Domains of Representatives of Linear Operators.

By

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In a previous paper,[1] one of the authors investigated the representatives of linear operators. Let $\mathfrak{H}$ be a complete linear vector space, where the inner product is defined, and let $\mathfrak{q}(U)$ be a completely additive vector valued differential set function, such that $\{\mathfrak{q}(U)\}$ is complete in $\mathfrak{H}$. Taking $\mathfrak{q}(U)$ as the basic of representation, we can represent $\mathfrak{H}$ by the space of differential set functions $\mathfrak{L}_2(\sigma)$, where $\sigma(U) = \|\mathfrak{q}(U)\|^2$. Let $\boldsymbol{T}$ be a linear operator in $\mathfrak{H}$ which transforms $\mathfrak{f}$ to $\mathfrak{g}$. That is

$$\mathfrak{g} = \boldsymbol{T}\mathfrak{f}.$$

Corresponding to this operator, we have an operator in $\mathfrak{L}_2(\sigma)$, which transforms the representative $\xi(U)$ of $\mathfrak{f}$ to the representative $\eta(U)$ of $\mathfrak{g}$. We may denote it by the same symbol $\boldsymbol{T}$ so that

$$\eta(U) = \boldsymbol{T}\xi(U).$$

On the other hand, let $\mathfrak{K}(U, U') = \big(\boldsymbol{T}\mathfrak{q}(U'), \mathfrak{q}(U)\big)$ be the representative of $\boldsymbol{T}$. Then we have

$$\eta(U) = \int_V \frac{\mathfrak{K}(U, dU')\,\xi(dU')}{\sigma(dU')}.$$

But the last integral is an integral operator in $\mathfrak{L}_2(\sigma)$, which we denote by $\boldsymbol{T}_{\mathfrak{K}}$, so that

$$\eta(U) = \boldsymbol{T}_{\mathfrak{K}}\xi(U).$$

Thus we have the relation:

$$\boldsymbol{T} \subseteqq \boldsymbol{T}_{\mathfrak{K}}.$$ [2]

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(1) F. Maeda, "Representations of Linear Operators by Differential Set Functions," this journal, **6** (1936), 115–137.

(2) This means that $\boldsymbol{T}_{\mathfrak{K}}$ is an extension of $\boldsymbol{T}$.

Now there is a problem concerning what case the equality $\boldsymbol{T} = \boldsymbol{T}_{\mathfrak{R}}$ holds. Of course, when $\boldsymbol{T}$ is a bounded linear operator with domain $\mathfrak{H}$, then this equality holds. The same problem occurs, when the basic of representation is composed of more than one completely additive vector valued set function. When $\mathfrak{H}$ is non-separable, it may occur that the basic of representation $\{\mathfrak{q}_\alpha(U)\}$ is a non-enumerable system. In the preceding paper, this case was not considered. Hence, in this paper, we discuss the representation theory in this case in detail. And next, we prove that there exists a basic system such that

$$\boldsymbol{T}_f \mathfrak{f} = \int_V f(\lambda)\, \boldsymbol{E}(dU)\, \mathfrak{f}$$

is represented in a diagonal form, and show that in this representation,

$$\boldsymbol{T}_f = \boldsymbol{T}_{\mathfrak{R}}$$

holds.(1)

Representations of Operators.

1. Let $\mathfrak{H}$ be a space of vectors, which satisfies the following axioms:

(i) $\mathfrak{H}$ is a linear space.
(ii) In $\mathfrak{H}$ an inner product is defined.
(iii) $\mathfrak{H}$ is complete.

Let $\mathfrak{N}$ be a multiplicative system of sets in an abstraet space V, which contains V itself. And let $\mathfrak{q}(U)$ be a completely additive differential set function defined for all sets of a differential set system $\mathfrak{N}\mathfrak{D}V$, whose functional values are vectors in $\mathfrak{H}$. And put $\sigma(U) = \|\mathfrak{q}(U)\|^2$.(2) Denote by $\mathfrak{M}'(\mathfrak{q})$ the linear manifold determined by the system $\{\mathfrak{q}(U)\}$, U being any set in $\mathfrak{N}\mathfrak{D}V$.

