On Some Reducible Quadratic Differential Forms in n-Dimensional Space.

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Introduction.

We put the question[1]: In an n-dimentional space, giving its fundamental differential form:

$$ds^2 = g_{\mu\nu} dx_\mu dx_\nu \tag{1}$$

on what condition can (1) be transformed into the form:

$$ds^2 = \sum_{i,k=1}^{n-1} \rho \bar{g}_{ik} dX_i dX_k + \theta dX^2 \tag{2}$$

where $\bar{g}_{ik}$ does not contain X, and ρ and θ are functions of X_1, X_2, ... X_{n-1}, X?

Notations and Formulae.

In this paper, we adopt the notation and formulae as follows:

$$ds^2 = g_{\mu\nu} dx_\mu dx_\nu$$

denotes the fundamental quadratic differential form of the n-dimensional space;

$$\Gamma_\nu A_\mu = \frac{\partial A_\mu}{\partial x_\nu} - \left\{\mu\nu, \alpha\right\} A_\alpha \tag{3}$$

(1) Eisenhalt obtained the condition in a geometrical form. The condition is that the space admits of a family of hypersurfaces with indeterminate lines of curvature. L.P. Eisenhalt. *Riemannian Geom.* (1926) p. 182.

the covariant derivative of a vector A_μ, where

$$\{\mu\nu, \alpha\}=\frac{1}{2}g^{\alpha\beta}\left(\frac{\partial g_{\mu\beta}}{\partial x_\nu}+\frac{\partial g_{\nu\beta}}{\partial x_\mu}-\frac{\partial g_{\mu\nu}}{\partial x_\beta}\right);$$

and $B_{\mu\nu\sigma\lambda}$, the curvature-tensor (or Riemann-Christoffel's tensor) of the space expressed as follows:

$$B_{\mu\nu\sigma\lambda}=g_{\lambda\varepsilon}B^{\varepsilon}_{\mu\nu\sigma}$$

where

$$B^{\varepsilon}_{\mu\nu\sigma}=\{\mu\sigma, \alpha\}\{\alpha\nu, \epsilon\}-\{\mu\nu, \alpha\}\{\alpha\sigma, \epsilon\}+\frac{\partial}{\partial x_\nu}\{\mu\sigma, \epsilon\}-\frac{\partial}{\partial x_\sigma}\{\mu\nu, \epsilon\}.$$

General Statement.

We are going to find how (1) can be transformed into the form (2).

Assume that such a transformation exists. Then, form (2), we have

$$g_{\mu\nu}=\sum_{i,k=1}^{n-1}\rho\bar{g}_{ik}\frac{\partial X_i}{\partial x_\mu}\frac{\partial X_k}{\partial x_\nu}+\theta\frac{\partial X}{\partial x_\mu}\frac{\partial X}{\partial x_\nu},$$

therefore,

$$\left.\begin{aligned} g^{\mu\nu}\frac{\partial X_i}{\partial x_\mu}\frac{\partial X_j}{\partial x_\nu}&=\frac{1}{\rho}\bar{g}_{ij} \\ g^{\mu\nu}\frac{\partial X}{\partial x_\mu}\frac{\partial X_i}{\partial x_\nu}&=0 \end{aligned}\right\}, \quad i, j=1, 2, \ldots\ldots, n-1, \tag{4}$$

From the assumption that $\bar{g}_{ij}$ does not contain X, we have

$$\frac{\partial}{\partial X}\left(\rho g^{\mu\nu}\frac{\partial X_i}{\partial x_\mu}\frac{\partial X_j}{\partial x_\nu}\right)=0,$$

or
$$g^{\alpha\beta}\frac{\partial X}{dx_\alpha}\frac{\partial}{\partial x_\beta}\left(\rho g^{\mu\nu}\frac{\partial X_i}{\partial x_\mu}\frac{\partial X_j}{\partial x_\nu}\right)=0^{(1)},$$

(1) From the last equation of (4), we get $g^{\mu\nu}\frac{\partial X}{\partial x_\mu}\Big/\left(\text{the minor of }\frac{\partial X}{\partial x_\nu}\text{ in }J\right)=\Delta(X,X)\Big/J$, where J is the Jacobian $\frac{\partial(X_1 X_2\ldots X_{n-1} X)}{\partial(x_1 x_2\ldots x_n)}$ and $\Delta(X,X)=g^{\alpha\beta}\frac{\partial X}{\partial x_\alpha}\frac{\partial X}{\partial x_\beta}$; so we have $\frac{\partial x_\nu}{\partial X}=\frac{1}{\Delta(X,X)}g^{\mu\nu}\frac{\partial X}{\partial x_\mu}$; therefore, $\frac{\partial V}{\partial X}=\frac{1}{\Delta(X,X)}g^{\mu\nu}\frac{\partial X}{\partial x_\mu}\frac{\partial V}{\partial x_\nu}$.

so $$g^{\alpha\beta}\frac{\partial g^{\mu\nu}}{\partial x_\beta}\frac{\partial X}{\partial x_\alpha}\frac{\partial X_i}{\partial x_\mu}\frac{\partial X_j}{\partial x_\nu}+g^{\alpha\beta}g^{\mu\nu}\frac{\partial X_j}{\partial x_\nu}\frac{\partial X}{\partial x_\alpha}\frac{\partial^2 X_i}{\partial x_\beta\partial x_\mu}+g^{\alpha\beta}g^{\mu\nu}\frac{\partial X_i}{\partial x_\mu}\frac{\partial X}{\partial x_\alpha}\frac{\partial^2 X_j}{\partial x_\beta\partial x_\nu}$$

$$+g^{\alpha}\,g^{\mu\nu}\frac{\partial \log\rho}{\partial x_\beta}\frac{\partial X}{\partial x_\alpha}\frac{\partial X_i}{\partial x_\mu}\frac{\partial X_j}{\partial x_\nu}=0\ ,$$

