On Certain Functional Inequalities.

By

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I

Mr. Jensen[1] defined *a convex function* as a real one-valued function which satisfies the inequality

$$\frac{\varphi(x)+\varphi(y)}{2} \geqq \varphi\left(\frac{x+y}{2}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

independently of the values of variables x and y in the given interval (α, β) and has proved that the inequality

$$\sum_{i=1}^{n} p_i\varphi(x_i) \Big/ \sum_{i=1}^{n} p_i \geqq \varphi\left(\sum_{i=1}^{n} p_i x_i \Big/ \sum_{i=1}^{n} p_i\right) \quad \ldots\ldots\ldots \quad (2)$$

occurs for any positive quantities p_i. Conversely I wish to show, in this paper, that if (2) holds good independently of the variables x_i for certain p_i's and n, then (1) also must hold, thus the inequality (2) takes place for and only for a convex function; in other words, the solution of the given functional inequality (2) is convex.

If the sum of some p_i's ($p_i+p_j+\ldots\ldots+p_k = p$ say) is half the total sum $\sum_{i=1}^{n} p_i$, then by putting

$$x_i = x_j = \ldots\ldots = x_k = x \text{ and the remaining } x\text{'s} = y$$

in (2), we clearly obtain (1). Even if this assumption does not hold, we can solve (2) by putting

$$x_1=x,\ x_2=x_3=\ldots\ldots=x_n=y \quad \text{and} \quad p_1=p,\ p_2+p_3+\ldots\ldots p_n=q,$$

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(1) Acta Math. (1906).

since (2) turns into the case where $n=2$ and moreover in which case we can prove the

Teorem A: For a real one-valued function $\varphi(x)$ which is defined in an interval (α, β), and is bounded upperly there, if an inequality

$$\frac{p\varphi(x)+q\varphi(y)}{p+q} \geqq \varphi\left(\frac{px+qy}{p+q}\right)$$

holds good independently of the values x and y in (α, β), p and q being any assigned positive quantities, then

i. *$\varphi(x)$ is continuous in an open interval (α, β) and*
ii. *an inequality*

$$\frac{r\varphi(x)+s\varphi(y)}{r+s} \geqq \varphi\left(\frac{rx+sy}{r+s}\right)$$

holds good for any real positive quantities r and s.

Proof: **i.** We shall show that $\varphi(x)$ is continuous at $x=a$ within the open interval (α, β). Let ρ be a positive number such that the interval $(a-\rho, a+\rho)$ is wholly included in the given interval, and g be an upper bound of $\varphi(x)$. Determine the positive integer n so that

$$l^n > \lambda\{g-\varphi(a)\}/\varepsilon$$

for any given $\varepsilon > 0$, where $l = 1+\lambda = 1+\dfrac{p}{q}$ (such a choice of n is always allowable since $l > 1$, and furthermore we may assume $\lambda \geqq 1$ without loss of generality). If δ is so chosen as to satisfy $0 < \delta < \rho/l^n$, then for any d such that $|d| < \delta$ the numbers

$$a \lesseqgtr a+d \lesseqgtr a+ld \lesseqgtr \ldots\ldots \lesseqgtr a+l^n d \quad \text{(according as } o \lesseqgtr d\text{)}$$

lies within $(a-\rho, a+\rho)$ since $|l^n d| < l^n\delta < \rho$, and hence, of course, within (α, β). Now since

$$\begin{aligned} \varepsilon/\lambda &> \{g-\varphi(a)\}/l^n > \{\varphi(a+l^n d)-\varphi(a)\}/l^n \\ &= \left\{\frac{\varphi(a+l^n d)+\lambda\varphi(a)}{l} - \varphi(a)\right\}\Big/ l^{n-1} \geqq \left\{\varphi(a+l^{n-1}d)-\varphi(a)\right\}\Big/ l^{n-1} \\ &\geqq \ldots\ldots \geqq \varphi(a+d)-\varphi(a). \end{aligned}$$

$$\therefore \quad \varepsilon > \varepsilon/\lambda > \varphi(a+d)-\varphi(a) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3).$$

