of U', then $\{J^nJ'^n\}$ is $\sigma_{\mathfrak{f}}$-admissible sequence of UU'.*

This is evident from the fact that

$$| \sigma_{\mathfrak{f}}(J^n J'^n) - \sigma_{\mathfrak{f}}(UU') | = \sigma_{\mathfrak{f}}(J^n J'^n - UU') \leqq \sigma_{\mathfrak{f}}(J^n - U) + \sigma_{\mathfrak{f}}(J'^n - U').$$

§ 4 Put

$$\boldsymbol{E}(\varDelta) = E(b) - E(a) \qquad \text{for} \qquad \varDelta = [a, b),$$

and

$$E(+\infty) = 1, \qquad E(-\infty) = 0.$$

From § 2 we have

$$\| \boldsymbol{E}(\varDelta)\mathfrak{f} \|^2 = \sigma_{\mathfrak{f}}(\varDelta),$$

then we can write (1), in the Introduction, in the form

$$\text{(1)} \qquad \boldsymbol{E}(\varDelta)\boldsymbol{E}(\varDelta') = \boldsymbol{E}(\varDelta\varDelta'),^{(1)} \qquad \text{for} \qquad \varDelta = (-\infty, \lambda), \varDelta' = (-\infty, \mu).$$

If an interval $\varDelta$ consists of two intervals $\varDelta_1, \varDelta_2$, that is, $\varDelta = \varDelta_1 + \varDelta_2$, we obviously have

$$\boldsymbol{E}(\varDelta) = \boldsymbol{E}(\varDelta_1) + \boldsymbol{E}(\varDelta_2).$$

Therefore if $\varDelta_1$ is contained in $\varDelta_2$ and has a common end point, then $\varDelta = \varDelta_2 - \varDelta_1$ is also an interval, and we have

$$\text{(2)} \qquad \boldsymbol{E}(\varDelta) = \boldsymbol{E}(\varDelta_2) - \boldsymbol{E}(\varDelta_1).$$

LEMMA 7. *If $\varDelta$ and $\varDelta'$ are intervals, then*

$$\boldsymbol{E}(\varDelta)\boldsymbol{E}(\varDelta') = \boldsymbol{E}(\varDelta\varDelta').$$

Proof. Let

$$\varDelta = [a, b), \varDelta_1 = (-\infty, a), \varDelta_2 = (-\infty, b),$$

$$\varDelta' = [a', b'), \varDelta_1' = (-\infty, a'), \varDelta_2' = (-\infty, b'),$$

then we have

$$\varDelta = \varDelta_2 - \varDelta_1,$$

$$\varDelta' = \varDelta_2' - \varDelta_1'.$$

$$\boldsymbol{E}(\varDelta)\boldsymbol{E}(\varDelta') = \{\boldsymbol{E}(\varDelta_2) - \boldsymbol{E}(\varDelta_1)\}\{\boldsymbol{E}(\varDelta_2') - \boldsymbol{E}(\varDelta_1')\}$$

(1) This notation indicates that for any element $\mathfrak{f}$ of $\mathfrak{R}$, $\boldsymbol{E}(\varDelta)\boldsymbol{E}(\varDelta')\mathfrak{f} = \boldsymbol{E}(\varDelta\varDelta')\mathfrak{f}$, and similarly for analogous notations.

$$= E(\Delta_2\Delta_2') - E(\Delta_1\Delta_2') - E(\Delta_2\Delta_1') + E(\Delta_1\Delta_1') \qquad \text{by (1)}$$

$$= E(\Delta\Delta_2') - E(\Delta\Delta_1') \qquad \text{by (2)}$$

$$= E(\Delta\Delta_2' - \Delta\Delta_1') \qquad \text{by (2)}$$

$$= E(\Delta\Delta')$$

LEMMA 8. *If* $\Delta = \Delta_1 + \Delta_2 + \ldots.,$

$$E(\Delta) = E(\Delta_1) + E(\Delta_2) + \ldots.$$

Proof. Since $E(\Delta_n)E(\Delta_m) = 0$ for $m \neq n$ by lemma 7, we have

$$\| E(\Delta)\mathfrak{f} - \sum_{n=1}^{N} E(\Delta_n)\mathfrak{f} \|^2 = \| E(\Delta)\mathfrak{f} \|^2 - \sum_{n=1}^{N} \| E(\Delta_n)\mathfrak{f} \|^2$$
$$= \sigma_{\mathfrak{f}}(\Delta) - \sigma_{\mathfrak{f}}(\sum_{n=1}^{N} \Delta_n) ,$$

for any element $\mathfrak{f}$ of $\mathfrak{R}$.

