An Extension of Parallel Displacement by Matrices.

By

Kakutarô MORINAGA.

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I.

Let us suppose that $A(t)=((a_{ij}(t)))$ is a L-integrable matrix of t in an interval $(p\leqq t\leqq r)$, and Y^0 is any matrix.(1) Here the order of the product of matrices is read from right to left.(2) If we put

$$Y^{\nu}=\prod_{k=1}^{\nu}\left\{I+(t_k-t_{k-1})A(\xi_{k-1})\right\}Y^0 \quad \text{and} \quad t_{k-1}\leqq\xi_{k-1}<t_k\,,$$

we have

$$Y=\lim_{m\to\infty}Y^m=\overset{\frown}{\int_p^r}(I+A(t)dt)Y^0=\lim_{m\to\infty}\prod_{k=1}^{m}e^{(t_k-t_{k-1})A(\xi_{k-1})}Y^0\,. \qquad (1)$$

The above defined Y is called the product integral of Y^0 with respect to the matrix $A(t)$ from $t=p$ to r. Then we have as Caque-Fuchs's expansion of Y

$$\overset{\frown}{\int_p^r}(I+A(t)dt)Y^0=\left\{I+\int_p^r(A(t)\int_p^t A(t_1)dt_1)dt+\ldots\ldots\right\}Y^0\,. \qquad (2)$$

Now we shall express the product integral of matrices in the form of an expansion using differential operations instead of the integral operations shown in (2). Form (1),

$$Y=\lim_{m\to\infty}\left\{(I+A(t_{m-1})\Delta t)\ldots\ldots(I+A(t_2)\Delta t)(I+A(t_1)\Delta t)(I+A(t_0)\Delta t)\right\}$$

(1) L. Schlesinger, *Math. Zeits.* **33** (1931), 33-61; **35** (1932), 485-501.

(2) This is the opposite of the usual order.

where

$$\Delta t = \frac{r-p}{m} .$$

But since

$$I + A(t_0)\Delta t = I + A(t_1)\Delta t - \left(\frac{dA}{dt}\right)_{t=t_1} \Delta t + o(\Delta t)$$

$$= \left\{\left(I - \frac{d}{dt}\Delta t\right)(I + A(t)\Delta t)\right\}_{t=t_1} + o(\Delta t) , \text{ (1)}$$

therefore we have

$$(I+A(t_1)\Delta t)(I+A(t_0)\Delta t) = \left\{\left(I + \left(A - \frac{d}{dt}\right)\Delta t\right)(I + A\Delta t)\right\}_{t=t_1}$$

$$+ (I + A(t_1)\Delta t)\, o(\Delta t) ;$$

similarly

$$(I+A(t_2)\Delta t)(I+A(t_1)\Delta t)(I+A(t_0)\Delta t) = \left\{\left(I + \left(A - \frac{d}{dt}\right)\Delta t\right)^2 (I + A\Delta t)\right\}_{t=t_2}$$

$$+ (I + A(t_2)\Delta t)\, o(\Delta t)$$

$$+ (I + A(t_2)\Delta t)(I + A(t_1)\, o(\Delta t) .$$

And therefore we have

$$Y = \lim_{m\to\infty} \left\{ \left\{\left(I + \left(A - \frac{d}{dt}\right)\Delta t\right)^{m-1} (I + A\Delta t)\right\}_{t=t_{m-1}} \right.$$

$$+ (I + A(t_{m-1})\Delta t)\, o(\Delta t)$$

$$+ (I + A(t_{m-1})\Delta t)(I + A(t_{m-2})\Delta t)\, o(\Delta t)$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\left. + (I + A(t_{m-1})\Delta t) \ldots . (I + A(t_1)\Delta t)\, o(\Delta t) \right\}$$

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(1) Since the operations are read from right to left, it may be adequate to write $\left(A - \frac{d}{dt}\right)$ in the form $\left(A - \frac{d}{dt} I\right)$.

