Kernels of Transformations in the Space of Set Functions.

By

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Let $\beta(E)$ be a completely additive, non-negative set function defined for all Borel subsets of a Borel set A in a separable metric space. Let $\phi(E)$ be a complex valued set function which is absolutely continuous with respect to $\beta(E)$. When $\int_A |D_{\beta(E)}\phi(a)|^2 d\beta(E)$ is finite, $\phi(E)$ is said to belong to the class $\mathfrak{L}_2(\beta)$. Then $\mathfrak{L}_2(\beta)$ is a Hilbert space with the inner product.

$$(\phi, \psi) = \int_A D_{\beta(E)}\phi(a)\overline{D_{\beta(E)}\psi(a)}d\beta(E) .^{(1)}$$

In a previous paper,(2) I proved that all bounded linear transformations T defined in $\mathfrak{L}_2(\beta)$ can be expressed in the integral form

$$T\phi(E) = \int_A D_{\beta(E')}\mathfrak{K}(E, a')D_{\beta(E')}\phi(a')d\beta(E') , \qquad (1)$$

and the kernels of T are expressed as follows:

$$\mathfrak{K}(E, E') \, [=]_{E,E'} \sum_\nu \zeta_\nu(E)\overline{\psi_\nu(E')} ,^{(3)}$$

where $\{\psi_\nu(E)\}$ is a complete normalized orthogonal system in $\mathfrak{L}_2(\beta)$ and

$$\zeta_\nu(E) = T\psi_\nu(E) \qquad (\nu = 1, 2, \ldots\ldots) .$$

Of course, $\mathfrak{K}(E, E')$ belongs to $\mathfrak{L}_2(\beta)$ as a function of set E and of set E'. In this case, I say that $\mathfrak{K}(E, E')$ belongs to $\mathfrak{L}_2(\beta, \beta)$.

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(1) Cf. F. Maeda, this journal, **3** (1933), 243; and **4** (1934), 141–142.

(2) F. Maeda, this journal, **3** (1933), 244–251.

(3) This expression means that $\sum_\nu \zeta_\nu(E)\psi_\nu(E')$ converges strongly to $\mathfrak{K}(E, E')$ as functions of set E and of set E'.

In this paper, I have investigated in a more general manner the properties of the transformation expressed in the integral form (1) with kernel $\mathfrak{K}(E, E')$ belonging to $\mathfrak{L}_2(\beta, \beta)$, and what transformations can be expressed in the integral form (1). And next I have shown that every unitary transformation U is representable in the form

$$U\phi(E) = \int_A D_{\beta(E')}\mathfrak{U}(E, a')D_{\beta(E')}\phi(a')d\beta(E') \qquad (2)$$

with kernel $\mathfrak{U}(E, E')$ which satisfies the following conditions:

$$\int_A D_{\beta(E'')}\mathfrak{U}(a'', E)\overline{D_{\beta(E'')}\mathfrak{U}(a'', E')}d\beta(E'') = \beta(EE'),$$

$$\int_A D_{\beta(E'')}\mathfrak{U}^*(a'', E)\overline{D_{\beta(E'')}\mathfrak{U}^*(a'', E')}d\beta(E'') = \beta(EE'),$$

$$\mathfrak{U}^*(E, E') = \overline{\mathfrak{U}(E', E)};$$

and conversely, if $\mathfrak{U}(E, E')$ satisfies these conditions, then (2) represents a unitary transformation.[(1)]

In this paper, $\mathfrak{K}(E, _{(E')})$ means $\mathfrak{K}(E, E')$ considered as a function of set E, E' being a parameter. Similarly for $\mathfrak{K}(_{(E)}, E')$.

Kernels of General Transformations.