or, using (4), we have

$$g^{\alpha\beta}\frac{\partial g^{\mu\nu}}{\partial x_\beta}\frac{\partial X}{\partial x_\alpha}\frac{\partial X_i}{\partial x_\mu}\frac{\partial X_j}{\partial x_\nu}-g^{\mu\nu}\frac{\partial X_j}{\partial x_\nu}\frac{\partial}{\partial x_\mu}\left(g^{\alpha\beta}\frac{\partial X}{\partial x_\alpha}\right)\frac{\partial X_i}{\partial x_\beta}-g^{\mu\nu}\frac{\partial X_i}{\partial x_\mu}\frac{\partial}{\partial x_\nu}\left(g^{\alpha\beta}\frac{\partial X}{\partial x_\alpha}\right)\frac{\partial X_j}{\partial x_\beta}$$

$$+g^{\alpha\beta}g^{\mu\nu}\frac{\partial}{\partial x_\beta}\left(\log\rho\right)\frac{\partial X}{\partial x_\alpha}\frac{\partial X_i}{\partial x_\mu}\frac{\partial X_j}{\partial x_\nu}=0\ ,$$

therefore,

$$\Bigg[(g^{\beta\nu}g^{\alpha\mu}+g^{\mu\beta}g^{\alpha\nu})\frac{\partial^2 X}{\partial x_\alpha\partial x_\beta}-g^{\alpha\beta}g^{\mu\nu}\frac{\partial\log\rho}{\partial x_\beta}\frac{\partial X}{\partial x_\alpha}$$

$$-\left(g^{\alpha\beta}\frac{\partial g^{\mu\nu}}{\partial x_\alpha}-g^{\beta\nu}\frac{\partial g^{\alpha\mu}}{\partial x_\beta}-g^{\beta\mu}\frac{\partial g^{\alpha\nu}}{\partial x_\beta}\right)\frac{\partial X}{\partial x_\beta}\Bigg]\frac{\partial X_i}{\partial x_\mu}\frac{\partial X_j}{\partial x_\nu}\ . \tag{5}$$

But, from (2), we get

$$\frac{\partial^2 X}{\partial x_\alpha\partial x_\beta}=\Gamma_\alpha\Gamma_\beta X+\{\alpha\beta,\,\epsilon\}\Gamma_\varepsilon X\ ,$$

and $$\Gamma_\varepsilon X=\frac{\partial X}{\partial x_\varepsilon}\ ,$$

therefore, substituting the above into (5), we have

$$\Bigg[(g^{\beta\nu}g^{\alpha\mu}+g^{\mu\beta}g^{\alpha\nu})(\Gamma_\alpha\Gamma_\beta X+\{\alpha\beta,\,\epsilon\}\Gamma_\varepsilon X)-g^{\alpha\beta}g^{\mu\nu}\frac{\partial\log\rho}{\partial x_\beta}\Gamma_\alpha X$$

$$-\left(g^{\alpha\beta}\frac{\partial g^{\mu\nu}}{\partial x_\alpha}-g^{\beta\nu}\frac{\partial g^{\alpha\mu}}{\partial x_\beta}-g^{\beta\mu}\frac{\partial g^{\alpha\nu}}{\partial x_\beta}\right)\Gamma_\alpha X\Bigg]\frac{\partial X_i}{\partial x_\mu}\frac{\partial X_j}{\partial x_\nu}=0\ . \tag{6}$$

But on the other hand,

$$(g^{\beta\nu}g^{\alpha\mu}+g^{\beta\mu}g^{\alpha\nu})\{\alpha\beta,\,\epsilon\}$$

$$=\frac{1}{2}(g^{\beta\nu}g^{\alpha\mu}+g^{\beta\mu}g^{\alpha\nu})g^{\varepsilon\lambda}\left(\frac{\partial g_{\alpha\lambda}}{\partial x_\beta}+\frac{\partial g_{\beta\lambda}}{\theta x_\alpha}-\frac{\partial g_{\alpha\beta}}{\partial x_\lambda}\right)$$

$$=g^{\varepsilon\beta}\frac{\partial g^{\mu\nu}}{\partial x_\beta}-g^{\beta\nu}\frac{\partial g^{\varepsilon\mu}}{\partial x_\beta}-g^{\beta\mu}\frac{\partial g^{\varepsilon\nu}}{\partial x_\beta}\ ;$$

so that (5) becomes as follows :

$$\left\{(g^{\beta\nu}g^{\alpha\mu}+g^{\mu\beta}g^{\alpha\nu})\Gamma_\alpha\Gamma_\beta X-g^{\alpha\beta}g^{\mu\lambda}\frac{\partial \log\rho}{\partial x_\beta}\Gamma_\alpha X\right\}\frac{\partial X_i}{\partial x_\mu}\frac{\partial X_j}{\partial x_\nu}=0\ ,$$

$$i,j=1,2,\ \ldots.,\ n-1, \qquad (7)$$

But, on the other hand, we can prove the following theorem: *If $A^{\mu\nu}$ be a symmetric tensor and V^μ be any vector, satisfying the relations:*

$$\sum_{\mu,\nu=1}^{n}A^{\mu\nu}X^i_\mu X^j_\nu=0\ ,\quad \sum_{\mu=1}^{n}A^\mu X^i_\mu=0\ ,\qquad (i,j=1,2,\ \ldots.,\ n-1,)$$

and

$$\begin{vmatrix} X^1_1 & X^1_2 & \cdot\ \cdot & X^1_n \\ \cdot & \cdot & \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot\ \cdot & \cdot \\ X^{n-1}_1 & X^{n-1}_2 & \cdot\ \cdot & X^{n-1}_n \\ V^1 & V^2 & \cdot\ \cdot & V^n \end{vmatrix}\neq 0\ ,$$

then it must be that $\quad A^{\mu\nu}=\rho^\mu V^\nu+\rho^\nu V^\mu,$

where ρ^μ is a vactor.

The proof of this theorem will be given at the end of this paper,(1) in order not to confuse our discussion.