$$= \lim_{m\to\infty}\left\{\left\{\left(I+\left(A-\frac{d}{dt}\right)\varDelta t\right)^{m-1}(I+A\varDelta t)\right\}_{t=t_{m-1}}\right.$$

$$\left. + E_{m-1}\,o(\varDelta t)+E_{m-2}\,o(\varDelta t)\ldots\ldots+E_1\,o(\varDelta t)\right\},$$

where

$$E_i = (I+A(t_{m-1})\varDelta t)(I+A(t_{m-2})\varDelta t)\ldots\ldots(I+A(t_i)\varDelta t)\,.$$

Since from the assumption that E_1, E_2, ..., E_{m-1} are finite, we have

$$Y = \lim_{m\to\infty}\left\{\left(I+\left(A-\frac{d}{dt}\right)\varDelta t\right)^{m}\right\}_{t=t_m} = \left\{e^{\left(A-\frac{d}{dt}\right)(r-p)}\right\}_{t=r} Y^0_{(p)}\,. \tag{3}$$

When Y^0 is a function of t, we have similarly

$$Y = \left\{e^{\left(A-\frac{d}{dt}\right)(r-p)}\,Y^0(t)\right\}_{t=r}\,. \tag{4}$$

The formulae (3) and (4) have been written formally, therefore, we must prove the convergency of the series

$$B = \left\{I+\frac{(r-p)}{1!}\left(A-\frac{d}{dt}\right)+\frac{(r-p)^2}{2!}\left(A-\frac{d}{dt}\right)^2+\ldots\ldots\right\}_{t=r}\,. \tag{5}$$

We know that[(1)]

$$[B]\leq[I]+\frac{|r-p|}{1!}\left[A-\frac{d}{dt}\right]+\frac{|r-p|^2}{2!}\left[\left(A-\frac{d}{dt}\right)^2\right]+\ldots\ldots,$$

and the ratio of the $(q+1)^{th}$ term to the q^{th} term

(1) We adopt the notatian [ ] used in *Math. Zeits.* **33** (1931), 31. Also cf. the foot-note on p. 2 of this paper.

$$\frac{|r-p|}{q}\frac{\left[\left(A-\frac{d}{dt}\right)^{q}\right]}{\left[\left(A-\frac{d}{dt}\right)^{q-1}\right]}\leqq\frac{|r-p|}{q}\frac{[A]\left[\left(A-\frac{d}{dt}\right)^{q-1}+\left[\frac{d}{dt}\left(A-\frac{d}{dt}\right)^{q-1}\right]\right.}{\left[\left(A-\frac{d}{dt}\right)^{q-1}\right]}. \tag{6}$$

Now we assume that there exist certain positive numbers M and η, and define the functions $\frac{d^p}{dt^p}\bar{a}_{ij}$ so that they satisfy the relations

$$\left|\frac{d^{l+1}}{dt^{l+1}}\bar{a}_{ij}\right|\leqq\left|M\frac{d^l}{dt^l}\bar{a}_{ij}\right|, \tag{7}$$

where

$$\left|\frac{d^p}{dt^p}\bar{a}_{ij}\right|\geqq\left|\frac{d^p}{dt^p}a_{ij}\right| \qquad \text{when} \qquad \left|\frac{d^p}{dt^p}a_{ij}\right|\geqq\eta,$$

and

$$\left|\frac{d^p}{dt^p}\bar{a}_{ij}\right|=\eta \qquad \text{when} \qquad \left|\frac{d^p}{dt^p}a_{ij}\right|<\eta,$$

$$(l,\ p=N+1\ldots\ldots \qquad i,j=1,2,\ldots\ldots n).$$

Then we have from (6)

$$\frac{|r-p|}{q}\frac{\left[\left(A-\frac{d}{dt}\right)^{q}\right]}{\left[\left(A-\frac{d}{dt}\right)^{q-1}\right]}\leqq\frac{|r-p|}{q}[A]+|r-p|M.$$

Therefore we may conclude that the right hand side of (5) is absolutely and uniformly convergent in the interval of t defined by

$$|t-p|<\frac{1}{M}.$$

So we have the

theorem 1 : *If there exist M and η satisfying (7), then in the interval $|t-p|<\frac{1}{M}$ the following relation holds*

$$\left\{Y(t)\right\}_{t=r} = \overset{r}{\underset{p}{\widehat{\int}}} (I+Adt)\,Y^0(p) = \left\{e^{\left(A-\frac{d}{dt}\right)(r-p)}\,Y^0(t)\right\}_{t=r}$$

$$= \left\{\left\{I+\frac{(r-p)}{1!}\left(A-\frac{d}{dt}\right)+\frac{(r-p)^2}{2!}\left(A-\frac{d}{dt}\right)^2+\ldots\right\}Y^0(t)\right\}_{t=r}^{(1)}. \quad (8)$$