1. THEOREM 1. *Put* $\mathfrak{E}(E, E') = \beta(EE')$. *Then* $\mathfrak{E}(E, E')$ *is the kernel of identical transformation.*[(2)]

(1) Recently, S. Bochner obtained an analogous result in the space of complex-valued point functions $f(x)$ with the inner product $(f, g) = \int_0^\infty f(x)g(x)dx$: Every unitary transformation $g(x) = Uf(x)$ is representable in the form

$$\int_0^x g(\xi)d\xi = \int_0^\infty \overline{K(x, y)}f(y)dy,$$

where $K(x, y)$ satisfies the following conditions:

$$\int_0^\infty K(a, y)\overline{K(b, y)}dy = \min(a, b), \qquad \int_0^\infty K^*(a, y)\overline{K^*(b, y)}dy = \min(a, b),$$

$$\int_0^a K^*(b, y)dy = \int_0^b \overline{K(a, y)}dy;$$

and vice versa. (Annals of Math. (2) **35** (1934), 111–114.)

(2) This theorem has already been mentioned in my previous paper (this journal, **3** (1933), 245).

For, since

$$D_{\beta(E')}\mathfrak{E}(E, a') = 1 \qquad \text{when } a' \text{ is a point of } E,$$
$$= 0 \qquad \text{when } a' \text{ is not a point of } E,$$

almost everywhere (β), we have

$$\int_A D_{\beta(E')}\mathfrak{E}(E, a')D_{\beta(E')}\phi(a')d\beta(E') = \int_E D_{\beta(E')}\phi(a')d\beta(E') = \phi(E)$$

for all $\phi(E)$ in $\mathfrak{L}_2(\beta)$.

Now consider the system $\{\mathfrak{E}(E, {}_{(E')})\}$, E' being a parameter. And denote by $\mathfrak{L}_2'(\beta)$ the linear manifold determined by $\{\mathfrak{E}(E, {}_{(E')})\}$. Then $\mathfrak{L}_2'(\beta)$ is dense in $\mathfrak{L}_2(\beta)$. For, let $\phi(E)$ be any set function in $\mathfrak{L}_2(\beta)$, then, as above,

$$\left(\mathfrak{E}(E, {}_{(E')}), \phi(E)\right) = \overline{\phi(E')}.$$

Hence, any set function orthogonal to all set functions in $\{\mathfrak{E}(E, {}_{(E')})\}$ vanishes identically. Therefore, the closed linear manifold determined by $\{\mathfrak{E}(E, {}_{(E')})\}$ is $\mathfrak{L}_2(\beta)$; that is, $\mathfrak{L}_2'(\beta)$ is dense in $\mathfrak{L}_2(\beta)$.

2. Let $\mathfrak{K}(E, E')$ be a set function in $\mathfrak{L}_2(\beta, \beta)$. Denote by $\mathfrak{D}_{\mathfrak{K}}$ the aggregate of set functions $\phi(E)$ in $\mathfrak{L}_2(\beta)$ so that

$$\int_A D_{\beta(E')}\mathfrak{K}(E, a')D_{\beta(E')}\phi(a')d\beta(E') \tag{1}$$

is a set function in $\mathfrak{L}_2(\beta)$. Then, $\mathfrak{D}_{\mathfrak{K}}$ is a linear manifold, and (1) is a linear transformation with domain $\mathfrak{D}_{\mathfrak{K}}$. We denote this linear transformation by $\boldsymbol{T}_{\mathfrak{K}}$.

THEOREM 2. *$\boldsymbol{T}_{\mathfrak{K}}$ is a closed linear transformation, and its domain $\mathfrak{D}_{\mathfrak{K}}$ contains $\mathfrak{L}_2'(\beta)$.*

Proof. Let $\{\phi_n(E)\}$ be a sequence of set functions in $\mathfrak{D}_{\mathfrak{K}}$ such that

$$[\lim_{n\to\infty}]\, \phi_n(E) = \phi(E), \text{[1]} \tag{2}$$

$$[\lim_{n\to\infty}]\, \boldsymbol{T}_{\mathfrak{K}}\phi_n(E) = \psi(E), \tag{3}$$

(1) This means that $\{\phi_n(E)\}$ converges strongly to $\phi(E)$ in $\mathfrak{L}_2(\beta)$.