We first consider the case where $\{\mathfrak{q}(U)\}$ is complete in $\mathfrak{H}$, that is $\mathfrak{M}'(\mathfrak{q})$ is dense in $\mathfrak{H}$. In this case any vector $\mathfrak{f}$ in $\mathfrak{H}$ is represented by a completely additive differential set function $\xi(U)$ in $\mathfrak{L}_2(\sigma)$, where $\xi(U) = \big(\mathfrak{f}, \mathfrak{q}(U)\big)$. And conversely, to any set function $\xi(U)$ in $\mathfrak{L}_2(\sigma)$,

(1) This result can also be applied in the problem discussed in the paper, F. Maeda, "Kernels of Transformations in the Space of Set Functions," this journal, **5** (1935), 107–116.

(2) Cf. F. Maeda, "Space of Differential Set Functions," this journal, **6** (1936), 33.

there corresponds an element $\mathfrak{f}$ in $\mathfrak{H}$, such that $\mathfrak{f} = \int_V \frac{\xi(dU)\mathfrak{q}(dU)}{\sigma(dU)}$. And $\mathfrak{L}_2(\sigma)$ is isomorph to $\mathfrak{H}$.[(1)]

Let $\mathfrak{K}(U, U')$ be a differential set function defined for all U and U' in $\mathfrak{N}\mathfrak{D}V$, which belongs to $\mathfrak{L}_2(\sigma, \sigma)$.[(2)] Then, since $\mathfrak{K}(U, U')$ belongs to $\mathfrak{L}_2(\sigma)$ as a function of set U',

$$\int_V \frac{\mathfrak{K}(U, dU')\xi(dU')}{\sigma(dU')} \tag{1.1}$$

is finite for any $\xi(U) \in \mathfrak{L}_2(\sigma)$. For simplicity's sake, we denote this integral by $\mathfrak{K}\xi(U)$. Denote by $\mathfrak{D}_{\mathfrak{K}}$ the aggregate of all set functions $\xi(U)$ in $\mathfrak{L}_2(\sigma)$, so that $\mathfrak{K}\xi(U)$ belongs to $\mathfrak{L}_2(\sigma)$. Then (1.1) expresses a linear operator with domain $\mathfrak{D}_{\mathfrak{K}}$. We denote this linear operator by $\boldsymbol{T}_{\mathfrak{K}}$.

Let $\boldsymbol{T}$ be an operator in $\mathfrak{H}$ with domain $\mathfrak{D}$. And let

$$\mathfrak{g} = \boldsymbol{T}\mathfrak{f} \quad \text{where} \quad \mathfrak{f} \in \mathfrak{D}.$$

Let $\xi(U)$ and $\eta(U)$ be the representatives of $\mathfrak{f}$ and $\mathfrak{g}$ respectively. Then by this correspondence between $\mathfrak{H}$ and $\mathfrak{L}_2(\sigma)$, we have an operator which transforms $\xi(U)$ to $\eta(U)$. We denote this operator by the same symbol $\boldsymbol{T}$, so that

$$\eta(U) = \boldsymbol{T}\xi(U).$$

When $\boldsymbol{T}$ has its adjoint $\boldsymbol{T}^*$, and their domains $\mathfrak{D}$ and $\mathfrak{D}^*$ contain $\mathfrak{M}'(\mathfrak{q})$, we have already seen[(3)] that

$$\mathfrak{K}(U, U') = \big(\boldsymbol{T}\mathfrak{q}(U'), \mathfrak{q}(U)\big) \quad \text{and} \quad \mathfrak{K}^*(U, U') = \big(\boldsymbol{T}^*\mathfrak{q}(U'), \mathfrak{q}(U)\big)$$

are the representatives of $\boldsymbol{T}$ and $\boldsymbol{T}^*$ respectively. That is, when $\xi(U)$ and $\eta(U)$ are the representatives of $\mathfrak{f}\,(\in \mathfrak{D})$ and $\boldsymbol{T}\mathfrak{f}$ respectively,

$$\eta(U) = \mathfrak{K}\xi(U).$$

This means that

$$\boldsymbol{T} \subseteqq \boldsymbol{T}_{\mathfrak{K}}. \tag{1.2}$$

Similarly

$$\boldsymbol{T}^* \subseteqq \boldsymbol{T}_{\mathfrak{K}^*}.$$

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(1) Cf. F. Maeda, this journal, **4** (1934), 69–75; **6** (1936), 35.