Thus, from (7), and taking into account the independence of the quantities $X_1, \ldots., X_{n-1}, X$, and the relation :

$$\left(g^{\alpha\beta}\frac{\partial X}{\partial x_\beta}\right)\frac{\partial X_i}{\partial x_\alpha}=0\ ,\qquad i=1,2,\ \ldots.,\ n-1\ ,$$

we can apply the above theorem in our case. Hence, we have

$$(g^{\beta\nu}g^{\alpha\mu}+g^{\mu\beta}g^{\alpha\nu})\Gamma_\alpha\Gamma_\beta X-g^{\alpha\beta}g^{\mu\nu}\frac{\partial \log\rho}{\partial x_\beta}\Gamma_\alpha X=2A^\mu\Gamma^\nu X+2A^\nu\Gamma^\mu X\ ,$$

(1) See p. 41.

or $$\nabla_\mu \nabla_\nu X = (g_{\mu\nu} g^{\alpha\beta} \nabla_\alpha \sigma + A_\mu g^\beta_\nu + A_\nu g^\beta_\mu) \nabla_\beta X\,, \tag{8}$$

where $\sigma = \frac{1}{2} \log \rho$ and A_λ is a certain vector.

Thus we know that if there exists the transformation of the form:

$$X_1 = X_1(x_1 \dots x_n),\ \dots,\ X_{n-1} = X_{n-1}(x_1 \dots x_n),\ X = X(x_1 \dots x_n),$$

then X must be a solution of the equation (8).

Next, we will find the condition of integrability of the equation (8).

From (8), we have

$$\begin{aligned}\nabla_\omega \nabla_\mu \nabla_\nu X = \Big\{ & g_{\mu\nu} g^{\alpha\xi} (\nabla_\omega \nabla_\alpha \sigma + \nabla_\omega \sigma \nabla_\alpha \sigma + A_\omega \nabla_\alpha \sigma) \\ & + g^{\alpha\xi} g_{\omega\nu} A_\mu \nabla_\alpha \sigma + g^{\alpha\xi} g_{\omega\mu} A_\nu \nabla_\alpha \sigma + A_\mu A_\omega g^\xi_\nu + 2 A_\mu A_\nu g^\xi_\omega \\ & + A_\nu A_\omega g^\xi_\mu + g_{\mu\nu} A^\alpha \nabla_\alpha \sigma g^\xi_\omega + \nabla_\omega A_\mu g^\xi_\nu + \nabla_\omega A_\nu g^\xi_\mu \Big\} \nabla_\xi X,\end{aligned}$$

so we have $$\nabla_\omega \nabla_\mu \nabla_\nu X - \nabla_\mu \nabla_\omega \nabla_\nu X = L^\xi_{\nu\mu\omega} \nabla_\xi X\,,$$

where $$\begin{aligned}L^\xi_{\nu\mu\omega} = {} & g_{\mu\nu} g^{\alpha\xi} (\nabla_\omega \nabla_\alpha \sigma + \nabla_\omega \sigma \nabla_\alpha \sigma) - g_{\omega\nu} g^{\alpha\xi} (\Delta_\mu \nabla_\alpha \sigma + \nabla_\mu \sigma \nabla_\alpha \sigma) \\ & + g_{\mu\nu} A^\alpha \nabla_\alpha \sigma g^\xi_\omega - g_{\omega\nu} A^\alpha \nabla_\alpha \sigma\, g^\xi_\mu + A_\mu A_\nu g^\xi_\omega - A_\omega A_\nu g^\xi_\mu \\ & + (\nabla_\omega A_\mu - \nabla_\mu A_\omega) g^\xi_\nu + \nabla_\omega A_\nu g^\xi_\mu - \nabla_\mu A_\nu g^\xi_\omega\,.\end{aligned}$$

So we have

$$(B^\xi_{\nu\mu\omega} - L^\xi_{\nu\mu\omega}) \nabla_\xi X = 0\ ; \tag{9}$$

this is the necessary condition for integrability of the equation (8).

Next, we are going to show that this is also the sufficient condition.

For, if (8) and (9) hold good, then the equation (8) is integrable in virtue of (9) and we can determine a solution X; and then from the equation:

$$\left(g^{\mu\nu} \frac{\partial X}{\partial x_\nu}\right) \frac{\partial \xi}{\partial x_\nu} = 0\,,$$

$(n-1)$ independent solutions of ξ (say, $X_1, X_2, \dots X_{n-1}$), can be obtained. Then, by the transformation:

$$X_1 = X_1(x_1, \dots, x_n),\ \dots,\ X_{n-1} = X_{n-1}(x_1, \dots, x_n),$$
$$X_n = X(x_1, \dots, x_n),$$

ds^2 will be transformed into the form :

$$ds^2 = \sum_{i,j=1}^{n-1} g'_{ij}\, dX_i\, dX_j + g'_{nn}\, dX_n^2 \tag{10}$$

and the equation (8) becomes as follows :

$$-\left\{\mu\nu,\, n\right\} = g'_{\mu\nu}\, g'^{\alpha n}\, \Gamma_\alpha\, \sigma + A_\mu\, g'^{n}_{\nu} + A_\nu g'^{n}_{\mu} \tag{11}$$

If we put $\mu,\ \nu \neq n$ in the above, then

$$\frac{1}{2}\frac{\partial g'_{\mu\nu}}{\partial X_n} = g'_{\mu\nu}\, \Gamma_n\, \sigma\,,$$

or

$$\frac{1}{2}\frac{1}{g'_{\mu\nu}}\frac{\partial g'_{\mu\nu}}{\partial X_n} = \frac{\partial \sigma}{\partial X_n}\,;$$

so we have

$$g'_{\mu\nu} = e^{2\sigma}\, \bar{g}_{\mu\nu} \tag{12}$$

where $\bar{g}_{\mu\nu}$ is a tensor expressed in terms of $X_1,\, X_2,\, \ldots,\, X_{n-1}$.