$\phi(E)$, $\psi(E)$ being set functions in $\mathfrak{L}_2(\beta)$. By (2)

$$\lim_{n\to\infty} \boldsymbol{T}_{\mathfrak{R}}\phi_n(E) = \lim_{n\to\infty}\left(\mathfrak{R}(_{(E)}, E'), \phi_n(E')\right) = \left(\mathfrak{R}(_{(E)}, E'), \phi(E')\right).$$

Hence, by (3)[1]

$$\int_A D_{\beta(E')}\mathfrak{R}(E, a')D_{\beta(E')}\phi(a')d\beta(E') = \psi(E).$$

That is, $\phi(E)$ belongs to $\mathfrak{D}_{\mathfrak{R}}$, and

$$\boldsymbol{T}_{\mathfrak{R}}\phi(E) = \psi(E).$$

Consequently, $\boldsymbol{T}_{\mathfrak{R}}$ is closed.

Since $$\boldsymbol{T}_{\mathfrak{R}}\mathfrak{E}(E, _{(E')}) = \mathfrak{R}(E, _{(E')}),$$

$\mathfrak{E}(E, _{(E')})$ belongs to $\mathfrak{D}_{\mathfrak{R}}$; that is,

$$\mathfrak{D}_{\mathfrak{R}} \geqq \mathfrak{L}_2'(\beta).$$

Put $$\mathfrak{R}^*(E, E') = \overline{\mathfrak{R}(E', E)};$$

then, as above, $\boldsymbol{T}_{\mathfrak{R}^*}$ is a closed linear transformation with domain $\mathfrak{D}_{\mathfrak{R}^*} \geqq \mathfrak{L}_2'(\beta)$.

3. Since $\mathfrak{D}_{\mathfrak{R}}$ is dense in $\mathfrak{L}_2(\beta)$, the adjoint transformation $\boldsymbol{T}_{\mathfrak{R}}^*$ of $\boldsymbol{T}_{\mathfrak{R}}$ exists. Its domain $\mathfrak{D}_{\mathfrak{R}}^*$ consists of those and only those set functions $\psi(E)$ such that the relation

$$(\boldsymbol{T}_{\mathfrak{R}}\phi, \psi) = (\phi, \xi) \tag{1}$$

holds for all set functions $\phi(E)$ in $\mathfrak{D}_{\mathfrak{R}}$ and some set function $\xi(E)$ in $\mathfrak{L}_2(\beta)$; and for such a set function,

$$\boldsymbol{T}_{\mathfrak{R}}^*\psi(E) = \xi(E).$$ [2]

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(1) In $\mathfrak{L}_2(\beta)$, if $[\lim_{n\to\infty}]\,\phi_n(E) = \phi(E)$, then $\lim_{n\to\infty}\phi_n(E) = \phi(E)$. For, by the Schwarzian inequality

$$\left|\int_E\{D_{\beta(E)}\phi_n(a) - D_{\beta(E)}\phi(a)\}d\beta(E)\right|^2 \leqq \beta(E)\int_E \left|D_{\beta(E)}\phi_n(a) - D_{\beta(E)}\phi(a)\right|^2 d\beta(E);$$

hence $$|\phi_n(E) - \phi(E)|^2 \leqq \beta(E)\,\|\phi_n - \phi\|^2.$$

(2) M. H. Stone, *Linear Transformations in Hilbert Space*, (1932), 42.

THEOREM 3. $\mathfrak{L}_2'(\beta) \subseteqq \mathfrak{D}_{\mathfrak{K}}^* \subseteqq \mathfrak{D}_{\mathfrak{K}^*}$,

and $\boldsymbol{T}_{\mathfrak{K}}^*\psi(E) = \boldsymbol{T}_{\mathfrak{K}^*}\psi(E)$ *for all* $\psi(E)$ *in* $\mathfrak{D}_{\mathfrak{K}}^*$.

That is, $\boldsymbol{T}_{\mathfrak{K}^*}$ is an extension of $\boldsymbol{T}_{\mathfrak{K}}^*$. I will express this relation by $\boldsymbol{T}_{\mathfrak{K}}^* \subseteqq \boldsymbol{T}_{\mathfrak{K}^*}$.[1]

Proof. Since the relation (1) holds for

$$\psi(E) = \mathfrak{E}(E, {}_{(E')}), \qquad \xi(E) = \mathfrak{K}^*(E, {}_{(E')}),$$

we have $\mathfrak{L}_2'(\beta) \subseteqq \mathfrak{D}_{\mathfrak{K}}^*$.