On the other hand, as we have $|-d/\lambda| < |d| < \delta$, we can substitute $-\dfrac{d}{\lambda}$ for d in (3); thus we obtain

$$\varepsilon/\lambda \geqq \varphi\left(a-\frac{d}{\lambda}\right)-\varphi(a)\,,$$

or
$$\varepsilon \geqq \lambda\varphi\left(a-\frac{d}{\lambda}\right)-\lambda\varphi(a) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4),$$

but
$$\varphi(a+d)-\varphi(a) \geqq \lambda\varphi(a)-\lambda\varphi\left(a-\frac{d}{\lambda}\right) \quad \ldots\ldots\ldots\ldots \quad (5),$$

and finally from (3) (4), (5) we get

$$\varepsilon > \varphi(a+d)-\varphi(a) > -\varepsilon\,. \qquad \text{Q.E.D.}$$

ii. Take two points $X(x\,,\,\varphi(x))$ and $Y(y\,,\,\varphi(y))$ on the curve $\varphi(x)$ and obtain a set of points by repeating the following process;

1′. Consider a set M_1 of points* x and y, and the point which divides the segment $\overline{xy}$ in the ratio $q:p$.

2′. Obtain M_2 consisting of all points of M_1 and of points which separate the segments between any two neighbouring points in M_1 in the given ratio $q:p$.

3′. Obtain M_3 ....

....

n'. Obtain M_n consisting of all points of M_{n-1} and of such points as divide the segments determined by any two neighbouring points in M_{n-1} in the given ratio $q:p$.

And so on. Then evidently

1° the set of points $M = M_1+M_2+M_3+$ .... is everywhere dense, and moreover we can show that

2° if ξ be a point in M, then the point $(\xi\,,\,\varphi(\xi))$ does not lie upward the segment $\overline{XY}$. In order to prove 2° by the method of complete induction, let us suppose that it holds good for any points belonging to M_r and let a be a point in M_{r+1} obtained as a point which divides the

* Point x represents the point with coordinates $(x\,,\,0)$.

segment $\overline{x_r y_r}$, determined by two points x_r and y_r in M_r, in the assigned ratio. Then by the assumption, we have

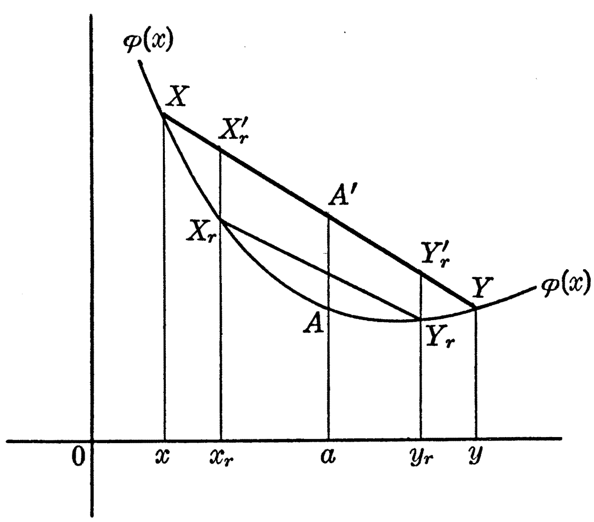
$$x_r X_r' \geqq x_r X_r = \varphi(x_r)\,, \quad \text{and} \quad y_r Y_r' \geqq y_r Y_r = \varphi(y_r)\,.$$

The points X_r, Y_r, A ; X_r', Y_r', A' are illustrated in the accompanying figure. Hence

$$aA' = \frac{px_r X_r' + qy_r Y_r'}{p+q} \geqq \frac{p\varphi(x_r) + q\varphi(y_r)}{p+q} \geqq \varphi\left(\frac{px_r + qy_r}{p+q}\right)$$
$$= \varphi(a) = aA$$

$$\therefore \qquad aA' \geqq aA\,.$$

Thus also the fact is true for any point in M_{r+1}, but since this occurs for $r = 1$, the enunciation is generally justified.