If we put $\mu = n$ in (11), we have

$$-\frac{1}{2}\frac{1}{g'_{nn}}\frac{\partial g'_{nn}}{\partial X_\nu} = A_\nu\,,\quad \nu = 1,\ 2,\ \ldots,\ n-1, \tag{13}$$

and

$$-\frac{1}{2}\frac{1}{g'_{nn}}\frac{\partial g'_{nn}}{\partial X_n} = \Gamma_n\, \sigma + 2A_n \tag{14}$$

But if we put $\nu = \omega = \mu\ (\neq n)$ in (9), we have

$$\Gamma_n\, A_\mu - 2\,\Gamma_\mu\, A_n - \Gamma_\mu\, \Gamma_n\, \sigma + A_\mu A_n - \Gamma_\mu\, \sigma\, \Gamma_n\, \sigma = 0\,, \tag{15}$$

But on the other hand,

$$\Gamma_n\, A_\mu = \frac{\partial A_\mu}{\partial X_n} - \left(-A_\mu\, \Gamma_n\, \sigma + \frac{1}{2}\frac{1}{g'_{nn}}\frac{\partial g'_{nn}}{\partial X_\mu} A_n\right)$$

and

$$\Gamma_\mu\, A_n = \frac{\partial A_n}{\partial X_\mu} - \left(-A_\mu\, \Gamma_n\, \sigma + \frac{1}{2}\frac{1}{g'_{nn}}\frac{\partial g'_{nn}}{\partial X_\mu} A_n\right);$$

hence, substituting (13) and (14) in the above,

$$\nabla_n A_\mu = \frac{\partial A_\mu}{\partial X_n} - A_\mu \nabla_n \sigma + A_\mu A_n ,$$

$$\nabla_\mu A_n = \frac{\partial A_\mu}{\partial X_\mu} - A_\mu \nabla_n \sigma + A_\mu A_n ,$$

$$\nabla_\mu \nabla_n \sigma = \frac{\partial^2 \sigma}{\partial X_\mu \partial X_n} - \nabla_n \sigma \nabla_\mu \sigma + A_\mu \nabla_n \sigma$$

Therefore, (15) can be written in the form:

$$\frac{\partial A_\mu}{\partial X_n} - \frac{\partial}{\partial X_\mu}(\nabla_n \sigma + 2A_n) = 0, \qquad \mu \neq n , \tag{16}$$

which shows that (13) and (14) are integrable and we can determine a function A, so that

$$A_\nu = \frac{\partial A}{\partial X_\nu}, \quad (\nu \neq n) ; \qquad \frac{\partial \sigma}{\partial X_n} + 2A_n = \frac{\partial A}{\partial X_n} ;$$

therefore, we have $\qquad g'_{nn} = e^{-2A}$;

so we have $\qquad ds^2 = e^{-2A} dX_n + e^{2\sigma} \sum_{i,j}^{n-1} \bar{g}_{ij} \, dX_i \, dX_j ,$

which shows that our condition is sufficient.

So we have the

Theorem: *The fundamental quadratic differential form can be transformed into the form* (2), *when and only when a function X exists satisfying the equation:*

$$\nabla_\mu \nabla_\nu X = (g_{\mu\nu} g^{\alpha\beta} \nabla_\alpha \sigma + g^\beta_\nu A_\mu + g^\beta_\mu A_\nu) \nabla_\beta X ,$$

and $\qquad (B^\xi_{\nu\mu\omega} - L^\xi_{\nu\mu\omega}) \nabla_\xi X = 0 ,$

where $\quad L^\xi_{\nu\mu\omega} = g_{\mu\nu} g^{\alpha\xi} (\nabla_\omega \nabla_\alpha \sigma + \nabla_\omega \sigma \nabla_\alpha \sigma) - g_{\omega\nu} g^{\alpha\xi} (\nabla_\mu \nabla_\alpha \sigma + \nabla_\mu \sigma \nabla_\alpha \sigma)$

$$+ g_{\mu\nu} A^\alpha \nabla_\alpha \sigma \, g^\xi_\omega - g_{\omega\nu} A^\alpha \nabla_\alpha \sigma \, g^\xi_\mu + A_\mu A_\nu g^\xi_\omega - A_\omega A_\nu g^\xi_\mu$$

$$+ (\nabla_\omega A_\mu - \nabla_\mu A_\omega) g^\xi_\nu + \nabla_\omega A_\nu \, g^\xi_\mu - \nabla_\mu A_\nu \, g^\xi_\omega .$$

And then the fundamental quadratic form is transformed into the form:

$$ds^2=e^{-2A}dX^2+e^{2\delta}\sum_{i,j=1}^{n-1}\bar{g}_{ij}\,dx_i dx_j\,,$$

where $\bar{g}_{ij}$ *is a function of* X_{n-1}.

Special Details.

The above is the general statement; here we are going to examine it in special details.

Assuming that the fundamental quadratic form is

$$ds^2=g_{\mu\nu}\,dx_\mu\,dx_\nu\,,$$

if the equation:

$$\nabla_\mu\nabla_\nu X=(g_{\mu\nu}g^{\alpha\beta}\nabla_\alpha\sigma+g^\beta_\nu A_\mu+g^\beta_\mu A_\nu)\nabla_\beta X \tag{17}$$

has m independent solutions: Y_1, Y_2, ..., Y_m, then for the condition of integrability we have the relation:

$$(B^\xi_{\nu\mu\omega}-L^\xi_{\nu\mu\omega})\nabla_\xi Y_i=0,\quad i=1,\ 2,\ \ldots.m, \tag{18}$$

Apply the transformation by which the fundamental form is transformed into the form:

$$ds^2=e^{-2A}dY_1+e^{2\gamma}\sum_{i,j}^{n-1}g'_{ij}\,dx'_i\,dx'\,; \tag{19}$$

Now, if we denote by $\nabla'\theta$ the covariant derivative of θ with respect to

$$ds'^2=\sum_{i,j}^{n-1}g'_{ij}\,dx'_i\,dx'_j\,,$$

then we have

$$\left\{\mu\nu,\,\alpha\right\}=g'^\alpha_\mu\nabla_\nu\sigma+g'^\alpha_\nu\nabla_\mu\sigma-g'^{\alpha\beta}g'_{\mu\nu}\nabla_\beta\sigma+\left\{\mu\nu\alpha\right\}'\,,$$

$$\left\{\mu\nu,\,n\right\}=-g^{nn}g'_{\mu\nu}\nabla_n\sigma,\qquad \mu,\,\nu,\,\alpha,\,\beta\neq n\,, \tag{20}$$

therefore,

$$\begin{aligned}\nabla_\mu\nabla_\nu X=&\nabla'_\mu\nabla'_\nu X-(-g'_{\mu\nu}g'^{\alpha\beta}\nabla'_\alpha\sigma+\nabla'_\mu\sigma\,g'^\beta_\nu+\nabla'_\nu\sigma\,g'^\beta_\mu)\nabla'_\beta X\\ &+g_{\mu\nu}g^{nn}\nabla_n\sigma\nabla_n X,\qquad \mu,\,\nu,\,\alpha,\,\beta\neq n,\end{aligned} \tag{21}$$