But since
$$\lim_{N\to\infty} \sigma_{\mathfrak{f}}(\sum_{n=1}^{N} \Delta_n) = \sigma_{\mathfrak{f}}(\Delta) ,$$

hence we have the lemma.

Let J be an elementary set, say $\sum_n \Delta_n$, then we define $E(J)$ as follows

$$E(J) = E(\Delta_1) + E(\Delta_2) + \ldots.$$

Since $E(\Delta_n)E(\Delta_m) = 0$ for $n \neq m$, $E(J)$ is a projective transformation[1] and $\| E(J)\mathfrak{f} \|^2 = \sigma_{\mathfrak{f}}(J)$.

LEMMA 9. *If J is expressed in two forms, say, $\sum_n \Delta_n, \sum_n \Delta_m'$, then we have*

$$\sum_n E(\Delta_n) = \sum_m E(\Delta_m') .$$

Proof. It is sufficient to show that $\sum_n E(\Delta_n) = \sum_{m,n} E(\Delta_n\Delta_m')$. For any element $\mathfrak{f}$ of $\mathfrak{R}$ we have

$$\| \sum_{n=1}^{N} E(\Delta_n)\mathfrak{f} - \sum_{n=1}^{N} \sum_{m=1}^{M} E(\Delta_n\Delta_m')\mathfrak{f} \|^2$$
$$= \sum_{n=1}^{N} \| E(\Delta_n)\mathfrak{f} \|^2 - \sum_{n=1}^{N} \sum_{m=1}^{M} \| E(\Delta_n\Delta_m')\mathfrak{f} \|^2 .$$

(1) J. v. Neumann, loc. cit., 77.

When $n \to \infty$, $m \to \infty$, the last term converges to zero. Therefore we have the lemma.

LEMMA 10. *Let J, J' be two elementary sets, then*

$$E(J)E(J') = E(JJ')$$

Proof. It is evident from the fact that

$$E(\sum_{n=1}^{N} \Delta_n)E(\sum_{m=1}^{M} \Delta'_m) = E(\sum_{n=1}^{N} \sum_{m=1}^{M} \Delta_n \Delta'_m) \qquad \text{by lemma 7.}$$

§ 5 Let U be a $\sigma_{\mathfrak{f}}$-measurable set for a fixed element $\mathfrak{f}$ of $\mathfrak{R}$. If $\{J^n\}$ is a $\sigma_{\mathfrak{f}}$-admissible sequence of U,

$$\| E(J^n)\mathfrak{f} - E(J^m)\mathfrak{f} \|^2 = \sigma_{\mathfrak{f}}(J^n) + \sigma_{\mathfrak{f}}(J^m) - 2\sigma_{\mathfrak{f}}(J^n J^m)\,.$$

Therefore we have

$$\lim_{\substack{n\to\infty \\ m\to\infty}} \| E(J^n)\mathfrak{f} - E(J^m)\mathfrak{f} \| = 0\,.$$

Consequently $E(J^n)\mathfrak{f}$ converges strongly to an element, say $E(U)\mathfrak{f}$, of $\mathfrak{R}$. We show that $E(U)\mathfrak{f}$ is independent of a $\sigma_{\mathfrak{f}}$-admissible sequence of U. Let $\{J'^n\}$ be another $\sigma_{\mathfrak{f}}$-admissible sequence of U, then evidently $J', J'^1, J^2, J'^2, \ldots.$ is also a $\sigma_{\mathfrak{f}}$-admissible sequence of U and $\{J^n\}$ and $\{J'^n\}$ are subsequences of this sequence, so that we have the conclusion.

THEOREM 1. *If U, U' are $\sigma_{\mathfrak{f}}$-measurable, then*

$$E(U)E(U')\mathfrak{f} = E(UU')\mathfrak{f}$$

Proof. Let $\{J^n\}$, $\{J'^n\}$ be a $\sigma_{\mathfrak{f}}$-admissible sequence of U, U' respectively, then by § 4 lemma 10 we have

$$E(J^n)E(J'^n)\mathfrak{f} = E(J^n J'^n)\mathfrak{f}\,.$$

The right hand side of this equation converges to $E(UU')\mathfrak{f}$ by § 3 lemma 6 when $n \to \infty$. Since

$$\| E(J^n)E(J'^n)\mathfrak{f} - E(J^n)E(U')\mathfrak{f} \| \leqq \| E(J'^n)\mathfrak{f} - E(U')\mathfrak{f} \|\,,$$

and the right hand side of the inequality converges to zero when $n \to \infty$, it is sufficient to show that $\{J^n\}$ is a $\sigma_{E(U')\mathfrak{f}}$-admissible sequence of U.