Since the points on the curve $\varphi(x)$ corresponding to the points in M, which is everywhere dense, do not lie upward the segment $\overline{XY}$, we see by i that the point on the curve $\varphi(x)$ corresponding to any point on the segment $\overline{xy}$, do not lie upward the segment $\overline{XY}$, that is, we have, in expression, as required

$$\frac{r\varphi(x) + s\varphi(y)}{r+s} \geqq \varphi\left(\frac{rx+sy}{r+s}\right)$$

Remark I: If r and s are real numbers of different sign, then

$$\frac{r\varphi(x)+s\varphi(y)}{r+s} \leqq \varphi\left(\frac{rx+sy}{r+s}\right)$$

Remark II: If we put $r=s=1$, we have the required result in the original problem.
Remark III: From the above argument it follows that a convex function with an upper bound must have a lower bound, but not necessarily an upper bound, even though it has a lower bound.

II

If we generalize the inequality (1) from the point of view that $\frac{1}{2}(x+y)$ is the root of the derived equation of a quadratic equation with roots x and y, we have

$$\frac{1}{n}\sum_{k=1}^{n}\varphi(x_k) \geqq \frac{1}{n-1}\sum_{k=1}^{n-1}\varphi(y_k) \quad \ldots\ldots\ldots\ldots\ldots \quad (6)$$

where $\{y_k\}$ is the derived system[1] of $\{x_k\}$.

A few years ago, Mr. H. Bray[2] showed that $x's$ being all non-negative (thus also all of y's), $\varphi(x)=x^m$ is a particular solution of (6) with non-negative integer m, and the sign of equality occurs when and only when either $m=1$ or 0, or $x_1=x_2=\ldots\ldots=x_n$. Recently, Prof. S. Kakeya[3] has extended the result of Mr. Bray by a proof, which is apparently simpler, enunciating that $\varphi(x)=x^m$ can still be a solution of (6) even when m is any real number and is greater than unity or less than zero. And furthermore he has proved that when $1>m>0$, $\varphi(x)=x^m$ satisfies the inequality

$$\frac{1}{n}\sum_{k=1}^{n}\varphi(x_k) \leqq \frac{1}{n-1}\sum_{k=1}^{n-1}\varphi(y_k)$$

and that the sign of equality occurs when and only when $m=1$ or 0, or $x_1=x_2=\ldots\ldots=x_n$. In the following, I search for the general solution of (6) in the theorems B and C.

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(1) S. Kakeya, Proc. of the Physico-Math. Soc. of Japan (1933), 149.
(2) Amer. Jour. of Math. (1931).
(3) Loc. cit.

Theorem B: Let $\varphi(x)$ be a real one-valued function defined in an open interval (α, β). If the inequality (6) always holds good independently of the values $x_1, x_2, \ldots, x_n$ in (α, β), $y_1, y_2, \ldots, y_{n-1}$ being the derived system of $\{x_k\}$, then $\varphi(x)$ must necessarily be a convex function.

Proof: (6) must hold independently of the values x and y even when we put $x_1=x_2=\ldots=x_q=x$, $x_{q+1}=\ldots x_n=y$. In this case, $y_1=\ldots=y_{q-1}=x$, $y_q=px+qy\big/p+q$, $y_{q+1}=\ldots=y_{n-1}=y$; where $n=p+q$.

$$\Delta\varphi\equiv\frac{1}{n}\left\{q\varphi(x)+p\varphi(y)\right\}$$

$$-\frac{1}{n-1}\left\{(q-1)\varphi(x)+\varphi\left(\frac{px+qy}{p+q}\right)+(p-1)\varphi(y)\right\}$$

$$\equiv\frac{1}{n-1}\left\{\frac{p\varphi(x)+q\varphi(y)}{p+q}-\varphi\left(\frac{px+qy}{p+q}\right)\right\}\geqq 0$$

$$\therefore\quad \frac{p\varphi(x)+q\varphi(y)}{p+q}\geqq\varphi\left(\frac{px+py}{p+q}\right).$$

Thus by the above theorem A, $\varphi(x)$ must be a convex function.

Remark: If n is even, we are able to dispense with theorem A by putting $p=q=\frac{n}{2}$ directly.