Since Y_i is a solution of (17), we have

$$\Gamma'_\mu \Gamma'_\nu Y_i = \left\{ g_{\mu\nu} g^{\alpha\beta} \Gamma_\alpha \sigma + A_\mu g^\beta_\nu + A_\nu g^\beta_\mu \right\} \Gamma_\beta Y_i ,$$

therefore, substituting the above into (21), we have

$$\Gamma'_\mu \Gamma'_\nu Y_i - \left\{ (A_\mu + \Gamma_\mu \sigma) g'^\beta_\nu + (A_\nu + \Gamma'_\mu \sigma) g'^\beta_\mu \right\} \Gamma'_\beta Y_i , \qquad (22)$$

$$\mu,\ \nu,\ \beta \neq n ;$$

and also from (20) and (18),

$$(B'^\varepsilon_{\nu\mu\omega} - L'^\varepsilon_{\nu\mu\omega}) \Gamma_\varepsilon Y_i = 0 , \qquad \mu,\ \nu,\ \omega,\ \epsilon \neq n , \qquad (23)$$

where $$L'^\varepsilon_{\nu\mu\omega} = A'_\mu A'_\nu g^\varepsilon_\omega - A'_\omega A'_\nu g'^\varepsilon_\mu + (\Gamma'_\omega A'_\mu - \Gamma'_\mu A'_\omega) g'^\varepsilon_\nu$$
$$+ \Gamma'_\omega A'_\nu g'^2_\mu - \Gamma'_\mu A'_\nu g'^\varepsilon_\omega ,$$

and $$A'_\lambda = A_\lambda + \Gamma'_\lambda \sigma, \qquad A_\lambda = \frac{\partial A}{\partial x'_\lambda} \quad (\lambda \neq n) .$$

Therefore, after the nearly same process by which we have obtained (19) from (18), we see that ds'^2 is transformed into the form:

$$ds'^2 = e^{-2(A+\sigma)+f(Y_1)} d\overset{2}{Y_2} + \sum_{i,j}^{n-2} g''_{ij} dx''_i dx''_j ,$$

From the assumption, $-2(A+\sigma)+f(Y_1)$ must not contain Y_1; so that

$$-2(A+\sigma)+f(Y_1) = F(x'_1, \ldots\ldots, x'_{\mu-1}) .$$

Here, if we put

$$2A = 2\theta + f(Y_1) , \qquad 2\sigma = -2\theta - F(x'_1, \ldots\ldots, x'_{n-1})$$

where θ is an arbitrary function of $x_1 , \ldots\ldots , x_n$, then (19) becomes as follows:

$$ds^2 = e^{-2\theta} \cdot e^{f(Y_1)} dY_1^2 + e^{F(x'_1 \ldots, x'_n-1)} \cdot \sum_{i,j}^{n-1} g'_{i,j} dx'_i dx'_j .$$

So, replacing Y_1 for $\int e^{\frac{1}{2}f(Y_1)} dY_1$, we may assume, without loss of generality, that

$$A = -\sigma = \theta .$$

Hence the expressions for $L^{\xi}_{\nu\mu\omega}$ and (18) become, respectively, as follows:

$$L^{\xi}_{\nu\mu\omega} = g\,[_{\mu\,[\nu}A_{\omega]}\,\delta\,]\,g^{\xi\delta} \quad ^{(1)}$$

where $$A_{\alpha\beta} = -\frac{1}{2} g_{\alpha\beta}\nabla^{\xi}A\,\nabla_{\xi}A + \nabla_{\alpha}A\,\nabla_{\beta}A - \nabla_{\alpha}\nabla_{\beta}A ,$$

and $$C^{\xi}_{\nu\mu\omega}\nabla_{\xi}\,Y_i = 0 , \quad i=1,\ 2,\ \ldots\ldots,\ m,$$

where $$C^{\xi}_{\nu\mu\omega} = B^{\xi}_{\nu\mu\omega} - L^{\xi}_{\nu\mu\omega} .$$

The tensor $C^{\xi}_{\nu\mu\omega}$ is what is called the conformal curvature tensor.

Next, consider another quadratic differential form expressed in the form:

$$d\bar{s}^2 = \bar{g}_{\mu\nu}dx_{\mu}dx_{\nu} \tag{24}$$

which is related with the original form ds^2 by the equation:

$$\bar{g}_{\mu\nu} = e^{2A}g_{\mu\nu} \tag{25}$$

and if we denote, respectively, by $\bar{\nabla}v$ and $\bar{B}^{\xi}_{\nu\mu\omega}$ the convariant derivative and the curvature-tensor with respect to the form (24), then we have the relations between the old and new tensor:

$$\bar{\nabla}_{\mu}v = \frac{\partial v}{\partial x_{\mu}} = \nabla^{\mu}v$$

$$\bar{\nabla}_{\mu}\,w_{\lambda} = \nabla_{\mu}\,w_{\lambda} - (w_{\mu}A_{\lambda} + w_{\lambda}A_{\mu}) + A^{\alpha}w_{\alpha}g_{\mu\lambda} \tag{26}$$

$$\bar{B}_{\nu w\mu\lambda} = B_{\nu w\mu\lambda} - L_{\nu w\mu\lambda} \tag{27}$$

So we have

$$\bar{B}_{\nu\omega\mu\nu} = C_{\nu\omega\mu\lambda} \tag{28}$$

Also, if we put $\nabla_{\lambda}Y_i$ ($i=1, 2, \ldots\ldots, m$) for W_{λ} in (26), we have

$$\bar{\nabla}_{\mu}\nabla_{\lambda}\,Y_i = \nabla_{\mu}\nabla_{\lambda}Y_i - (\nabla_{\mu}Y_i\,A_{\lambda} + \nabla_{\lambda}Y_i\,A_{\mu}) + A^{\alpha}\nabla_{\alpha}Y_i\,g_{\mu\lambda} ;$$