(2) That is, $\mathfrak{K}(U, U')$ belongs to $\mathfrak{L}_2(\sigma)$ as a function of set U and as a function of set U'.

(3) F. Maeda, this journal, **6** (1936), 119–120.

2. The representation that we have used in the preceding section, is the simple case, namely, when the basic of the representation is composed with one completely additive vector valued differential set function. In the general case, let $\{\mathfrak{q}_\alpha(U)\}$ (for all $\alpha \in \mathfrak{A}$), α being the parameter, be a finite or enumerably infinite or non-enumerably infinite system of completely additive vector valued differential set functions defined in $\mathfrak{R}\mathfrak{D}V$, where $\mathfrak{A}$ is the aggregate of the suffices α. Let $\mathfrak{M}(\mathfrak{q}_\alpha)$ be the closed linear manifold determined by the system $\{\mathfrak{q}_\alpha(U)\}$ (for all $U \in \mathfrak{R}\mathfrak{D}V$), U being the parameter. When for any different indices α and α', $\mathfrak{M}(\mathfrak{q}_\alpha)$ and $\mathfrak{M}(\mathfrak{q}_{\alpha'})$ are orthogonal, then we say that $\{\mathfrak{q}_\alpha(U)\}$ (for all $\alpha \in \mathfrak{A}$) is an orthogonal system. If there exists no element in $\mathfrak{H}$, except a null element, orthogonal to all the closed linear manifolds $\mathfrak{M}(\mathfrak{q}_\alpha)$, then we say that the orthogonal system $\{\mathfrak{q}_\alpha(U)\}$ is complete in $\mathfrak{H}$.(1) In this case we can take $\{\mathfrak{q}_\alpha(U)\}$ as the basic of the representation. In the preceding paper,(2) we investigated only the case where $\{\mathfrak{q}_\alpha(U)\}$ is a finite or enumerably infinite system. Hence here we discuss the case where $\{\mathfrak{q}_\alpha(U)\}$ is a non-enumerably infinite system in detail.

Let $\mathfrak{f}$ be any element in $\mathfrak{H}$, and denote by $\mathfrak{f}_\alpha$ its component in $\mathfrak{M}(\mathfrak{q}_\alpha)$. Then $\mathfrak{f}_\alpha = \int_V \frac{\xi_\alpha(dU)\,\mathfrak{q}_\alpha(dU)}{\sigma_\alpha(dU)}$ where $\xi_\alpha(U) = \big(\mathfrak{f}, \mathfrak{q}_\alpha(U)\big)$.(3) $\xi_\alpha(U)$ belongs to $\mathfrak{L}(\sigma_\alpha)$, and $\| \mathfrak{f}_\alpha \| = \| \xi_\alpha \|$. Let $\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots$ be any enumerably infinite sequence of suffices α ($\in \mathfrak{A}$). Then we have

$$\sum_n \| \xi_{\alpha_n} \|^2 \leqq \| \mathfrak{f} \|^2.$$

Hence $\| \xi_\alpha \| = 0$ that is $\xi_\alpha(U) = 0$ except at most enumerably infinite system of $\{\xi_\alpha(U)\}$. Hence we may write

$$\mathfrak{f} \,[=]\, \sum_{\alpha \in \mathfrak{A}} \int_V \frac{\xi_\alpha(dU)\,\mathfrak{q}_\alpha(dU)}{\sigma_\alpha(dU)}.^{(4)}$$

(1) The existence of the orthogonal system $\{\mathfrak{q}_\alpha(U)\}$ which is complete in the non-separable space $\mathfrak{H}$, can be proved as O. Teichmüller did. (Cf. Journal für die reine u. angew. Math., **174** (1935), 78.) But in this paper, in sec. 4, we prove the existence of $\{\mathfrak{q}_\alpha(U)\}$ which satisfies certain conditions.

(2) F. Maeda, this journal, **6** (1936), 123-127.

(3) F. Maeda, this journal, **4** (1934), 73.

(4) Here the summation takes place for all α such that $\xi_\alpha(U) \neq 0$. [=] means the strong convergence of the series.