Let $\psi(E)$ be any set function in $\mathfrak{D}_{\mathfrak{K}}^*$, and

$$\boldsymbol{T}_{\mathfrak{K}}^*\psi(E) = \xi(E).$$

Then (1) must hold when $\phi(E) = \mathfrak{E}(E, {}_{(E')})$. And the left hand side of (1) becomes

$$\big(\boldsymbol{T}_{\mathfrak{K}}\mathfrak{E}(E, {}_{(E')}), \psi(E)\big) = \big(\mathfrak{K}(E, {}_{(E')}), \psi(E)\big)$$

$$= \int_A D_{\beta(E)}\mathfrak{K}^*(E', a) D_{\beta(E)}\psi(a) d\beta(E),$$

but the right hand side of (1) becomes

$$\big(\mathfrak{E}(E, {}_{(E')}), \xi(E)\big) = \xi(E').$$

Hence $\psi(E)$ belongs to $\mathfrak{D}_{\mathfrak{K}^*}$ and

$$\boldsymbol{T}_{\mathfrak{K}^*}\psi(E') = \xi(E').$$

Consequently, $\mathfrak{D}_{\mathfrak{K}}^* \subseteqq \mathfrak{D}_{\mathfrak{K}^*}$

and $\boldsymbol{T}_{\mathfrak{K}}^*\psi(E) = \boldsymbol{T}_{\mathfrak{K}^*}\psi(E)$ for all $\psi(E)$ in $\mathfrak{D}_{\mathfrak{K}}^*$.

Now apply the following lemma proved by J. v. Neumann and J. D. Tamarkin :[2]

LEMMA. If $\boldsymbol{T}$ is defined throughout the Hilbert space $\mathfrak{H}$, and if the domain of $\boldsymbol{T}^*$ is dense in $\mathfrak{H}$, then $\boldsymbol{T}$ and $\boldsymbol{T}^*$ are both bounded and defined everywhere.

(1) Cf. M. H. Stone, ibid. 36.

(2) Cf. S. Bochner, Annals of Math. (2) **35** (1934), 114; J. v. Neumann, Annals of Math. (2) **33** (1932), 310.

Then we have the following

COROLLARY 1. *If the domain* $\mathfrak{D}_{\mathfrak{K}}$ of $\boldsymbol{T}_{\mathfrak{K}}$ *is* $\mathfrak{L}_2(\beta)$, *then* $\boldsymbol{T}_{\mathfrak{K}}$ *and* $\boldsymbol{T}_{\mathfrak{K}^*}$ *are bounded, and* $\mathfrak{D}_{\mathfrak{K}^*} = \mathfrak{L}_2(\beta)$.

4. Let $\boldsymbol{T}$ be a transformation which has its adjoint $\boldsymbol{T}^*$, and their domains $\mathfrak{D}$ and $\mathfrak{D}^*$ contain $\mathfrak{L}_2'(\beta)$. Put

$$\boldsymbol{T}\mathfrak{E}(E, {}_{(E')}) = \mathfrak{K}(E, {}_{(E')}),$$

$$\boldsymbol{T}^*\mathfrak{E}(E, {}_{(E')}) = \mathfrak{K}^*(E, {}_{(E')}).$$

Then $\mathfrak{K}(E, {}_{(E')})$ and $\mathfrak{K}^*(E, {}_{(E')})$ belong to $\mathfrak{L}_2(\beta)$.

Since $\mathfrak{E}(E, {}_{(E')})$ belongs to $\mathfrak{D}$ and $\mathfrak{D}^*$,

$$\left(\boldsymbol{T}\mathfrak{E}(E, {}_{(E')}), \mathfrak{E}(E, {}_{(E'')})\right) = \left(\mathfrak{E}(E, {}_{(E')}), \boldsymbol{T}^*\mathfrak{E}(E, {}_{(E'')})\right),$$

that is, $$\mathfrak{K}(E'', E') = \overline{\mathfrak{K}^*(E', E'')}.$$

Hence $\mathfrak{K}(E, E')$ and $\mathfrak{K}^*(E, E')$ belong to $\mathfrak{L}_2(\beta, \beta)$.