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(1) $A_{[\alpha\beta]} \equiv A_{\alpha\beta} - A_{\beta\alpha}$, $\qquad A_{[\alpha[\beta\gamma]\delta]} \equiv A_{\alpha[\beta\gamma]\delta} - A_{\delta[\beta\gamma]\alpha}$.

the right-hand side of the above vanishes in virtue of (17), so we have

$$\bar{\Gamma}_\mu \Gamma_\lambda Y_i = 0, \qquad i=1, 2, \ldots\ldots, m, \tag{29}$$

therefore,

$$\bar{\Gamma}_\mu(\bar{g}^{\alpha\beta} \Gamma_\alpha Y_i \, \Gamma_\beta Y_j) = \bar{g}^{\alpha\beta}(\bar{\Gamma}_\mu \Gamma_\alpha Y_i \, \Gamma_\beta Y_j + \bar{\Gamma}_\mu \Gamma_\beta Y_j \Gamma_\alpha Y_i) = 0,$$

so that

$$\bar{g}^{\alpha\beta} \Gamma_\alpha Y_i \Gamma_\beta Y_j = \bar{c}_{ij} (=\text{const.}), \quad i, j = 1, 2, \ldots\ldots, m.$$

From the independence of $Y_1, Y_2, \ldots\ldots, Y_m$, we know that

$$\begin{vmatrix} \bar{c}_{11} & \bar{c}_{21} & \cdot & \cdot & \bar{c}_{m1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \bar{c}_{1m} & \bar{c}_{2m} & \cdot & \cdot & \bar{c}_{um} \end{vmatrix} \neq 0 \tag{30}$$

so, applying a linear transformation of $Y_1, \ldots\ldots, Y_m$, we can render it to the case where $\bar{c}_{ii}=1$, $\bar{c}_{ij}=0$ $(i \neq j)$; hence, without loss of generality, we may assume from the first that $Y_1, \ldots\ldots, Y_m$ satisfy the following relations:

$$\bar{g}^{\alpha\beta} \Gamma_\alpha Y_i \Gamma_\beta Y_i = 1, \; \bar{g}^{\alpha\beta} \Gamma_\alpha Y_i \Gamma_\beta Y_j = 0 \qquad (i \neq j) \tag{31}$$

Now, considering the system of equations:

$$\Gamma_\mu Y_i \bar{\Gamma}^\mu s = 0, \; i=1, 2, \ldots\ldots, m, \tag{32}$$

we observe that it forms a complete system; for, all the quantities:

$$\Gamma_\mu Y_i \bar{\Gamma}^\mu \Gamma_\lambda Y_j \, Y_j - \Gamma_\mu Y_j \bar{\Gamma}^\mu \Gamma_\lambda \, Y_i$$

vanish in virtue of (29). So, $n-m$ independent solutions can be obtained from (32), say, $Z_1, Z_2, \ldots\ldots, Z_{n-m}$; and also we know that each of the Z_s is independent of $Y_1, \ldots\ldots, Y_m$; for, if there exists a relation such that $Z_1 = F(Y_1, \ldots\ldots, Y_m)$, then we have

$$\bar{\Gamma}^\lambda \, Y_i \Gamma_\lambda Z_1 = \sum_{j=1}^{m} \frac{\partial F}{\partial Y_j} \bar{\Gamma}^\lambda \, Y_i \Gamma_\lambda \, Y_j;$$

so that
$$0 = \frac{\partial F}{\partial Y_j},$$

which is a contradiction. So the n functions $Y_1, \ldots\ldots, Y_m, Z_1, \ldots\ldots, Z_{n-m}$ are independent of each other.

Next, considering the transformation:

$$\left.\begin{array}{ll} Y_1 = Y_1(x_1, \ldots\ldots, x_n), & Z_1 = Z_1(x_1, \ldots\ldots, x_n), \\ \cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots \\ Y_m = Y_m(x_1, \ldots\ldots, x_n), & Z_{n-m} = Z_{n-m}(x_1, \ldots\ldots, x_n), \end{array}\right\} \quad (33)$$

we note that

$$\bar{g}^{\alpha\beta} \Gamma_\alpha Y_i \Gamma_\beta Y_i = 1, \qquad \bar{g}^{\alpha\beta} \Gamma_\alpha Y_i \Gamma_\beta Y_j = 0,$$

$$\bar{g}^{\alpha\beta} \Gamma_\alpha Y_i \Gamma_\beta Z_j = 0, \qquad \bar{g}_{\alpha\beta} \Gamma_\alpha Z_i \Gamma_\beta Z_j = g'_{ij},$$

so the quadratic form (24) is transformed by the transformation (33) into the form:

$$d\bar{s}^2 = dY_1^2 + \ldots\ldots + dY_m^2 + \sum_{i,j=1}^{n-m} g'_{ij}\, dZ_i\, dZ_j,$$

therefore, ds^2 is transformed into the form:

$$ds^2 = e^{-2A}(dY_1^2 + \ldots\ldots + dY_m^2 + \sum_{i,j=1}^{n-m} g'_{ij}\, dZ_i\, dZ_j).$$

Further, we can easily see that the function g'_{ij} does not contain any of $Y_1, \ldots\ldots, Y_m$.

For
$$\frac{\partial}{\partial Y_1} g'_{ij} = \frac{\partial}{\partial Y_1}(\bar{g}^{\alpha\beta} \Gamma_\alpha Z_i \Gamma_\beta Z_j),$$

the right-hand side of the above is, as we have seen before,[1] proportional to

$$\bar{g}^{\varepsilon\delta} \Gamma_\varepsilon Y_1 \bar{\Gamma}_\delta (\bar{g}^{\alpha\beta} \Gamma_\alpha Z_i \Gamma_\beta Z_j),$$

or

$$\bar{g}^{\alpha\beta} \bar{g}^{\varepsilon\delta} \Gamma_\varepsilon Y_1 (\Gamma_\beta Z_j \bar{\Gamma}_\delta \Gamma_\alpha Z_j + \Gamma_\alpha Z_i \bar{\Gamma}_\delta \Gamma_\beta Z_i) \qquad (34)$$

But, from (32),

$$\bar{g}^{\varepsilon\delta} \Gamma_\varepsilon Y_1 \Gamma_\delta Z_j = 0,$$

(1) Cf. the foot-note in p. 26.

and applying the operation $\bar{\Gamma}_\alpha$ to the above and taking (29) into account, we have

$$\bar{g}^{\varepsilon\delta}\Gamma_\varepsilon Y_1 \bar{\Gamma}_\alpha \Gamma_\delta Z_j=0\,, \qquad \text{or} \qquad \bar{g}^{\varepsilon\delta}\Gamma_\varepsilon Y_1 \bar{\Gamma}_\delta \Gamma_\alpha Z_j=0\,,$$

so (34) vanishes, therefore, we know that $\bar{g}_{ij}$ does not contain Y_1.