Thus to any element $\mathfrak{f}$ in $\mathfrak{H}$ there corresponds a system of set functions $\{\xi_\alpha(U)\}$, such that

$$\xi_\alpha(U) \in \mathfrak{L}_2(\sigma_\alpha) \qquad \text{for all } \alpha \in \mathfrak{A},$$

and $\| \xi_\alpha \| = 0$ except at most enumerably infinite system of $\xi_\alpha(U)$, and $\sum_{\alpha \in \mathfrak{A}} \| \xi_\alpha \|^2$ converges to a finite value. Now for simplicity's sake, denote such a system of set functions $\{\xi_\alpha(U)\}$ which satisfy above conditions, by ξ, and the aggregate of them by $\mathfrak{L}_2(\sigma)$. And in $\mathfrak{L}_2(\sigma)$, define the sum and the inner product as follows:

$$a\xi + b\eta = \{a\xi_\alpha(U) + b\eta_\alpha(U)\} \qquad \text{and} \qquad (\xi, \eta) = \sum_{\alpha \in \mathfrak{A}} (\xi_\alpha, \eta_\alpha),$$

where $$\xi = \{\xi_\alpha(U)\}, \qquad \eta = \{\eta_\alpha(U)\}.$$

Next, take any element $\{\xi_\alpha(U)\}$ in $\mathfrak{L}_2(\sigma)$. And put $\mathfrak{f}_\alpha = \int_V \frac{\xi_\alpha(dU)\,\mathfrak{q}_\alpha(dU)}{\sigma_\alpha(dU)}$. Then, since $(\mathfrak{f}_\alpha, \mathfrak{f}_\beta) = 0$ when $\alpha \neq \beta$, and $\sum_{\alpha \in \mathfrak{A}} \| \mathfrak{f}_\alpha \|^2 = \sum_{\alpha \in \mathfrak{A}} \| \xi_\alpha \|^2$ converges, $\sum_{\alpha \in \mathfrak{A}} \mathfrak{f}_\alpha$ expresses an element in $\mathfrak{H}$.

Thus, there exists a one-to-one correspondence between $\mathfrak{H}$ and $\mathfrak{L}_2(\sigma)$.

And since $$(\mathfrak{f}, \mathfrak{g}) = \sum_{\alpha \in \mathfrak{A}} (\xi_\alpha, \eta_\alpha) = (\xi, \eta)$$

where $$\mathfrak{f} [=] \sum_{\alpha \in \mathfrak{A}} \int_V \frac{\xi_\alpha(dU)\,\mathfrak{q}_\alpha(dU)}{\sigma_\alpha(dU)}, \qquad \mathfrak{g} [=] \sum_{\alpha \in \mathfrak{A}} \int_V \frac{\eta_\alpha(dU)\,\mathfrak{q}_\alpha(dU)}{\sigma_\alpha(dU)},$$

this correspondence is isomorph. Therefore we can say that $\xi = \{\xi_\alpha(U)\}$ is the representative of $\mathfrak{f}$.

Let $\boldsymbol{T}$ be a linear operator in $\mathfrak{H}$ which has its adjoint $\boldsymbol{T}^*$, and their domains $\mathfrak{D}$ and $\mathfrak{D}^*$ contain $\mathfrak{M}'(\mathfrak{q}_\alpha)$ for all $\alpha \in \mathfrak{A}$. Put

$$\mathfrak{K}_{\alpha\beta}(U, U') = \big(\boldsymbol{T}\mathfrak{q}_\beta(U'), \mathfrak{q}_\alpha(U)\big), \qquad \mathfrak{K}^*_{\alpha\beta}(U, U') = \big(\boldsymbol{T}^*\mathfrak{q}_\beta(U'), \mathfrak{q}_\alpha(U)\big).$$

Since $\{\mathfrak{K}_{\alpha\beta}(U, U')\}$ $(\alpha \in \mathfrak{A})$ is the representative of $\boldsymbol{T}\mathfrak{q}_\beta(U')$, $\mathfrak{K}_{\alpha\beta}(U, U')$ vanishes except at most enumerably infinite system of α. Similarly for β. Let $\{\xi_\alpha(U)\}$ and $\{\eta_\alpha(U)\}$ be the representatives of $\mathfrak{f}\ (\in \mathfrak{D})$ and $\boldsymbol{T}\mathfrak{f}$ respectively. Then