Now we will proceed to apply this theorem to the theory of parallel displacement of vectors in general manifolds. In the manifold, let $\bar{v}^{\lambda}_{x+dx}$ be a vector at the point $x^{\lambda}+dx^{\lambda}$ which is parallel to the vector v^{λ} at the point x^{λ}. The equation of the parallelism is given by

$$\bar{v}^{\lambda}_{x+dx} = \left\{\left\{\delta^{\lambda}_{\mu} - \left(\frac{\partial}{\partial x^{\nu}}\delta^{\lambda}_{\mu} + \Gamma^{\lambda}_{\mu\nu}\right)dx^{\nu}\right\}v^{\mu}\right\}_{x=x+dx}. \quad (9)$$

We will call such a parallelism a *point-parallelism* in distinction from that called a *non-point-parallelism* which will be treated in the latter part of this paper.

Using theorem 1, we will derive the formula for the parallelism at a finite distance along a curve $x^i = f_i(t)$ starting from a given initial point P. Substituting the equation $x^i = f_i(t)$ into (9), we have

$$\bar{v}^{\lambda}_{t+dt} = \left\{\left\{\delta^{\lambda}_{\mu} - \left(\frac{d}{dt}\delta^{\lambda}_{\mu} + \Gamma^{\lambda}_{\mu\nu}f'_{\nu}\right)dt\right\}v^{\mu}\right\}_{t=t+dt}^{(2)}.$$

Therefore, if we put

$$-((\Gamma^{\lambda}_{\mu\nu}f'_{\nu})) = A(t) \qquad \text{and} \qquad ((v^{\mu})) = Y^0, \text{ (3)}$$

(1) When $Y^0(t)$ is a constant matrix, (8) can also be obtained from the following equations:

$$\begin{cases} \dfrac{dY}{dt} - AY = 0, \\ Y(p) = Y(r) + \dfrac{(p-r)}{1!}\left(\dfrac{dY}{dt}\right)_{t=r} + \dfrac{(p-r)^2}{2!}\left(\dfrac{d^2Y}{dt^2}\right)_{t=r} + \ldots, \\ Y(p) = Y^0. \end{cases}$$

(2) $\Gamma^{\lambda}_{\mu\nu}f_{\nu}$ means $\sum_{i=1}^{n}\Gamma^{\lambda}_{\mu i}f_i$

(3) By our assumption p. I, $((v^{\mu})) = (v^1\,v^2\ldots.v^n)$

we know that the expression for the parallel displacement of vector from $P(t=p)$ to $Q(t=r)$ along the curve $x^i=f_i(t)$ is given by matrix equation

$$\overline{V}(P,\,Q,f_i(t)) = \left\{ \widehat{\int_p^r} \left(I - ((\Gamma^\lambda_{\mu\nu} f'_\nu))dt \right) \right\}_{t=r} V(p)\,, \tag{10}$$

where the notation $\overline{V}(P,\,Q,f_i(t))$ represents a vector obtained by the parallel displacement from $P(t=p)$ to $Q(t=r)$ along the curve.

Then by theorem 1, (10) is written in the form

$$\overline{V}(P,\,Q,f_i(t)) = \left\{ e^{((\frac{d}{dt}\delta^\lambda_\mu + \Gamma^\lambda_{\mu\nu} f'_\nu))(p-r)} \right\}^{(1)}_{t=r} V(p)\,, \tag{11}$$

or

$$\overline{V}(P,\,Q,f_i(t)) = \left\{ e^{((\frac{d}{dt}\delta^\lambda_\mu + \Gamma^\lambda_{\mu\nu} f'_\nu))(p-r)} V(t) \right\}_{t=r}. \tag{12}$$

So we have the

theorem 2 : *Given a vector field* $\mathbf{V}(t)$ *on a curve* $x^i=f_i(t)$, *the vector at* $t=r$ *parallel to* $\mathbf{V}(p)$ *along the curve is given by the equation (11) or (12).*

The formulae (11) and (12) may correspond to Taylor's expansion in the case of a parallel vector along the curve $x^i=f_i(t)$[2].