$$\sigma_{E(U')\mathfrak{f}}(\varDelta) = \| E(\varDelta)E(U')\mathfrak{f} \|^2 = \lim_{n\to\infty} \| E(\varDelta)E(J'^n)\mathfrak{f} \|^2$$
$$= \lim_{n\to\infty} \| E(J'^n)E(\varDelta)\mathfrak{f} \|^2 \leqq \| E(\varDelta)\mathfrak{f} \|^2 = \sigma_{\mathfrak{f}}(\varDelta) .$$

Therefore $\mathfrak{f}$ is a majorant of $E(U')\mathfrak{f}$, hence by § 3 lemma 4 $\{J^n\}$ is also a $\sigma_{E(U')\mathfrak{f}}$-admissible sequence.

LEMMA 11. *If U is $\sigma_{\mathfrak{f}}$-measurable and $\mathfrak{g}$, $\mathfrak{h}$ are minorants of $\mathfrak{f}$, then*

$$\big(E(U)\mathfrak{g}, \mathfrak{h}\big) = \big(\mathfrak{g}, E(U)\mathfrak{h}\big) = \big(E(U)\mathfrak{g}, E(U)\mathfrak{h}\big) .$$

Proof. If $\{J^n\}$ is a $\sigma_{\mathfrak{f}}$-admissible sequence of U, then by § 3 lemma 4 $\{J^n\}$ is also a $\sigma_{\mathfrak{g}}$- and $\sigma_{\mathfrak{h}}$-admissible sequence of U. Since $E(J^n)$ is a projective transformation we have

$$\big(E(J^n)\mathfrak{g}, \mathfrak{h}\big) = \big(\mathfrak{g}, E(J^n)\mathfrak{h}\big) = \big(E(J^n)\mathfrak{g}, E(J^n)\mathfrak{h}\big) .$$

When $n \to \infty$, we have the lemma.

THEOREM 2. *When U is the sum of $\sigma_{\mathfrak{f}}$-measurable sets $\{U_n\}$, that is,*

$$U = U_1 + U_2 + \dots ,$$
$$E(U)\mathfrak{f} \,[=]\, E(U_1)\mathfrak{f} + E(U_2)\mathfrak{f} + \dots$$

Proof. Since $E(U_n)\mathfrak{f}$ is a minorant of $\mathfrak{f}$ as shown in the proof of theorem 1, we have

$$\big(E(U)\mathfrak{f}, E(U_i)\mathfrak{f}\big) = \big(E(U_i)E(U)\mathfrak{f}, E(U_i)\mathfrak{f}\big) \quad \text{by above lemma}$$
$$= \big(E(U_iU)\mathfrak{f}, E(U_i)\mathfrak{f}\big) \quad \text{by theorem 1}$$
$$= \sigma_{\mathfrak{f}}(U_i)$$

Similarly $\big(E(U_i)\mathfrak{f}, E(U_j)\mathfrak{f}\big) = \sigma_{\mathfrak{f}}(U_iU_j)$

It follows easily that

$$\| E(U)\mathfrak{f} - \sum_{n=1}^{N} E(U_n)\mathfrak{f} \|^2 = \sigma_{\mathfrak{f}}(U) - \sigma_{\mathfrak{f}}(\sum_{n=1}^{N} U_n) .$$

When $n \to \infty$, we have the theorem.

§ **6** Consider the family of sets which are $\sigma_{\mathfrak{f}}$-measurable for all $\mathfrak{f}$ of $\mathfrak{R}$. Since $\boldsymbol{E}(R_1) = 1$, if we show that $\boldsymbol{E}(U)$ is a self-adjoint transformation for every set of this family, which we shall call the $\boldsymbol{E}$ family, then $\boldsymbol{E}(U)$ is, by theorem 1 and 2, a resolution of identity in the sense of F. Maeda.