$$\eta_\alpha(U) = \big(\boldsymbol{T}\mathfrak{f}, \mathfrak{q}_\alpha(U)\big) = \big(\mathfrak{f}, \boldsymbol{T}^*\mathfrak{q}_\alpha(U)\big) = \sum_{\beta \in \mathfrak{A}} \int_V \frac{\xi_\beta(dU')\,\overline{\mathfrak{K}^*_{\beta\alpha}(dU', U)}}{\sigma_\beta(dU')},$$

since $\{\mathfrak{K}^*_{\beta\alpha}(U', U)\}$ $(\beta \in \mathfrak{A})$ is the representative of $\boldsymbol{T}^* \mathfrak{q}_\alpha(U)$. Thus we have

$$\eta_\alpha(U) = \sum_{\beta \in \mathfrak{A}} \mathfrak{K}_{\alpha\beta} \xi_\beta(U) \,. \tag{2.1}$$

Hence we may call $\{\mathfrak{K}_{\alpha\beta}(U, U')\}$ the representative of $\boldsymbol{T}$.

Let $\xi = \{\xi_\alpha(U)\}$ be an element in $\mathfrak{L}_2(\sigma)$ such that $\{\eta_\alpha(U)\}$, obtained by (2.1), belongs to $\mathfrak{L}_2(\sigma)$. Then (2.1) expresses an operator in $\mathfrak{L}_2(\sigma)$ which transforms $\xi = \{\xi_\alpha(U)\}$ to $\eta = \{\eta_\alpha(U)\}$. We denote this operator by $\boldsymbol{T}_{\mathfrak{K}}$, its domain by $\mathfrak{D}_{\mathfrak{K}}$, such that

$$\eta = \boldsymbol{T}_{\mathfrak{K}} \xi \qquad (\xi \in \mathfrak{D}_{\mathfrak{K}}) \,.$$

Then the fact that $\{\mathfrak{K}_{\alpha\beta}(U, U')\}$ is the representative of $\boldsymbol{T}$ means that

$$\boldsymbol{T} \subseteqq \boldsymbol{T}_{\mathfrak{K}} \,. \tag{2.2}$$

Similarly we have $\boldsymbol{T}^* \subseteqq \boldsymbol{T}_{\mathfrak{K}^*} \,.$

Domains of Diagonal Representatives.

3. Let $\mathfrak{R}$ be a closed family (σ-Körper) of sets in an abstract space V, which contains V itself. Let $\boldsymbol{E}(U)$ be a resolution of identity which is defined for all sets in $\mathfrak{R}$.(1)

Let $f(\lambda)$ be a complex valued point function defined in V, which is measurable ($\mathfrak{R}$).(2) Define a linear operator in $\mathfrak{H}$ by

$$\boldsymbol{T}_f \mathfrak{f} = \int_V f(\lambda) \boldsymbol{E}(dU) \mathfrak{f} \,.$$

The domain $\mathfrak{D}(f)$ of $\boldsymbol{T}_f$ is the set of elements $\mathfrak{f}$ in $\mathfrak{H}$ such that $\int_V |f(\lambda)|^2 \|\boldsymbol{E}(dU)\mathfrak{f}\|^2$ is finite. $\mathfrak{D}(f)$ is a linear manifold which is

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(1) Cf. F. Maeda, this journal, **6** (1936), 38, where $\mathfrak{R}$ is a multiplicative system. But here $\mathfrak{R}$ is restricted to be a closed family. Hence we can use the results in the paper; F. Maeda, this journal, **4** (1934), 85–86.

(2) A real valued point function $f(\lambda)$ is said to be measurable ($\mathfrak{R}$), when $V[f(\lambda) > r]$ belongs to $\mathfrak{R}$ for any r. A complex valued point function is measurable ($\mathfrak{R}$) when its real parts and imaginary part are measurable ($\mathfrak{R}$).

dense in $\mathfrak{H}$.[1] We define also $T_{\bar{f}}$ by

$$T_{\bar{f}}\mathfrak{f} = \int_V \overline{f(\lambda)} E(dU)\mathfrak{f}.$$

Then the domain $\mathfrak{D}(\bar{f})$ of $T_{\bar{f}}$ is identical to $\mathfrak{D}(f)$.

We have the relation

$$T_f^* = T_{\bar{f}},^{(2)} \tag{3.1}$$

and T_f is a normal operator.[3] From (3.1), T_f is closed.