III

We will next prove the converse

Theorem C: When $\varphi(x)$ is a convex function defined in (α, β), the inequality (6) always holds good.

Proof: Let $x_1<x_2<\ldots<x_r$ be all of the distinct roots of the given algebraic equation of n-th degree with respective multiplicities $m_1, m_2, \ldots, m_r$ (so that $m_1+m_2+\ldots+m_r=n$), all lying within the given interval (α, β) in which $\varphi(x)$ is convex. The derived system of $\{x_k\}$ is given in the following table

Elements of the Derived System	$x_1<y_1<x_2<y_2<\ldots<y_{r-1}<x_r$
the Multiplicities of the resp. Elements	m_1-1,1 , m_2 -1, 1, ..., 1, m_r -1

It is well known that $y_1, y_2, \ldots, y_{r-1}$ are the roots of the algebraic equation of $(r-1)$ th degree

$$\frac{m_1}{y-x_1}+\frac{m_2}{y-x_2}+\ldots\ldots+\frac{m_r}{y-x_r}=0.$$

Now let us momentarily suppose that x_1 is a variable $<x_2$, and that y_k's are functions of x_1, $y_k(x_1)$. We always have

$$\frac{\partial y_k}{\partial x_1}=\frac{m_1}{(y_k-x_1)^2}\Bigg/\sum_{i=1}^{r}\frac{m_i}{(y_k-x_i)^2}>0 \quad\ldots\ldots\ldots\ldots (7).$$

The fact that the mean centre of any system of real numbers is coincident with that of its derived system gives

$$\frac{1}{n}\sum_{i=1}^{r}m_ix_i=\frac{1}{n-1}\left\{\sum_{i=1}^{r}(m_i-1)x_i+\sum_{i=1}^{r-1}y_i\right\},$$

or

$$\sum_{i=1}^{r-1}y_i=\sum_{i=1}^{r}\left(1-\frac{m_i}{n}\right)x_i.$$

In order to prove the present theorem, put

$$R\left(\frac{x_1, x_2, \ldots, x_r}{m_1, m_2, \ldots, m_r}\right)=R(x_1)=\frac{1}{n}\sum_{i=1}^{r}m_i\varphi(x_i)$$

$$-\frac{1}{n-1}\left\{\sum_{i=1}^{r}(m_i-1)\varphi(x_i)+\sum_{i=1}^{r-1}\varphi(y_i)\right\}.$$

From (7) we see that $y_i(x_1)$ is an increasing function; hence

$$D_+\varphi\big[y_i(x_1)\big]=D_+\varphi(y_i)\frac{\partial y_i}{\partial x_1},$$

thus we have

$$D_+R(x_1)=\frac{1}{n}m_1D_+\varphi(x_1)$$

$$-\frac{1}{n-1}\left\{(m_1-1)D_+\varphi(x_1)+\sum_{i=1}^{r-1}D_+\varphi(y_i)\frac{\partial y_i}{\partial x_1}\right\}.$$

From Mr. Jensen[1] we know that convex functions are not only continuous but also have both left-handed and right-handed derivatives, and that the point set, in which the derivatives of both sides do not coincide, is at most denumerable, and further that the derivative is increasing, that is to say, if $x < y$, then

$$D_-\varphi(x) \leqq D_+\varphi(x) \leqq D_-\varphi(y) \leqq D_+\varphi(y) .$$

So that by using the foregoing table and the relation (7), we have

$$\sum_{i=1}^{r-1} D_+\varphi(y_i)\frac{\partial y_i}{\partial x_1} \geqq \sum_{i=1}^{r-1} D_+\varphi(x_1)\frac{\partial y_i}{\partial x_1} = D_+\varphi(x_1)\frac{\partial \sum_{i=1}^{r-1} y_i}{\partial x_1} = \left(1-\frac{m_1}{n}\right)D_+\varphi(x_1).$$

$$\therefore \quad D_+R(x_1) \leqq \frac{m_1}{n}D_+(x_1) - \frac{1}{n-1}\left\{(m_1-1)D_+\varphi(x_1) + \left(1-\frac{m_1}{n}\right)D_+\varphi(x_1)\right\} = 0$$

In an exactly similar manner we can show that $D_-R(x_1) \leqq 0$ for any $x_1 < x_2$.