Let $\phi(E)$ be any set function in $\mathfrak{D}$. Then

$$\left(\boldsymbol{T}\phi(E), \mathfrak{E}(E, {}_{(E')})\right) = \left(\phi(E), \boldsymbol{T}^*\mathfrak{E}(E, {}_{(E')})\right),$$

that is $$\boldsymbol{T}\phi(E') = \left(\phi(E), \mathfrak{K}^*(E, {}_{(E')})\right)$$

$$= \int_A D_{\beta(E)}\mathfrak{K}(E', a) D_{\beta(E)}\phi(a) d\beta(E).$$

Hence $$\boldsymbol{T} \subseteqq \boldsymbol{T}_{\mathfrak{K}}.$$

Since the domain $\mathfrak{D}^*$ of $\boldsymbol{T}^*$ is dense in $\mathfrak{L}_2(\beta)$, $\boldsymbol{T}^{**}$ exists, and

$$\boldsymbol{T}^{**} \supseteqq \boldsymbol{T}.$$

Hence $\mathfrak{D}^{**}$ contains $\mathfrak{L}_2'(\beta)$. If we apply the above reasoning to $\boldsymbol{T}^*$, we have

$$\boldsymbol{T}^* \subseteqq \boldsymbol{T}_{\mathfrak{K}^*}.$$

Consequently, we have

THEOREM 4. *If* $\boldsymbol{T}$ *is a transformation which has its adjoint* $\boldsymbol{T}^*$ *and their domains* $\mathfrak{D}$ *and* $\mathfrak{D}^*$ *contain* $\mathfrak{L}_2'(\beta)$, *then*

$$\boldsymbol{T} \subseteqq \boldsymbol{T}_{\mathfrak{K}} \quad \textit{and} \quad \boldsymbol{T}^* \subseteqq \boldsymbol{T}_{\mathfrak{K}^*},$$

where $\qquad T\mathfrak{C}(E, {}_{(E')}) = \mathfrak{K}(E, {}_{(E')})\,, \qquad \mathfrak{K}^*(E, E') = \overline{\mathfrak{K}(E', E)}\,.$

When $T = T^*$, we have

Corollary 2. *If* T *is a self-adjoint transformation whose domain* $\mathfrak{D}$ *contains* $\mathfrak{L}_2'(\beta)$, *then*

$$T \subseteqq T_{\mathfrak{K}}\,,$$

where $\quad T\mathfrak{C}(E, {}_{(E')}) = \mathfrak{K}(E, {}_{(E')}) \quad$ *and* $\quad \mathfrak{K}(E, E') = \overline{\mathfrak{K}(E', E)}\,.$

When T is a bounded linear transformation with domain $\mathfrak{D} \supseteqq \mathfrak{L}_2'(\beta)$, then T^* exists and its domain is $\mathfrak{L}_2(\beta)$.[1] Hence, by theorem 3,

$$T \subseteqq T_{\mathfrak{K}}\,, \qquad T^* \subseteqq T_{\mathfrak{K}^*}\,.$$

But since T^* has no proper extension, we have

Corollary 3. *When* T *is a bounded linear transformation with domain* $\mathfrak{D} \supseteqq \mathfrak{L}_2'(\beta)$, *then*

$$T \subseteqq T_{\mathfrak{K}}\,, \qquad T^* = T_{\mathfrak{K}^*}\,,$$

and the domain of $T_{\mathfrak{K}^*}$ *is* $\mathfrak{L}_2(\beta)$.

In the special case, when $\mathfrak{D} = \mathfrak{L}_2(\beta)$, then we have

Corollary 4. *When* T *is a bounded linear transformation with domain* $\mathfrak{L}_2(\beta)$, *then*

$$T = T_{\mathfrak{K}} \qquad \text{and} \qquad T^* = T_{\mathfrak{K}^*}\,.$$

This property has already been proved in my previous paper.[2]

5. Theorem 5. *If* $\mathfrak{K}_1(E, E')$ *and* $\mathfrak{K}_2(E, E')$ *are kernels of transformations both with domain* $\mathfrak{L}_2(\beta)$, *then*

$$\int_A D_{\beta(E'')}\mathfrak{K}_1(E, a'')D_{\beta(E'')}\mathfrak{K}_2(a'', E')d\beta(E'') = \mathfrak{K}_3(E, E') \qquad (1)$$

is likewise so, and

$$T_{\mathfrak{K}_1}T_{\mathfrak{K}_2} = T_{\mathfrak{K}_3}\,.$$

(1) M. H. Stone, loc. cit., 64.
(2) Cf. p. 107.