It is already known that T_f and $E(U)$ are permutable. That is

$$E(U)T_f \subseteqq T_f E(U).^{(4)}$$

Now we have the following theorem:

Let T be an operator such that

$$T_f \subseteqq T. \tag{3.2}$$

If

$$E(U)T \subseteqq TE(U), \tag{3.3}$$

then

$$T = T_f.$$

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(1) F. Maeda, this journal, **4** (1934), 86.

(2) Cf. M. H. Stone, *Linear Transformations in Hilbert Space*, (1932), 229. Since $E(U)$ is defined for all sets U in $\mathfrak{R}$, we can prove this relation by the following simple method: Denote by $\mathfrak{D}^*$ the domain of T_f^*. When $\mathfrak{g} \in \mathfrak{D}(\bar{f})$, then

$$(T_f\mathfrak{f}, \mathfrak{g}) = \int_V f(\lambda)\big(E(dU)\mathfrak{f}, \mathfrak{g}\big) = (\mathfrak{f}, T_{\bar{f}}\mathfrak{g}) \qquad \text{for all } \mathfrak{f} \in \mathfrak{D}(f).$$

Hence $T_{\bar{f}} \subseteqq T_f^*$. Of course $T_{\bar{f}} = T_f^*$ when $|f(\lambda)|$ is bounded. Next put $V_N = V[|f(\lambda)| < N]$. Since $T_f E(V_N)\mathfrak{f} = \int_{V_N} f(\lambda)E(dU)\mathfrak{f}$, we have $[T_f E(V_N)]^*\mathfrak{f} = \int_{V_N} \overline{f(\lambda)} E(dU)\mathfrak{f}$. But $E(V_N)T_f^* \subseteqq [T_f E(V_N)]^*$. Hence we have

$$E(V_N)T_f^*\mathfrak{f} = \int_{V_N} \overline{f(\lambda)} E(dU)\mathfrak{f} \qquad \text{when} \qquad \mathfrak{f} \in \mathfrak{D}^*.$$

Let $N \to \infty$, then we have $T_f^* \subseteqq T_{\bar{f}}$. Consequently $T_f^* = T_{\bar{f}}$.

(3) Cf. M. H. Stone, ibid., 312. Conversely, any normal operator T is expressed as follows: $T\mathfrak{f} = \int_{R_2} zE(dU)\mathfrak{f}$, where $E(U)$ is the resolution of identity defined in the space R_2 of complex numbers. Cf. F. Maeda, this journal, **4** (1934), 91.

(4) F. Maeda, this journal, **4** (1934), 88.

For the proof, put $V[|f(\lambda)|<N] = V_N$, N being any positive integer. Let $\mathfrak{f}$ be any element in the domain of $\boldsymbol{T}$. Then, since

$$\int_V |f(\lambda)|^2 \| \boldsymbol{E}(dU)\boldsymbol{E}(V_N)\mathfrak{f}\|^2 = \int_{V_N} |f(\lambda)|^2 \|\boldsymbol{E}(dU)\mathfrak{f}\|^2 \leqq N^2 \|\mathfrak{f}\|^2,$$

we have $\boldsymbol{E}(V_N)\mathfrak{f} \in \mathfrak{D}(f)$. Then from (3.3)

$$\boldsymbol{E}(V_N)\boldsymbol{T}\mathfrak{f} = \boldsymbol{T}\boldsymbol{E}(V_N)\mathfrak{f} = \boldsymbol{T}_f\boldsymbol{E}(V_N)\,\mathfrak{f}\,.$$

Since $\quad \underset{N\to\infty}{[\lim]}\,\boldsymbol{E}(V_N)\boldsymbol{T}\mathfrak{f} = \boldsymbol{T}\mathfrak{f}$,[1] $\quad \underset{N\to\infty}{[\lim]}\,\boldsymbol{E}(V_N)\mathfrak{f} = \mathfrak{f}$,

and $\boldsymbol{T}_f$ is closed, we have $\mathfrak{f} \in \mathfrak{D}(f)$ and $\boldsymbol{T} \subseteqq \boldsymbol{T}_f$. Hence from (3.2) $\boldsymbol{T} = \boldsymbol{T}_f$.