As an application of the formulae (11) and (12), we shall consider the problem concerning the curvature tensors. When we calculate the curvature tensors in the usual manner, a difficulty arises[3] but it will be seen that the difficulty does not occur when we apply the formulae (11) and (12).

Let $\overline{\mathbf{V}}$ be the vector which returns to the initial point after moving around the circuit $\overrightarrow{EFGHE}$ parallel to $\mathbf{V}$. If we put

$$A_1 = \left(\left(\frac{\partial}{\partial u}\delta^\lambda_\mu + \Gamma^\lambda_{\mu\nu}(u,\,v)\frac{\partial x^\nu}{\partial u} \right)\right),$$

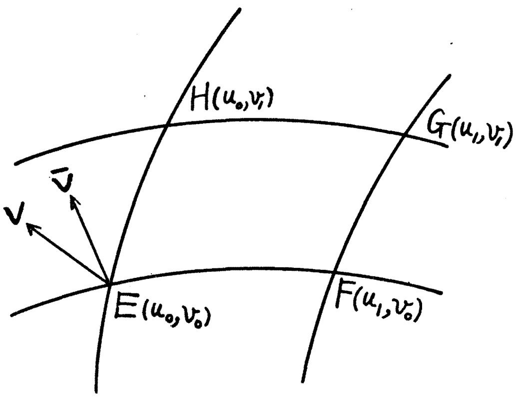
(1) The equation (11) is obtained by a method similar to that of the footnote on p. 5.

(2) Ruse: *Proceedings of the London. Math. Society*, **31** (1930), 225.

(3) See the foot-note in the next page.

$$B_1 = \left(\left(\frac{\partial}{\partial v}\delta_\mu^\lambda + \Gamma_{\mu\nu}^{\lambda}(u, v)\frac{\partial x^\nu}{\partial v}\right)\right),$$

$$a = (u_1 - u_0)A_1 \qquad \text{and} \qquad b = (v_1 - v_0)B_1\,,$$



we easily obtain

$$\bar{\mathbf{V}} - \mathbf{V} = \left\{e^{b}e^{a}e^{-b}e^{-a}\right\}_{\substack{u=u_0\\ v=v_0}}\mathbf{V} - \mathbf{V} = \left\{I + (ba - ab) + \ldots\ldots\right\}_{\substack{u=u_0\\ v=v_0}}\mathbf{V} - \mathbf{V}$$

$$= \left(\left(R_{\omega\mu\nu}^{\cdot\cdot\cdot\lambda}\frac{\partial x^\omega}{\partial u}\frac{\partial x^\mu}{\partial v}(u_1 - u_0)(v_1 - v_0) + \ldots\ldots\right)\right)_{\substack{u=u_0\\ v=v_0}}\mathbf{V}^{(1)} \qquad (13)$$

where the dotted terms are higher than the 3rd degree in $(u_1 - u_0)$ and $(v_1 - v_0)$. When the circuit becomes infinitesimal, we have for (13) by putting $(u_1 - u_0) = du$ and $(v_1 - v_0) = dv$

(1) The result corresponds to equations (18), (20) in *Math. Zeits.* **35** (1932), 496, by I. Schlesinger. But if we calcute this after the usual manner (Schouten: Ricci Kalkül, etc.) we have another result.