Now since $R(x_1)$ is continuous not only because of the continuity of $\varphi(x)$ but also by reason of the fact that y's are also continuous functions of x_1, we see that the curve $R(x_1)$ consist of an at most denumerable number of smooth continuous arcs, each of which is non-increasing. Hence $R(x_1)$ is itself non-increasing in (x_1, x_2), thus we have, owing to the continuity of $R(x_1)$, $R(x_1) \geqq R(x_2)$, or in detail

$$R\left(\frac{x_1, x_2, \ldots, x_r}{m_1, m_2, \ldots, m_r}\right) \geqq R\left(\frac{x_2, \quad x_3, \ldots, x_r}{m_1+m_2, m_3, \ldots, m_r}\right).$$

In a similar way we shall have

$$R\left(\frac{x_2, \quad x_3, \ldots, x_r}{m_1+m_2, m_3, \ldots, m_r}\right) \geqq R\left(\frac{x_3, \qquad x_4, \ldots, x_r}{m_1+m_2+m_3, m_4, \ldots, m_r}\right).$$

By such recurring inequalities we have finally

$$R\left(\frac{x_1, x_2, \ldots, x_r}{m_1, m_2, \ldots, m_r}\right) \geqq R\left(\frac{x_r}{n}\right) = 0 . \qquad \text{Q.E.D.}$$

(1) Loc. cit.

It only remains to examine the case in which the sign of equality occurs, $\varphi(x)$ being an assigned convex function in (x_1, x_r) at least. In the foregoing we have of course assumed that all x's are distinct.

(1) If x's are all equal, the sign of equality evidently occurs.

(2) If this is not the case, in order that the sign of equality may occur, the function $R(x_1)$ must be constant in the interval (x_1, x_2) as follows from the above argument, and $DR(x_1) = 0$. But this is possible when and only when

$$D\varphi(x_1) = D\varphi(y_1) = \dots\dots = D\varphi(y_{r-1}) .$$

Since $D\,\varphi(x)$ is non-decreasing and $x_1 < y_{r-1}$, $D\,\varphi(x)$ must be constant throughout the interval (x_1, y_{r-1}), that is to say, $\varphi(x)$ must be a linear function $ax+b$. On the other hand, from the assumption that the sign of equality occurs, we have $\varphi(x_r) = ax_r+b$. Thus the point $(x_r, \varphi(x_r))$ also lies on the prolongation of the straight line $ax+b$. But since $\varphi(x)$ is convex, we see that $\varphi(x)$ is identical with $ax+b$ in an enlarged interval (x_1, x_r).

Remark I: As regards the functional inequality $\frac{1}{n}\sum\varphi(x) \leqq \frac{1}{n-1}$ $\times\sum\varphi(y)$, we can enunciate the following proposition that this occurs for and only for concave functions, since we may change it into $\frac{1}{n}\sum-\varphi(x) \geqq \frac{1}{n-1}\sum-\varphi(y)$.

Remark II: If we apply this result to the successive derived systems of $\{x_k\}$, we get Jensen's well known inequality

$$\frac{1}{n}\sum\varphi(x) \gtreqqless \frac{1}{n-1}\sum\varphi(y) \gtreqqless \frac{1}{n-2}\sum\varphi(z) \gtreqqless \dots\dots \gtreqqless \varphi\left(\frac{\sum x}{n}\right)$$

according as $\varphi(x)$ is convex or concave.

Remark III: It is worthy of notice that in spite of the generality of the result, the proof here given is far simpler than those of the two former writers.

Example I: Put $\varphi(x) = x^m$, then we have the result obtained by Prof. Kakeya.