Let U be a set of $\boldsymbol{E}$ family, and $\mathfrak{f}$, $\mathfrak{g}$ any two elements of $\mathfrak{R}$. We can take a $\sigma_{\mathfrak{f}}$- and $\sigma_{\mathfrak{g}}$-admissible sequence $\{J^n\}$ by § 3 lemma 5. Since $\boldsymbol{E}(J^n)$ is a projective transformation, we have

$$\left(\boldsymbol{E}(J^n)\mathfrak{f}, \mathfrak{g}\right) = \left(\mathfrak{f}, \boldsymbol{E}(J^n)\mathfrak{g}\right)$$

When $n \to \infty$, we have

$$\left(\boldsymbol{E}(U)\mathfrak{f}, \mathfrak{g}\right) = \left(\mathfrak{f}, \boldsymbol{E}(U)\mathfrak{g}\right)$$

§ 7 Let $E_1(a), E_2(b)$ be two permutable resolutions of identity which are defined in R_1. If we put $E(a, b) = E_1(a)E_2(b) = E_2(b)E_1(a)$, $E(a, b)$ has the following properties[2];

(1) $\quad E(a, b)E(a', b') = E\left(\min(a, a'), \min(b, b')\right).$

(2) $\quad \lim\limits_{\substack{a' \to a-0 \\ b' \to b-0}} E(a', b') = E(a, b)$

(3) $\quad \lim\limits_{a \to -\infty} E(a, b) = 0, \quad \lim\limits_{b \to -\infty} E(a, b) = 0, \quad$ and $\quad \lim\limits_{\substack{a \to +\infty \\ b \to +\infty}} E(a, b) = 1$

where 1 represents an identical transformation.

When we consider (a, b) as a point in the complex plane R_2, a family of projective transformation, $E(a, b)$, is called a complex resolution of identity.

As in the case of $E(\lambda)$, starting from $E(a, b)$ we can define a resolution of identity $\boldsymbol{E}(U)$, which depends on a plane set U. I shall here give in outline, the method of defining $\boldsymbol{E}(U)$.

Let $\mathfrak{f}$ be an element of $\mathfrak{R}$, and put

$$\alpha_{\mathfrak{f}}(a, b) = \| E(a, b)\mathfrak{f} \|^2$$

Then $\alpha_{\mathfrak{f}}(a, b)$ is non-negative and bounded for fixed $\mathfrak{f}$. It is easy to prove that $\alpha_{\mathfrak{f}}(a, b)$ has the monotone property;

(1) See introduction.

(2) J. v. Neumann, Math. Ann., **102** (1929), 411.

$$\alpha_{\mathfrak{f}}(a, b)-\alpha_{\mathfrak{f}}(a, b')-\alpha_{\mathfrak{f}}(a', b)+\alpha_{\mathfrak{f}}(a', b') \geqq 0 \qquad a \geqq a', \quad b \geqq b'.$$

Hence by (2) we have

$$\lim_{\substack{a' \to a-0 \\ b' \to b-0}} \alpha_{\mathfrak{f}}(a', b') = \alpha_{\mathfrak{f}}(a, b).$$

By these two properties we can determine a completely additive set function $\sigma_{\mathfrak{f}}(U)$ by the method mentioned in § 2[1]; it is sufficient to consider the "interval" $\Delta = (a', a\,;\, b', b)$ as the set of points (λ, μ) which satisfy the relation

$$a' \leqq \lambda < a, \qquad b' \leqq \mu < b,$$

and

$$\sigma_{\mathfrak{f}}(\Delta) \quad \text{as} \quad \sigma_{\mathfrak{f}}(a, b)-\alpha_{\mathfrak{f}}(a, b')-\alpha_{\mathfrak{f}}(a', b)+\alpha_{\mathfrak{f}}(a', b'),$$

If we put $\quad E(\Delta) = E(a, b)-E(a, b')-E(a', b)+E(a', b')$,

then we can define $E(U)$ by the process described in the preceding §§.

J. v. Neumann's method of determining $E(U)$ of self-adjoint transformation.

§ 8 For every bounded self-adjoint transformation A we can find a resolution of identity $E(\lambda)$ by Riesz' method[2] which seems to me, from the analytical view-point, most suggestive so that we can find the resolution of identity $E(U)$ corresponding to $E(\lambda)$ by the preceding §§.

Divide R_1 by the intersection of points, such that

$$-\infty < \ldots\ldots < \lambda_{-n} < \ldots\ldots < \lambda_0 < \ldots\ldots < \lambda_n < \ldots\ldots < +\infty$$

and

$$\lambda_{n+1} - \lambda_n < \varepsilon,$$

where ε is a given positive number.

Let Δ_n be $(\lambda_n, \lambda_{n+1})$, and $\varphi_{\Delta_n}(\lambda)$ its characteristic function, then corresponding to $\varphi_{\Delta_n}(\lambda)$, there exists a projective transformation $E(\Delta_n) = \varphi_{\Delta_n}(A)$.