$$\bar{\mathbf{V}} - \mathbf{V} = (I + b)(I + a)(I + b)(I - a)\mathbf{V} - \mathbf{V}$$
$$= (ba - ab)V - (b^2 + a^2)\mathbf{V} + \ldots\ldots$$
$$= \left(\left(R_{\omega\mu\nu}^{\cdot\cdot\cdot\lambda}\frac{\partial x^\omega}{\partial u}\frac{\partial x^\mu}{\partial v}(u_1 - u_0)(v_1 - v_0)\right)\right)_{\substack{u=u_0\\ v=v_0}}\mathbf{V} - (b^2 + a^2)\mathbf{V} + \ldots\ldots$$

$$\overline{V}-V=\left(\left(R_{\omega\mu\nu}^{\cdot\cdot\cdot\lambda}\frac{\partial x^{\omega}}{\partial u}\frac{\partial x^{\mu}}{\partial v}dudv\right)\right)_{\substack{u=u_0\\ v=v_0}}V=\left(\left(R_{\omega\mu\nu}^{\cdot\cdot\cdot\lambda}d_1x^{\omega}d_2x^{\mu}\right)\right)_{x^{\xi}=x_0^{\xi}}V. \quad (14)$$
[1]

which holds good within the infinitesimals of the 3rd order.

As shown in foot-not (1) p. 133, equation (14) can not be obtained by carrying round a vector along a small circuit, by means of equation (9) alone.

II.

In the case of point parallelism, the expression for the two parallel vectors at any neighbouring points on a curve is given by functions of the two end points only.

Now in this section, we will generalize the idea of parallelism. The parallelism between two vectors on a curve $x^i=f_i(t)$ *even for an infinitesimal distance* depends not only on the end points but on the behavior of the functions expressing the equation of curve—for example, it depends on the parameter of the curve, $f_i(t)$ $(i=1, \ldots . n)$ and their successive derivatives, and other factors. To be more precise, if $x^i=f_i(t)$ be the base curve, the expression for parallelism between two vectors V_{t_0}, V_t at two points on the curve $t=t_0$ and $t=t$ respectively (not necessarily for an infinitesimal distance) is generally written in the following form

$$V_t^{\lambda}=F^{\lambda}\,|\,[t_0,\, t\,|\,f_i(u),\, V_{t_0}]\,|,$$

where F^{λ} is a functionals depending on the end points, functions f_i and V_{t_0}.

We will call such a parallelism a *non-point-parallelism* corresponding to the term "point-parallelism." But it must be noticed that *this parallelism becomes different when the expression of the base curve is changed although the curve is same;* for, by changing the parameter $t=h(s)$ the values of F^{λ} become different.

(1) Eisenhart obtained this equation in his *Non Riemmanian Geometry* by the method of power series.

In order to promote our discussion reasonably, we make the following assumptions :

$$\left.\begin{array}{ll}
\text{(i)} & \text{if } \overset{1}{V}_{t_1} // \overset{1}{\overline{V}}_{t_2} \text{ and } \overset{2}{V}_{t_1} // \overset{2}{\overline{V}}_{t_2}, \text{ then } \overset{1}{V}_{t_1} + \overset{2}{V}_{t_1} // \overset{1}{\overline{V}}_{t_2} + \overset{2}{\overline{V}}_{t_2}, \\
\text{(ii)} & F^{\lambda} \,|\, [t_1, t_1 \,|\, f_i(u), V_{t_1}] \,| = V^{\lambda}_{t_1}, \\
\text{(iii)} & \text{if } V_{t_1} // \overline{V}_{t_2} \text{ and } \overline{V}_{t_2} // \overline{\overline{V}}_{t_3}, \text{ then } V_{t_1} // \overline{\overline{V}}_{t_3},
\end{array}\right\} \quad (15)$$

along the base curve with same parameter.
Then from the assumption (i), F^{λ} must be linear with respect to V_{t_1} i.e.

$$\overline{V}^{\lambda}_t = F^{\lambda} \,|\, [t_1, t \,|\, f_i(u), V_{t_1}] \,| = F^{\lambda}_{\mu} \,|\, [t_1, t \,|\, f_i(u)] \,|\, V^{\mu}_{t_1},$$

$$\text{or} \quad = F^{\lambda}_{\mu} \,|\, [t_1, t \,|\, f_i(u)] \,|\, V^{\mu}_{t}.$$

Or using the matrix symbol, if we put $((F^{\lambda}_{\mu} \,|\, [''] \,|)) = F$, the above equations are written in the following forms :

(i)' $V_t = F \,|\, [t_1, t \,|\, f_i(u)] \,|\, V_{t_1}$,

(ii)' $F \,|\, [t_1, t_1 \,|\, f_i(u)] \,| = I$,

(iii)' $F \,|\, [t_2, t_3 \,|\, f_i(u)] \,|\, F \,|\, [t_1, t_2 \,|\, f_i(u)] \,| = F \,|\, [t_1, t_3 \,|\, f_i(u)] \,|$.