Similarly, it may be proved that the same does not contain any of $Y_1, \ldots\ldots, Y_m$.

So we have the

Theorem: *If the equation:*

$$\Gamma_\mu \Gamma_\nu X=(g_{\mu\nu}g^{\alpha\beta}\Gamma_\alpha\,\sigma+g^\beta_\nu A_\mu+g^\beta_\mu A_\nu)\Gamma_\beta X$$

has m independent solutions: $Y_1, \ldots\ldots, Y_m$, then the fundamental quadratic differential form can be transformed into the form:

$$ds^2=e^{2\gamma}(dY_1^2+\ldots.+dY_m^2+\sum_{i,j=1}^{n-m}g'_{ij}\,dZ_i\,dZ_j)\,,$$

where g'_{ij} does not contain $Y_1, \ldots\ldots, Y_m$. And the conformal curvature tensor vanishes in the m directions of $\Gamma_\varepsilon Y_i$ $(i=1, 2, \ldots., m)$.

Next, as a special case of the above, consider the space in which the equation:

$$\Gamma_\mu \Gamma_\nu X=(-g_{\mu\nu}g^{\alpha\beta}\Gamma_\alpha A+\Gamma_\mu A\, g^\beta_\nu+\Gamma_\nu A\, g^\beta_\mu)\Gamma_\beta X$$

is always integrable; i. e. the space where the condition (18) is satisfied independent of X.

In this case, from (18), we have

$$B_{\nu\mu\omega\lambda}=L_{\nu\mu\omega\lambda} \tag{35}$$

so the space is conformal euclidian.

If we consider the quadratic form:

$$d\bar{s}^2=\bar{g}_{\mu\nu}dx_\mu dx_\nu \tag{36}$$

where

$$\bar{g}_{\mu\nu}=e^{2A}g_{\mu\nu}\,,$$

then, as seen in (27), we get

$$\bar{B}_{\nu\mu\omega\lambda}=C_{\nu\mu\omega\lambda}\,;$$

therefore, from (35),

$$\bar{B}_{\nu\mu\omega\lambda}=0$$

which shows that the quadratic form (36) represents the linear element of an euclidian space.

Therefore, it can be transformed into the form :

$$d\bar{s}^2=dX_1^2+\ldots.+dX_n^2,$$

so we have

$$ds^2=e^{-2A}(dX_1^2+\ldots.+dX_n^2).$$

We can also prove that the function X_i in the above satisfies the equation :

$$\bar{\Gamma}_\mu\,\bar{\Gamma}_\nu\,X_i=0\,;$$

for, since $d\bar{s}^2=dx_1{}^2+\ldots+dx_n^2$ is the fundamental quadratic form, we must have

$$\bar{\Gamma}_\nu\,Xi=\frac{\partial X_i}{\partial X_\nu}=\bar{g}_\nu^i\,;$$

so $\bar{\Gamma}_\mu\,\bar{\Gamma}_\nu\,X_i=0$; therefore, this holds good in general.

So that returning to the original fundamental quadratic form ds^2 and taking the relation into account :

$$\bar{\Gamma}_\nu\,X_i=\frac{\partial X_i}{\partial x^\nu}=\Gamma_\nu\,X_i$$

and form (26), we have

$$\Gamma_\mu\,\Gamma_\nu\,X_i=(-g_{\mu\nu}g^{\alpha\beta}\Gamma_\alpha\,A+\Gamma_\mu A g_\nu^\beta+\Gamma_\nu\,A g_\mu^\beta)\Gamma_\beta X_i\,.$$

So we have the

Theorem : *In a conformal euclidian space whose curvature-tensor is*

$$g^\varepsilon{}_{[\mu}\,{}_{[\nu}A_{\omega]}\,{}_{\lambda]}$$

where $$A_{\alpha\beta}=-\frac{1}{2}g_{\alpha\beta}\,\Gamma^\varepsilon\,A\,\Gamma_\varepsilon+\Gamma_\alpha A\,\Gamma_\beta A-\Gamma_\alpha\,\Gamma_\beta\,A\,,$$

the fundamental quadratic differential form can be transformed into the form :
$$ds^2 = e^{-2A}(dx_1^2 + \dots + dX_n^2),$$
where the functions $X_1, \dots\dots, X_n$ *are* n *independent solutions of the equation :*

$$\nabla_\mu \nabla_\nu X = (g_{\mu\nu} g^{\alpha\beta} \nabla_\alpha A + \nabla_\mu A\, g_\nu^\beta + \nabla_\nu A\, g_\mu^\beta) \nabla_\beta X.$$

As a spectal case, if we put

$$A = \log \frac{X}{R}$$

R being a constant, then we have

$$B_{\nu\omega\mu\lambda} = g_{[\mu\,[\nu} A_{\omega]\,\lambda]}$$

and
$$A_{\alpha\beta} = -\frac{1}{2X^2} g_{\alpha\beta} \nabla^\varepsilon X \nabla_\varepsilon X + 2\frac{1}{X^2} \nabla_\alpha X \nabla_\beta X - \frac{1}{X} \nabla_\alpha \nabla_\beta X_i.$$

and, for the differential equation of X, we have

$$\nabla_\alpha \nabla_\beta X = -\frac{1}{X} g_{\alpha\beta} \nabla^\varepsilon X \nabla_\varepsilon X + 2\frac{1}{X} \nabla_\alpha X \nabla_\beta X. \tag{38}$$

But, since

$$\nabla^\varepsilon \nabla_\varepsilon X = e^{2A} = \frac{X}{R^2},$$

we have

$$B_{\nu\omega\mu\lambda} = \frac{1}{2R^2} g[_{\mu\,[\nu} g_{\omega]\,\lambda}],$$

which shows that the space has the constant curvature $\frac{1}{R^2}$.