Example II: Put $\varphi(x) = \log x =$ concave,

$$\frac{1}{n}\sum \log x \leqq \frac{1}{n-1}\sum \log y \leqq \frac{1}{n-2}\sum \log z \leqq \ldots \leqq \log \frac{\sum x}{n},$$

or $$(x_1 x_2 \ldots . x_n)^{\frac{1}{n}} \leqq (y_1 y_2 \ldots . y_{n-1})^{\frac{1}{n-1}} \leqq \ldots . \leqq (x_1+x_2+\ldots+x_n)/n,$$

or $$\left(x_1 x_2 \ldots . x_n \Big/ \binom{n}{n}\right)^{\frac{1}{n}} \leqq \ldots . \leqq \left(\sum xx \Big/ \binom{n}{2}\right)^{\frac{1}{2}} \leqq \left(\sum x \Big/ \binom{n}{1}\right).$$

IV

In the same paper, Prof. Kakeya remarked that it is desirable to examine the sign of the expression

$$F_m = \frac{1}{n}\sum x^m - \frac{2}{n-1}\sum y^m + \frac{1}{n-2}\sum z^m,$$

where $\{y_k\}$ and $\{z_k\}$ are the successive derived systems of $\{x_k\}$.

If we study the function $\varphi(x)$ which makes the expression

$$F_r\varphi(f) = \binom{r}{0} M_\varphi(f) - \binom{r}{1} M_\varphi(f') + \ldots . + (-1)^r \binom{r}{r} M_\varphi(f^{(r)}),$$

where $M_\varphi(f) = \frac{1}{n}\sum_{i=1}^{n} \varphi(x_i)$, $x_1, \ldots ., x_n$ being the roots of an algebraic equation $f(x) = 0$, of constant sign independently of the values of $\{x_k\}$, under the assumption that $\varphi(x)$ has continuous $\varphi^{(n)}(x)$ $(n = 1, 2, \ldots ., r+2)$ in the interval (α, β) in which $\varphi(x)$ is considered, then the result is contrary to our expectation that $F_r\varphi(f)$ should be of constant sign for certain functions.

Lemma: If $n_k \geqq r$ $(k = 1, 2, \ldots, p)$, and $r \geqq m_1 \geqq m_2 \geqq \ldots \geqq m_p \geqq m_{p+1} \geqq 0$, but the equalities $r = m_1$, and $m_{p+1} = 0$ do not occur at one time, then

$$N = \sum_{i=0}^{r} (-1)^i \binom{r}{i} \binom{n_1 - i}{m_1 - m_2} \binom{n_2 - i}{m_2 - m_3} \ldots \binom{n_p - i}{m_p - m_{p+1}} = 0.$$

Proof: N is equal to the coefficient of the term $x_1^{m_1-m_2} x_2^{m_2-m_3} \ldots . x_p^{m_p-m_{p+1}}$ in the expression $\sum (-1)^i \binom{r}{i} (1+x_1)^{n_1-i} \ldots (1+x_p)^{n_p-i}$;

or in $(1+x_1)^{n_1-r}\dots(1+x_p)^{n_p-r}\sum(-1)^i\binom{r}{i}P^{r-i} = (1+x_1)^{n_1-r}\dots\dots$ $(1+x_p)^{n_p-r}(P-1)^r$, where $P=\prod_{j=1}^{p}(1+x_j)$. But this expression consists of terms of degree not less than r about x's, while the degree of $x_1^{m_1-m_2}\dots x_p^{m_p-m_{p+1}}$ is less than r. Q.E.D.

Corollary:
$$\sum_{i=0}^{r}(-1)^i\binom{r}{i}(n-i)^k = 0 \qquad (k=0,1,\dots,r-1).$$
$$\sum_{i=0}^{r}(-1)^i\binom{r}{i}(n-i)^r = r\,!\,.$$
$$\sum_{i=0}^{r}(-1)^i\binom{r}{i}(n-i)^{r+1} = (r+1)\,!\left(n-\frac{r}{2}\right).$$

Let $f(x)=\sum_{i=0}^{n}\binom{n}{i}a_ix^{n-i}=0$ be the given equation of degree n, then the successive derived equations are given by

$$\frac{1}{n\dots(n-k+1)}f^{(k)}(x)=\sum_{i=0}^{n-k}\binom{n-k}{i}a_ix^{n-k-i}=0 \quad (k=1,2,\dots,r).$$