Since

$$\left| \lambda - \sum_n l_n \varphi_{\Delta_n}(\lambda) \right| \leqq \varepsilon$$

(1) Radon, loc. cit., II.

(2) F. Riesz, loc. cit., 31–37.

we have $$\| A\mathfrak{f} - \sum_n l_n E(\Delta_n)\mathfrak{f} \| \leqq \varepsilon \| \mathfrak{f} \|,$$

for any element $\mathfrak{f}$ of $\mathfrak{R}$. Consequently we have[1]

$$A\mathfrak{f} = \int_{R_1} \lambda dE(U)\mathfrak{f}.$$

§ 9 Let H be a self-adjoint transformation and $\mathfrak{f}$ any element of the domain of definition $\mathfrak{D}$. And put

$$H\mathfrak{f} + i\mathfrak{f} = \mathfrak{x}, \tag{1}$$

$$H\mathfrak{f} - i\mathfrak{f} = U\mathfrak{x}. \tag{2}$$

J. v. Neumann has shown[2] that U is a unitary transformation and that $U\mathfrak{x} = \mathfrak{x}$ means $\mathfrak{x} = 0$.

Put

$$A = \frac{1}{2}(U + U^*) \tag{3}$$

$$B = \frac{1}{2}(U - U^*) \tag{4}$$

Then A and B are self-adjoint transformations and permutable. Therefore $E_1(a), E_2(b)$, corresponding to A and B respectively, are permutable. From the preceding § we have

$$A\mathfrak{f} = \int_{R_1} a d\boldsymbol{E}_1(U)\mathfrak{f},$$

$$B\mathfrak{f} = \int_{R_1} b d\boldsymbol{E}_2(U)\mathfrak{f},$$

where $\boldsymbol{E}_1(U)$ and $\boldsymbol{E}_2(U)$ are resolutions of identity corresponding to $E_1(a)$ and $\boldsymbol{E}_2(b)$ respectively.

From the definition of integral[3] we have

$$\int_{R_1} a d\boldsymbol{E}_1(U)\mathfrak{f} = \int_{R_2} a d\boldsymbol{E}(U)\mathfrak{f},$$

(1) For the meanings of integral see F. Maeda, loc. cit., 60-69.
(2) J. v. Neumann, loc. cit., 91.
(3) F. Maeda, loc. cit., 62.

where R_2 denotes the complex plane and $\boldsymbol{E}(U)$ is a resolution of identity corresponding to $E(a, b) = E_1(a)E_2(b)$.

Therefore we have $$A\mathfrak{f} = \int_{R_2} ad\boldsymbol{E}(U)\mathfrak{f}.$$

Similarly $$B\mathfrak{f} = \int_{R_2} bd\boldsymbol{E}(U)\mathfrak{f}.$$

From (3) (4) $\boldsymbol{U} = A + iB$, hence we have

$$\text{(5)} \qquad \boldsymbol{U}\mathfrak{f} = \int_{R_2} zd\boldsymbol{E}(U)\mathfrak{f},$$

where $$z = a + bi.$$

Let C be a unit circle with centre origin, and C_- and C_+ its inner and outer parts in R_2 respectively. Since $\boldsymbol{U}$ is a unitary transformation,

$$\| \boldsymbol{E}(C_-)\mathfrak{f} \|^2 = \| \boldsymbol{U}\boldsymbol{E}(C_-)\mathfrak{f} \|^2.$$

But, since

$$\| \boldsymbol{E}(U)\boldsymbol{E}(C_-)\mathfrak{f} \|^2 = \| \boldsymbol{E}(UC_-)\mathfrak{f} \|^2 = \sigma_{\mathfrak{f}}(UC_-)$$

we have $$\sigma_{\mathfrak{f}}(C_-) = \int_{R_2} | z |^2 d\sigma_{\mathfrak{f}}(UC_-)^{(1)} \qquad \text{by (5)};$$

that is, $$\int_{C-} (| z |^2 - 1)d\sigma_{\mathfrak{f}}(U) = 0$$

This equation holds for all $\mathfrak{f}$ of $\mathfrak{R}$, and $|z|^2 - 1$ is negative in C_-, therefore $\sigma_{\mathfrak{f}}(C_-) = 0$ for all $\mathfrak{f}$, that is $\boldsymbol{E}(C_-) = 0$ identically. By the same reasoning we shall have $\boldsymbol{E}(C_+) = 0$. Hence (5) can be written as follows:

$$\text{(6)} \qquad \boldsymbol{U}\mathfrak{f} = \int_{C} zd\boldsymbol{E}(U)\mathfrak{f},$$

where C is a unit circle with centre origin, and $\boldsymbol{E}(U)$ is a resolution of identity which depends on a subset of this unit circle.
But $\boldsymbol{U}\mathfrak{x} = \mathfrak{x}$ means $\mathfrak{x} = 0$; hence by (6)

$$\int_{C} zd\boldsymbol{E}(U)\mathfrak{x} = \mathfrak{x},$$

(1) F. Maeda, loc. cit., 65.

that is
$$\int_C |z-1|^2 d\sigma_{\mathfrak{x}}(U) = 0 .$$[1]

Since $|z-1|^2$ is positive except for $z=1$, if we denote by p_0 the point $z=1$ in R_2, then $\sigma_{\mathfrak{x}}(C-p_0)=0$, that is $E(C-p_0)\mathfrak{x}=0$. But $E(C)=1$, hence we have

$$E(p_0)\mathfrak{x} = \mathfrak{x} .$$

From this equation, we can not infer that $\mathfrak{x}=0$ unless $E(p_0)=0$, so that (6) can be written in the form

$$U\mathfrak{f} = \int_{C-p_0} z dE(U)\mathfrak{f} \tag{7}$$

§ 10 From § 9 (1) (2) we have

$$H\mathfrak{f} = \frac{1}{2}(\mathfrak{x}+U\mathfrak{x}) ,$$

$$\mathfrak{f} = \frac{1}{2i}(\mathfrak{x}-U\mathfrak{x}) ,$$

hence by § 9 (7) we have

$$H\mathfrak{f} = \int_{C-p_0} \frac{1+z}{2} dE(U)\mathfrak{x} , \tag{1}$$

$$\mathfrak{f} = \int_{C-p_0} \frac{1-z}{2i} dE(U)\mathfrak{x} . \tag{2}$$

Let C_ε be the set of points of C such that $|z-1|>\varepsilon$, then $\frac{2i}{1-z}$ is bounded on C_ε. By the theory of integration,

$$\int_{C_\varepsilon} \frac{2i}{1-z} dE(U)\mathfrak{f} \tag{3}$$

exists. But from (2) we have

$$E(U)\mathfrak{f} = \int_U \frac{1-z}{2i} dE(U')\mathfrak{x}$$[2]

Hence (3) becomes

(1) F. Maeda, loc. cit., 65.
(2) F. Maeda, loc. cit., 79.

$$\int_{C_\varepsilon} \frac{2i}{1-z} d\int_U \frac{1-z}{2i} dE(U')\mathfrak{x}$$

$$= \int_{C_\varepsilon} dE(U)\mathfrak{x}^{(1)} = E(C_\varepsilon)\mathfrak{x}.$$

Evidently $\lim_{\varepsilon \to 0} C_\varepsilon = C - p_0$, Since $E(p_0) = 0$, we have by $\varepsilon \to 0$[2]

(4) $$\mathfrak{x} = \int_{C-p_0} \frac{2i}{1-z} dE(U)\mathfrak{f}$$

When we substitute (4) into (1), we have by the same reasoning as above

(5) $$H\mathfrak{f} = \int_{C-p_0} i\frac{1+z}{1-z} dE(U)\mathfrak{f}.$$

The expression $\lambda = i\dfrac{1+z}{1-z}$ represents a transformation which transforms a unit circle into a real axis. And its inversion is represented by $z = \dfrac{\lambda - i}{\lambda + i}$. Let U be transformed into a set $\overline{U}$ on the real axis R_1, and let it be denoted symbolycally

$$i\frac{1+z}{1-z}U = \overline{U}, \qquad U = \frac{\lambda - i}{\lambda + i}\overline{U}.$$

Then (5) becomes

$$H\mathfrak{f} = \int_{R_1} \lambda d\overline{E}(\overline{U})\mathfrak{f},$$

where $\overline{E}(\overline{U}) = E\left(\dfrac{\lambda - i}{\lambda + i}\overline{U}\right)$.

Thus we obtain the resolution of identity $\overline{E}(\overline{U})$ corresponding to the self-adjoint transformation H.

In conclusion, the writer wishes to express his hearty thanks to Prof. F. Maeda for his kind guidance.

(1) F. Maeda, loc. cit., 76–78.
(2) F. Maeda, loc. cit., 68.