From these assumptions we can obtain, just as in the case of point-parallelism, the expression for the general non-point-parallelism in the finite distance by iterating the infinitesimal displacement $(I+Hdt)$ successively, i.e.

$$F \,|\, [t_1, t \,|\, f_i(u)] \,| = \overset{t}{\underset{t_1}{\widehat{\int}}} (I + Hdt)$$

where H may be regarded as a matrix, each element ${}^{p}_{q}H$ being an arbitrary functionals

$${}^{p}_{q}H \,|\, [t, f_i(u)] \,| \qquad (L\text{-integrabel}).$$

And by theorem I, we have

$$\overline{V}_t = \left\{ e^{\left(H - \frac{d}{dt} I\right)(t - t_1)} V \right\}_{t=t} . \qquad (16)$$

Since we have assumed that f_i and its derivatives with respect to its parameter are contained explicitly in ${}^r_q H$, now we will see how the parallelism may change its form by transformation of the parameter.

For this purpose, consider an infinitesimal transformation of the parameter

$$t = t' + \eta(t')\delta\alpha .$$

Then the variation of $({}^r_q H dt)$(1) is

$$\begin{aligned} \delta({}^r_q H dt) = \Big\{ & -{}^r_q H \eta'(t) - \frac{\partial H}{\partial t} \eta(t) + \sum_i {}'^r_q H_{f_i'(t)} f_i'(t) \eta'(t) + \dots . \\ & + \sum_i \int_{..}^{..} {}'^r_q H_{f_i(v)} \,|\, [t \,|\, f_j(u) \,|\, v] \,|\, f_i'(v)\, \eta(v) dv \\ & + \sum_i \int_{..}^{..} {}'^r_q H_{f_i'(v)} \,|\, [''] \,|\, (f_i'(v)\, \eta(v))' dv \\ & + \dots\dots\dots\dots\dots\dots \Big\} dt\, \delta\alpha . \end{aligned} \qquad (17)$$

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(1) $\delta(Hdt)$ is calculated from the following equations:

$$\left.\begin{aligned} \frac{df_i}{dt} &= \frac{dg_i(t')}{dt'} \frac{dt'}{dt} , \\ \frac{d^2 f_i}{dt^2} &= \frac{d^2 g_i(t')}{dt'^2} \left(\frac{dt'}{dt} \right)^2 + \frac{dg_i\, t')}{dt'} \frac{d^2 t'}{dt^2} , \\ &\dots\dots\dots\dots \\ &\dots\dots\dots\dots \end{aligned}\right\} (A) \qquad \left.\begin{aligned} f_i(t) = g(t') &= f_i(t') + \frac{df_i(t')}{dt'} \eta(t')\delta_\alpha + \dots , \\ \frac{dg_i(t')}{dt'} &= \frac{df_i(t')}{dt'} + (f_i'(t')\eta(t'))'\delta_\alpha + \dots , \\ \frac{df_i}{dt} &= \frac{dg_i(t')}{dt'} (1 - \eta'(t')\delta_\alpha) + \dots\dots , \\ &\dots\dots\dots\dots \\ &\dots\dots\dots\dots \end{aligned}\right\} (B)$$

The 3rd, 4th and 5th terms of (17) are derived from the 3rd, 1st and 2nd equations of (B), and the 2nd term of (18) is derived from (A).

As important special cases, consider the two following:

(I) When $\delta({}^r_qHdt)=0$ by any infinitesimal transformation of parameter, that is, the case in which the parallelism is independent of the change of parameter.