Denote $M_\varphi(f)$ by $M_m(f)$ when $\varphi(x)=x^m$, then we shall have the

Theorem D:

$$\left.\begin{aligned}&\sum_{i=0}^{r}(-1)^i\binom{r}{i}\binom{n_1-i}{m_1-m_2}\cdot\dots\cdot\binom{n_p-i}{m_p-m_{p+1}}M_{m_{p+1}}(f^{(i)})=0\\ &\qquad\qquad (m_{p+1}=1,2,\dots,m_p-1)\end{aligned}\right\}\quad(8)$$

Proof: We shall prove this by the method of complete induction under the assumption that

$$\left.\begin{aligned}&\sum_{i=0}^{r}(-1)^i\binom{r}{i}\binom{n_1-i}{m_1-m_2}\dots\binom{n_p-i}{m_p-m_{p+1}}\binom{n-i}{m_{p+1}-m}M_m(f^{(i)})=0\\ &\qquad\qquad (m=1,2,\dots,m_{p+1}-1)\end{aligned}\right\}\quad(9)$$

By Newton's formula we have, putting $\binom{i}{j}=0$ if $i<j$,

$$\binom{n-1-i}{m_{p+1}-1}a_{m_{p+1}}+\binom{n-i}{m_{p+1}-1}a_{m_{p+1}-1}M_1(f^{(i)})+\ldots.$$
$$+\binom{n-i}{0}a_0\,M_{m_{p+1}}(f^{(i)})=0$$
$$(i=0,1,\ldots.,r).$$

$$\therefore\quad a_{m_{p+1}}\sum_i(-1)^i\binom{r}{i}\binom{n_1-i}{m_1-m_2}\cdots\binom{n_p-i}{m_p-m_{p+1}}\binom{n-1-i}{m_{p+1}-1}$$
$$+a_{m_{p+1}-1}\sum_i(-1)^r\binom{r}{i}\binom{n_1-i}{m_1-m_2}\cdots\binom{n-i}{m_{p+1}-1}M_1(f^{(i)})+\ldots.$$
$$+a_0\sum_i(-1)^i\binom{r}{i}\binom{n_1-i}{m_1-m_2}\cdots\binom{n-i}{0}M_{m_{p+1}}(f^{(i)})=0$$

Here the coefficient of the first term vanishes by our Lemma and furthermore those of $a_{m_{p+1}-i}$ $(i=1,2,\ldots,m_{p+1}-1)$ vanish by (9), thus (8) holds good. On the other hand, (9) reduces to the equality in the Lemma when m becomes unity, hence our theorem has been proved.

Corollary: $$\sum_{i=0}^{r}(-1)^i\binom{r}{i}M_m(f^{(i)})=0\quad(m=1,2,\ldots.,r).$$

This is soon proved by Newton's formula.

Now we shall begin to study the sign of $F_r\,\varphi(f)$:

1°. If $\varphi^{(r+1)}(x)\equiv 0$ in the given interval (α,β), then $\varphi(x)\equiv\sum_{m=0}^{r}a_{r-m}x^m$.

$$\therefore\quad F_r\varphi(f)=\sum_{m=0}^{r}a_{r-m}\sum_{i=0}^{r}(-1)^i\binom{r}{i}M_m(f^{(i)})=0\,.$$

2°. If $\varphi^{(r+1)}(x)\not\equiv 0$, then there exists a point $x=a$ at which $\varphi^{(r+1)}(a)\neq 0$. By Taylor's theorem we have

$$\varphi(x)=\sum_{k=0}^{r}\frac{\varphi^{(k)}(a)}{k!}(x-a)^k+\frac{\varphi^{(r+1)}(a)}{(r+1)!}(x-a)^{r+1}+\frac{\varphi^{(r+2)}(\xi)}{(r+2)!}(x-a)^{r+2},$$
$$a\lesseqgtr\xi\lesseqgtr x\,.$$