By corollary 1, $\boldsymbol{T}_{\mathfrak{K}_1}$ and $\boldsymbol{T}_{\mathfrak{K}_2}$ are bounded transformations defined throughout $\mathfrak{L}_2(\beta)$. Hence $\boldsymbol{T}_{\mathfrak{K}_1}\boldsymbol{T}_{\mathfrak{K}_2}$ is likewise so. But, by (1)

$$\boldsymbol{T}_{\mathfrak{K}_1}\boldsymbol{T}_{\mathfrak{K}_2}\mathfrak{E}(E, {}_{(E')}) = \boldsymbol{T}_{\mathfrak{K}_1}\mathfrak{K}_2(E, {}_{(E')}) = \mathfrak{K}_3(E, E').$$

Therefore, by corollary 4

$$\boldsymbol{T}_{\mathfrak{K}_1}\boldsymbol{T}_{\mathfrak{K}_2} = \boldsymbol{T}_{\mathfrak{K}_3}.$$

THEOREM 6. *Let* $\mathfrak{K}_1(E, E')$ *and* $\mathfrak{K}_2(E, E')$ *be two set functions in* $\mathfrak{L}_2(\beta, \beta)$. *If*

$$\mathfrak{K}_2(E, {}_{(E')}) \in \mathfrak{D}_{\mathfrak{K}_1} \qquad \text{and} \qquad \mathfrak{K}_1^*(E, {}_{(E')}) \in \mathfrak{D}^*_{\mathfrak{K}_2}, \tag{2}$$

then
$$\int_A D_{\beta(E'')}\mathfrak{K}_1(E, a'')D_{\beta(E'')}\mathfrak{K}_2(a'', E')d\beta(E'') = \mathfrak{K}_3(E, E') \tag{3}$$

belongs to $\mathfrak{L}_2(\beta, \beta)$, *and*

$$\boldsymbol{T}_{\mathfrak{K}_1}\boldsymbol{T}_{\mathfrak{K}_2} \subseteqq \boldsymbol{T}_{\mathfrak{K}_3}.$$

Denote by $\mathfrak{D}$ the domain of $\boldsymbol{T}_{\mathfrak{K}_1}\boldsymbol{T}_{\mathfrak{K}_2}$; then $\phi(E)$ belongs to $\mathfrak{D}$ when and only when

$$\phi(E) \in \mathfrak{D}_{\mathfrak{K}_2} \qquad \text{and} \qquad \boldsymbol{T}_{\mathfrak{K}_2}\phi(E) \in \mathfrak{D}_{\mathfrak{K}_1}.$$

Since, by (2) $\quad \boldsymbol{T}_{\mathfrak{K}_2}\mathfrak{E}(E, {}_{(E')}) = \mathfrak{K}_2(E, {}_{(E')}) \in \mathfrak{D}_{\mathfrak{K}_1}$,

we have
$$\mathfrak{L}_2'(\beta) \subseteqq \mathfrak{D}. \tag{4}$$

Since, by theorem 3, $\quad \mathfrak{L}_2'(\beta) \subseteqq \mathfrak{D}^*_{\mathfrak{K}_1} \quad$ and $\quad \boldsymbol{T}^*_{\mathfrak{K}_1} \subseteqq \boldsymbol{T}_{\mathfrak{K}_1^*}$,

$$\boldsymbol{T}^*_{\mathfrak{K}_1}\mathfrak{E}(E, {}_{(E')}) = \boldsymbol{T}_{\mathfrak{K}_1^*}\mathfrak{E}(E, {}_{(E')}) = \mathfrak{K}_1^*(E, {}_{(E')}) \in \mathfrak{D}^*_{\mathfrak{K}_2} \qquad \text{by (2).}$$