From the arbitrariness[1] of function $\eta(t)$, we haue from (17)

$$\left.\begin{aligned}
&\frac{\partial {}^r_qH}{\partial t}=0\,,\\
&-{}^r_qH+\sum_i\left({}'{}^r_qH_{f'_i(t)}f'_i(t)+2\,{}'{}^r_qH_{f''_i(t)}f''_i(t)+\ldots\right)=0\,,\\
&\qquad\qquad\ldots\ldots\ldots\ldots\\
&\qquad\qquad\ldots\ldots\ldots\ldots\\
&\sum_i\Big\{\int_{..}^{..}{}'{}^r_qH_{f_i(u_{\alpha_i})}\,|\,[t\,|\,f_j(u)\,|\,u_{\alpha_i}]\,|\,f'_i(u_{\alpha_i})\,\eta(u_{\alpha_i})\,du_{\alpha_i}\\
&\qquad+\int_{..}^{..}{}'{}^r_qH_{f'_i(u..)}\,|\,[\prime\prime]\,|\,(f'_i(u..)\,\eta(u..))'du+\ldots\Big\}=0\,.
\end{aligned}\right\}\tag{18}$$

The 2nd equation of (18) becomes by an easy calculation

$$\sum_m(m-1)\,{}^r_qA_m=0$$

where, if m is called *the index of the term* when it has the m_1^{th}, m_2^{th}, m_3^{th}, ... degree with respect to $f_i'(t)$, f_j'', $f_k'''(t)$, ... and $m=\sum_i im_i$, ${}^r_qA_m$ is a summation of all the terms of the index m in ${}^r_qH$.

(II) We will consider the case in which $\delta(Hdt)=0$ for all the forms of $\eta(t)$ and $f_i(t)$.

For this purpose we have express the form of H more concretely. The most general form of the matrix H will supposedly be the following expression, on the assumption that an element of the matrix is ${}^r_qH$:

$${}^r_qH=\sum_{(\alpha,j,b_j)}{}^r_qH_{\substack{b_{j_1}\ldots b_{j_\alpha}\\ j_1\ldots j_\alpha}}=\sum_\beta{}^r_qH_\beta\tag{20}$$

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(1) (18) is obtained by the usual method for the calculation of functionals: i. e. by substituting $\bar{\eta}(t)$, $\bar{\eta}'(t)$, ... for $\eta(t)$, $\eta'(t)$, ... , where $\bar{\eta}(t)$, $\bar{\eta}'(t)$, ... have any given values in the neighbourhood of point $\bar{t}$, and have the same values as $\eta(t)$, $\eta'(t)$, ... respectively almost everywhere at other points.

where

$$
{}_q^r H_\beta = \overset{(b_{j_1})}{f_{j_1}}(t) \ldots \overset{(b_{j\alpha})}{f_{j_\alpha}}(t) \sum_{(l, i_p, c_{i_p})} \int_{\bar{t}_{i_l}}^{\bar{\bar{t}}_{i_l}} \ldots
$$

$$
\ldots \int_{\bar{t}_{i_1}}^{\bar{\bar{t}}_{i_1}} {}_q^r \overset{c_{i_1} \cdots c_{i_l}}{\underset{i_1 \cdots i_l \beta}{K}} (t, u_1, \ldots u_l) \overset{(c_{i_1})}{f_{i_1}}(u_1) \ldots \overset{(c_{i_l})}{f_{i_l}}(u_l)\, du_1 \ldots du_l ,
$$

and $\bar{t}_{i_p}$ and $\bar{\bar{t}}_{i_p}$ may be, in general, functions of t and u_k $(k > p)$.

Next we denote by ${}_q^r H_{\beta m s_1 \cdots s_n}$ the summation of all the terms of the s_i th degree with respect to $f_i(u), f_i'(u) \ldots$ in ${}_q^r H_\beta$, and if we denote the index of ${}_q^r H_{\beta m s_1 \cdots s_n}$ by m, we easily see that $m = \sum_{j_i} b_{j_i}$.