$$\therefore \quad F_r\varphi(f) = \sum_{k=0}^{r} \frac{\varphi^{(k)}(a)}{k!} \sum_{i=0}^{r} (-1)^i \binom{r}{i} M_k(f^{(i)}(x+a))$$

$$+\frac{\varphi^{(r+1)}(a)}{(r+1)\,!} \sum_{i=0}^{r} (-1)^i \binom{r}{i} M_{r+1}(f^{(i)}(x+a))$$

$$+\frac{1}{(r+2)\,!} \sum_{i=0}^{r} (-1)^i \binom{r}{i} \frac{\sum \varphi^{(r+2)}(\zeta)(z-a)^{r+2}}{n-i},$$

where $z's$ and $\zeta's$ are the roots of $f^{(i)}(x)=0$ and the corresponding numbers to ξ respectively. This reduces, by the Corollary of theorem D, to

$$\frac{\varphi^{(r+1)}(a)}{(r+1)\,!} \sum (-1)^i \binom{r}{i} M_{r+1}(f^{(i)}(x+a))$$

$$+\frac{1}{(r+2)\,!} \sum (-1)^i \binom{r}{i} \frac{\sum \varphi^{(r+2)}(\zeta)(z-a)^{r+2}}{n-i}.$$

Now put $x_1 = x_2 = \dots\dots = x_{n-1} = a$ and $x_n = a+\epsilon$, then we have the following table for the derived systems of $\{x_k\}$:

	$f(x)=0$		$f'(x)=0$			$f^{(r)}(x)=0$	
roots	a	$a+\varepsilon$	a	$a+\frac{n-1}{n}\varepsilon$		a	$a+\frac{n-r}{n}\varepsilon$
multiplicities	n-1	1	n-2	1		n-r-1	1

In this case we have

$$\sum (-1)^i \binom{r}{i} M_{r+1}(f^{(i)}(x+a)) = \sum (-1)^i \binom{r}{i} (n-i)^r \left(\frac{\epsilon}{n}\right)^{r+1} = r!\left(\frac{\epsilon}{n}\right)^{r+1},$$

$$\sum (-1)^i \binom{r}{i} \frac{\sum \varphi^{(r+2)}(\zeta)(z-a)^{r+2}}{n-i} = \sum (-1)^i \binom{r}{i} \frac{\varphi^{(r+2)}(\zeta)\left(\frac{n-i}{n}\epsilon\right)^{r+2}}{n-i}$$

$$\risingdotseq (r+1)\,!\left(n-\frac{r}{2}\right)\varphi^{(r+2)}(a)\left(\frac{\epsilon}{n}\right)^{r+2}.$$

$$\therefore \quad F_r\varphi(f) \doteqdot \frac{\varphi^{(r+1)}(a)}{r+1}\left(\frac{\epsilon}{n}\right)^{r+1} + \frac{n-\dfrac{r}{2}}{r+2}\varphi^{(r+2)}(a)\left(\frac{\epsilon}{n}\right)^{r+2} .$$

If we make ϵ sufficiently small, then the sign of $F_r\varphi(f)$ is determined by the term $\frac{\varphi^{(r+1)}(a)}{r+1}\left(\frac{\epsilon}{n}\right)^{r+1}$, hence if $r+1$ is odd, $F_r\varphi(f)$ may be positive or negative according as our choice of the values of $\{x_k\}$. If $r+1$ is even, in order that $F_r\varphi(f)$ should be of constant sign, $\varphi^{(r+1)}(x)$ must be of constant sign in (α, β). Thus we have the

Theorem E: 1°. $r =$ even: If $\varphi(x)$ is a polynomial of degree r, $F_r\varphi(f) = 0$; and if this is not the case, $F_r\varphi(f)$ can not be of constant sign.

2°. $r =$ odd: In order that $F_r\varphi(f)$ should be of constant sign, it is necessary that $\varphi^{(r+1)}(x)$ must be of constant sign.

Remark: In the case where r is odd, it is desirable to examine whether the condition $\varphi^{(r+1)}(x) > 0$ is not only necessary but also sufficient in order to ascertain that $F_r\varphi(f)$ should be positive, as in the case $r = 1$, or not.