Hence
$$\boldsymbol{T}^*_{\mathfrak{K}_2}\boldsymbol{T}^*_{\mathfrak{K}_1}\mathfrak{E}(E, {}_{(E')}) = \boldsymbol{T}^*_{\mathfrak{K}_2}\mathfrak{K}_1^*(E, {}_{(E')})$$
$$= \boldsymbol{T}_{\mathfrak{K}_2^*}\mathfrak{K}_1^*(E, {}_{(E')}) = \mathfrak{K}_3^*(E, {}_{(E')}) \qquad \text{by (3).}$$

Therefore,

$$\mathfrak{L}_2'(\beta) \subseteqq \text{domain of } \boldsymbol{T}^*_{\mathfrak{K}_2}\boldsymbol{T}^*_{\mathfrak{K}_1} \subseteqq \text{domain of } (\boldsymbol{T}_{\mathfrak{K}_1}\boldsymbol{T}_{\mathfrak{K}_2})^*.\text{[1]} \tag{5}$$

Since (4) and (5) hold, we can apply theorem 4 to $\boldsymbol{T}_{\mathfrak{K}_1}\boldsymbol{T}_{\mathfrak{K}_2}$. Now

$$\boldsymbol{T}_{\mathfrak{K}_1}\boldsymbol{T}_{\mathfrak{K}_2}\mathfrak{E}(E, {}_{(E')}) = \boldsymbol{T}_{\mathfrak{K}_1}\mathfrak{K}_2(E, {}_{(E')}) = \mathfrak{K}_3(E, {}_{(E')}) \qquad \text{by (3).}$$

Hence, we have
$$\boldsymbol{T}_{\mathfrak{K}_1}\boldsymbol{T}_{\mathfrak{K}_2} \subseteqq \boldsymbol{T}_{\mathfrak{K}_3}.$$

(1) Since $\boldsymbol{T}_2^*\boldsymbol{T}_1^* \subseteqq (\boldsymbol{T}_1\boldsymbol{T}_2)^*$. Cf. M. H. Stone, loc. cit., 43.

Kernels of Unitary Transformations.

6. Let U be a unitary transformation in $\mathfrak{L}_2(\beta)$. And put

$$U\mathfrak{E}(E, {}_{(E')}) = \mathfrak{U}(E, {}_{(E')}),$$

$$\mathfrak{U}^*(E, E') = \overline{\mathfrak{U}(E', E)}. \qquad (1)$$

Then, by corollary 4,

$$U = T_{\mathfrak{U}}, \qquad U^* = T_{\mathfrak{U}^*}.$$

Since $\quad \big(U\mathfrak{E}(E, {}_{(E')}), U\mathfrak{E}(E, {}_{(E'')})\big) = \big(\mathfrak{E}(E, {}_{(E')}), \mathfrak{E}(E, {}_{(E'')})\big),$

we have $\quad \big(\mathfrak{U}(E, {}_{(E')}), \mathfrak{U}(E, {}_{(E'')})\big) = \beta(E'E'').$[(1)] $\qquad (2)$

Similarly, with respect to U^*, we have

$$\big(\mathfrak{U}^*(E, {}_{(E')}), \mathfrak{U}^*(E, {}_{(E'')})\big) = \beta(E'E''). \text{[(2)]} \qquad (3)$$

Hence we have

THEOREM 7. *Any unitary transformation U can be expressed in the integral form with kernel* $\mathfrak{U}(E, E')$ *in* $\mathfrak{L}_2(\beta, \beta)$ *which satisfies* (1), (2) *and* (3).