Using these notations and substituting $\rho_i f_i$ for f_i (ρ_i being any constant), and variating continuously the values of $f_i(t)$, $f_j'(t)$, $f_k''(t)$, ... to any assigned values in the neighbourhood of $t=t$ and leaving the values unchanged for the other points, we have from (18) the following relations

$$
\left.
\begin{aligned}
& \frac{\partial {}_q^r H_{\beta m s_1 \cdots s_n}}{\partial t} = 0 , \\
& (m-1)\, {}_q^r H_{\beta m s_1 \cdots s_n} = 0 , \\
& \sum_{(i_p, c_{i_p})} \left\{ \int_{\bar{t}_{i_p}}^{\bar{\bar{t}}_{i_p}} {}'{}_q^r H_{\beta m s_1 \cdots s_n} \overset{(c_{i_p})}{f_{i_p}}(u_p) \frac{d^{c_{i_p}}}{du_p^{c_{i_p}}} \Big(f_{i_p}(u_p)\, \eta(u_p) \Big) du_p \right\} = 0 .
\end{aligned}
\right\} \qquad (21)
$$

From the 2nd equation of (21) we see that

$$
H_{\beta m s_1 \cdots s_n} = 0 \qquad \text{for} \quad m \neq 1 ,
$$

and for $m=1$, $H_{\beta s_1 \cdots s_n}$ do not necessarily vanish and in this case, from the definition of the index m, we know that ${}_q^r H_{\beta s_1 \cdots s_n}$ can not contain $f_i''(t)$, $f_j'''(t)$, ... and must be a linear homogeneous function of $f_i'(t)$, $f_j'(t)$ .... .

Therefore ${}_q^rH$ must have the following form :

$$ {}_q^rH = \sum_{(j_p,\, b_{j_p})} {}_q^rH_{\substack{b_{j_1}\cdots b_{j_\alpha} \\ j_1\cdots j_\alpha}} \tag{22} $$

where $\sum_i b_{j_i} = 1$ and ${}_q^rH$ satisfy (21).

Then we have the

result: *In a non point-parallelism, which does not change whatever changes of parameter occur according to every curve taken to its base, the general form of the equation of the parallelism is expressed by*

$$ \overline{\mathbf{V}}_t = \left\{ e^{\left(H-\frac{d}{dt}\right)(t-t_1)}\, \mathbf{V} \right\}_{t=t} = \widehat{\int_{t_1}^{t}} (+Hdt)\, \mathbf{V}_{t_1} $$

where H is given by (22).

As a special case, consider in which functionals ${}_q^rH$ has the form as usually treated

$$ {}_q^rH = \sum_{(j_c,\, b_{j_c},\, i_p)} \overset{(b_{j_1})}{f_{j_1}}(t) \ldots \overset{(b_{j_\alpha})}{f_{j_\alpha}}(t) \int_{\bar{t}_{i_l}}^{\bar{\bar{t}}_{i_l}} \ldots $$

$$ \ldots \int_{\bar{t}_{i_1}}^{\bar{\bar{t}}_{i_1}} {}_q^rK_{\substack{b_{j_1}\cdots b_{j_\alpha} \\ j_1\cdots j_\alpha\, i_1\cdots i_l}} (t, u_1 \ldots u_l) f_{i_1}(u_1) \ldots f_{i_l}(u_l)\, du_1 \ldots du_l \tag{23} $$ [(1)]

where $\bar{t}_{i_p}$ and $\bar{\bar{t}}_{i_p}$ are functions of t.

Then from the last equation of (21), and taking account arbitrariness of $\eta(t)$ and $f_i(t)$, we can easily see that ${}_q^rH$ does not contain $f(u)$ and t explicitly.

Therefore from (22) and (23), we have

$$ {}_q^rH = \sum_{(\alpha,\, j,\, b_j)} {}_q^rK_{\substack{b_{j_1}\cdots b_{j_\alpha} \\ j_1\cdots j_\alpha}} \overset{(b_{j_1})}{f}_{j_1}(t) \ldots \overset{(b_{j_\alpha})}{f}_{j_\alpha}(t) $$

(1) I have presume that this is the most general form usually treated.

where $1=\sum_{j_k} b_{j_k}$ and ${}^{r}_{q}K^{b_{j_1}\cdots b_{j_\alpha}}_{j_1\cdots\cdots j_\alpha}$ are constants, which is nothing but $-\Gamma^{r}_{q\nu}f'_{\nu}(t)$ in the case of a point-parallelism.

So we have the

theorem 3: *When ${}^{r}_{q}H$ of a non point-parallelism is expressed by (23), the parallelism becomes a point-parallelism when and only when the parallelism is independent of all transformations of parameter of all curves taken to the base.*

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