And the fundamental form becomes as follows :

$$ds^2 = \frac{R^2}{X^2}(dx_1^2 + \dots + dx_{n-1}^2 + dx_n^2)$$

and here, if we put $Y = \frac{1}{X}$ in (38), we have the equation for Y:

$$\nabla_\alpha \nabla_\beta Y = \frac{1}{R^2}$$

So we have the result: *In the space of the constant curvature* $\frac{1}{R^2}$, *the linear element can be transformed into the form:*

$$ds^2 = \frac{R^2}{X^2}(dx^2 + dx_1^2 + \ldots\ldots + dx_{n-1}^2),$$

where X *is a solution of the equation:*

$$\Gamma_\alpha \Gamma_\beta \frac{1}{X} = \frac{1}{R^2}.$$

This result has been obtained by J.E. Campbell[1] in another way.

Remark: We have left the case where

$$A_\alpha = 0, \qquad \alpha = 1, 2, \ldots\ldots n.$$

But, as we have noticed before, we can prove that this case is included in the case in which $A_\alpha \neq 0$. For, if $A_\alpha = 0$, ($\alpha = 1, 2, \ldots\ldots, n$,) then the equation (8) becomes

$$\Gamma_\mu \Gamma_\nu X = g_{\mu\nu} g^{\alpha\beta} \Gamma_\alpha \sigma \Gamma_\beta X \tag{39}$$

and (13), (14) become respectively,

$$-\frac{1}{2}\frac{1}{\theta}\frac{\partial \theta}{\partial X_\nu} = 0, \qquad -\frac{1}{2}\frac{1}{\theta}\frac{\partial \theta}{\partial X} = \frac{\partial \sigma}{\partial X};$$

so that, without loss of generality, we may take the value of σ as follows:

$$\sigma = \log\sqrt{X}.$$

Substituting this into (39), we have

$$\Gamma_\mu \Gamma_\nu X = \frac{1}{2X} g_{\mu\nu} g^{\alpha\beta} \Gamma_\alpha X \Gamma_\beta X.$$

But, on the other hand, in the equation:

$$\Gamma_\mu \Gamma_\nu X' = (-g_{\mu\nu} g^{\alpha\beta} \Gamma_\alpha A + \Gamma_\mu A\ g_\nu^\beta + \Gamma_\nu A\ g_\mu^\beta) \Gamma_\beta X',$$

(1) J.E. Campbell, *Differential Geom.* (1926) p. 232.

if we put

$$A = X, \qquad Y = e^{-2X},$$

then we have the equation for Y:

$$\nabla_\mu \nabla_\nu Y = \frac{1}{2Y} g_{\mu\nu} g^{\alpha\beta} \nabla_\alpha Y \nabla_\beta Y,$$

which is the same equation as (40) ; so the former case is included in the latter case, where $A = \log \sqrt{\frac{1}{X}}$.

The proof of the theorem on p. 28.

On the way of discussion, we used the theorem without proof, that is :

If $A^{\mu\nu}$ is asymmetric tensor and A^μ, a vector satisfying the following relations :

$$\sum_{\mu,\nu=1}^{n} A^{\mu\nu} X_\mu^i X_\nu^j = 0 , \tag{i}$$

$$\sum_{\mu=1}^{n} A^\mu X_\mu^i = 0, \; i, j = 1, 2 \ldots n-1 , \tag{ii}$$

and

$$\begin{vmatrix} X_1^1 & X_2^1 & \cdot & \cdot & X_n^1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ X_1^{n-1} & X_2^{n-1} & \cdot & \cdot & X_n^{n-1} \\ A^1 & A^2 & \cdot & \cdot & A^n \end{vmatrix} \neq 0, \tag{iii}$$

then it must be true that

$$A^{\mu\nu} = \rho^\mu A^\nu + \rho^\nu A^\mu \text{(1)} .$$

The proof : From (iii), we may assume that

$$A^n \neq 0, \qquad \begin{vmatrix} X_1^1 & \cdot & \cdot & X_{n-1}^1 \\ \cdot & \cdot & \cdot & \cdot \\ X_1^{n-1} & \cdot & \cdot & X_{n-1}^{n-1} \end{vmatrix} \neq 0 ; \tag{iv}$$

(1) This is a slight generalization of A. Friedmann's theorem. Schouten : *Ricci Kalkül*, p. 59.

for, if we expand the determinant (iii) along the last line, we have

$$A^1 \times \text{cofactor of } A^1 + \ldots\ldots + A^n \times \text{cofactor of } A^n ,$$

and, therefore, in order that the above may not vanish, there must be at least one term which does not vanish, say, the last term ; so the assumption is true.

Therefore, we can solve $X_n^i \, (i=1, 2, \ldots\ldots, n-1)$ from (ii), and substituting this into (i), we have

$$\sum_{\mu,\nu=1}^{n-1}\left(A^{\mu\nu}-A^{\mu}\frac{A^{n\nu}}{A^n}-A^{\nu}\frac{A^{n\mu}}{A^n}+\frac{A^{nn}}{(A^n)^2}A^{\mu}A^{\nu}\right)X_{\mu}^{i}X_{\nu}^{j}=0 ,$$

$$i, j=1, 2, \ldots\ldots, n-1 ;$$

so from the assumption (iv), it must be true that

$$A^{\mu\nu}-A^{\mu}\frac{A^{n\nu}}{A^n}-A^{\nu}\frac{A^{n\mu}}{A^n}+\frac{A^{nn}}{(A^n)^2}A^{\mu}A^{\nu}=0 , \qquad \text{(v)}$$

from which

$$A^{\mu\nu}=A^{\mu}\rho^{\nu}+A^{\nu}\rho^{\mu} , \qquad \mu, \nu=1, 2, \ldots\ldots, n-1 .$$

But, if we put $\mu=n$ or $\nu=n$ in (v), the left-hand side vanishes identically ; so we have

$$A^{\mu\nu}=A^{\mu}\rho^{\nu}+A^{\nu}\rho^{\mu} , \qquad \mu, \nu=1, 2, \ldots\ldots, n .$$