Conversely,

THEOREM 8. *If* $\mathfrak{U}(E, E')$ *be a set function in* $\mathfrak{L}_2(\beta, \beta)$ *which satisfies* (1), (2) *and* (3), *then* $T_{\mathfrak{U}}$ *represents a unitary transformation.*

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(1) This result can also be obtained by the method shown in my previous paper (cf. p. 107). Let $\{\phi_\nu(E)\}$ be a complete normalized orthogonal system in $\mathfrak{L}_2(\beta)$. Put $\zeta_\nu(E) = U\phi_\nu(E)$ $(\nu = 1, 2, \ldots)$. Then $\{\zeta_\nu(E)\}$ is also a complete normalized orthogonal system in $\mathfrak{L}_2(\beta)$. And

$$\mathfrak{U}(E, E')\,[=]_{E, E}\, \Sigma\, \zeta_\nu(E)\phi_\nu(E').$$

Since $\quad \Sigma\, \phi_\nu(E')\overline{\phi_\nu(E'')} = \beta(E'E''), \quad$ we obtain (2).

(2) The expressions (2) and (3) show that $\{\mathfrak{U}(E, {}_{(E')})\}$ is a complete normalized orthogonal system in $\mathfrak{L}_2(\beta)$ in the extended sense. (Cf. F. Maeda, this journal, **3** (1933), 253.) And the expansion of $\phi(E)$ with respect to $\{\mathfrak{U}(E, {}_{(E')})\}$ is as follows:

$$\phi(E) = \int_A D_{\beta(E')}\mathfrak{U}(E, a')D_{\beta(E')}\zeta(a')d\beta(E') = U\zeta(E),$$

where $\quad \zeta(E) = (\phi(E), \mathfrak{U}(E, {}_{(E')})) = U^*\phi(E).$

Hence $\quad (U\zeta_1, U\zeta_2) = (\zeta_1, \zeta_2) \quad$ means the Parseval identity.

For, by theorem 2, $T_{\mathfrak{U}}$ is a linear transformation whose domain contains $\mathfrak{L}_2'(\beta)$. Any set function $\phi(E)$ in $\mathfrak{L}_2'(\beta)$ may be expressed as follows:

$$\phi(E) = \sum_{i=1}^{n} a_i \frac{\mathfrak{E}(E, {}_{(E_i)})}{\sqrt{\beta(E_i)}},$$

where $E_iE_j = 0$ when $i \neq j$. Since

$$T_{\mathfrak{U}}\mathfrak{E}(E, {}_{(E')}) = \mathfrak{U}(E, {}_{(E')}),$$

we have
$$T_{\mathfrak{U}}\phi(E) = \sum_{i=1}^{n} a_i \frac{\mathfrak{U}(E, {}_{(E_i)})}{\sqrt{\beta(E_i)}},$$

Since $\left\{\frac{\mathfrak{E}(E, {}_{(E_i)})}{\sqrt{\beta(E_i)}}\right\}$ is a normalized orthogonal system and by (2) $\left\{\frac{\mathfrak{U}(E, {}_{(E_i)})}{\sqrt{\beta(E_i)}}\right\}$ is likewise so, we have

$$\| T_{\mathfrak{U}}\phi \|^2 = \sum_{i=1}^{n} | a_i |^2 = \| \phi \|^2.$$

Hence, $T_{\mathfrak{U}}$ is bounded in $\mathfrak{L}_2'(\beta)$. Then by corollary 3 and 1, $T_{\mathfrak{U}^*}$, $T_{\mathfrak{U}}$ are bounded linear transformations with domain $\mathfrak{L}_2(\beta)$, and $T_{\mathfrak{U}}^* = T_{\mathfrak{U}^*}$. Then by theorem 5, $T_{\mathfrak{U}^*}T_{\mathfrak{U}}$ is equal to the transformation with kernel

$$\int_A D_{\beta(E'')}\mathfrak{U}^*(E, a'')D_{\beta(E'')}\mathfrak{U}(a'', E')d\beta(E'')$$

$$= \big(\mathfrak{U}(E'', {}_{(E')}), \mathfrak{U}(E'', {}_{(E)})\big) = \beta(EE') \qquad \text{by (2).}$$

Since, by theorem 1, $\beta(EE')$ is the kernel of identical transformation, we have

$$T_{\mathfrak{U}^*}T_{\mathfrak{U}} = 1.$$

Similarly
$$T_{\mathfrak{U}}T_{\mathfrak{U}^*} = 1.$$

That is, $T_{\mathfrak{U}}$ and $T_{\mathfrak{U}^*}$ are unitary transformations.