4. Now we find the basis of the representation which is contained in $\mathfrak{D}(f)$. For this purpose we prove the following theorem:

We can find an aggregate $\mathfrak{P}$ of elements in $\mathfrak{D}(f)$, such that

$$\{\mathfrak{q}_\mathfrak{b}(U)\} \qquad (\mathfrak{b} \in \mathfrak{P})$$

is an orthogonal system which is complete in $\mathfrak{H}$, where

$$\mathfrak{q}_\mathfrak{b}(U) = \boldsymbol{E}(U)\mathfrak{b}\,.$$

Give to $\mathfrak{D}(f)$ a normal order-type (Wohlordnung). We find the elements of $\mathfrak{P}$ by transfinite induction as follows: $\mathfrak{b}$ belongs to $\mathfrak{P}$ when and only when $\|\mathfrak{b}\| > 0$ and $\mathfrak{b}$ is orthogonal to $\mathfrak{M}(\mathfrak{q}_\mathfrak{a})$ for all $\mathfrak{a}$ $(\in \mathfrak{P})$ which have lower ranks than $\mathfrak{b}$ in the normal order-type. Since

$$\big(\boldsymbol{E}(U')\mathfrak{a}, \boldsymbol{E}(U)\mathfrak{b}\big) = \big(\boldsymbol{E}(UU')\mathfrak{a}, \mathfrak{b}\big) = 0\,,$$

it is evident that $\{\mathfrak{q}_\mathfrak{b}(U)\}$ is an orthogonal system.

Let $\mathfrak{f}$ be any element in $\mathfrak{D}(f)$. And denote by $\mathfrak{f}_\mathfrak{b}$ the component of $\mathfrak{f}$ in $\mathfrak{M}(\mathfrak{q}_\mathfrak{b})$ and by $\mathfrak{f}'$ the component of $\mathfrak{f}$ orthogonal to $\mathfrak{M}(\mathfrak{q}_\mathfrak{b})$. Then

$$\boldsymbol{E}(U)\mathfrak{f} = \boldsymbol{E}(U)\mathfrak{f}_\mathfrak{b} + \boldsymbol{E}(U)\mathfrak{f}'.$$

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(1) [lim] means the strong convergence.

Since $\boldsymbol{E}(U)\mathfrak{f}_{\mathfrak{b}}$ belongs to $\mathfrak{M}(\mathfrak{q}_{\mathfrak{b}})$,(1)

$$\big(\boldsymbol{E}(U)\mathfrak{f}_{\mathfrak{b}}, \boldsymbol{E}(U)\mathfrak{f}'\big) = \big(\boldsymbol{E}(U)\mathfrak{f}_{\mathfrak{b}}, \mathfrak{f}'\big) = 0$$

for all U. Hence

$$\| \boldsymbol{E}(U)\mathfrak{f} \|^2 = \| \boldsymbol{E}(U)\mathfrak{f}_{\mathfrak{b}} \|^2 + \| \boldsymbol{E}(U)\mathfrak{f}' \|^2.$$

But $\int_V |f(\lambda)|^2 \| \boldsymbol{E}(dU)\mathfrak{f} \|^2$ is finite, therefore $\int_V |f(\lambda)|^2 \| \boldsymbol{E}(dU)\mathfrak{f}_{\mathfrak{b}} \|^2$ is finite. That is $\mathfrak{f}_{\mathfrak{b}}$ belongs to $\mathfrak{D}(f)$.

For $\mathfrak{f} \in \mathfrak{D}(f)$, there exists at most enumerably infinite system of $\mathfrak{b}$ in $\mathfrak{P}$ such that $\mathfrak{f}_{\mathfrak{b}}$ are non-null. Let us denote them by $\mathfrak{f}_{\mathfrak{b}_1}, \mathfrak{f}_{\mathfrak{b}_2}, \dots,$ $\mathfrak{f}_{\mathfrak{b}_i}, \dots$. Since

$$\sum_{i=1}^{\nu} \| \boldsymbol{E}(U)\mathfrak{f}_{\mathfrak{b}_i} \|^2 \leqq \| \boldsymbol{E}(U)\mathfrak{f} \|^2,$$

$$\sum_{i=1}^{\nu} \int_V |f(\lambda)|^2 \| \boldsymbol{E}(dU)\mathfrak{f}_{\mathfrak{b}_i} \|^2 \leqq \int_V |f(\lambda)|^2 \| \boldsymbol{E}(dU)\mathfrak{f} \|^2.$$

Hence $\sum_{i=1}^{\nu} \int_V \mathfrak{f}(\lambda)\boldsymbol{E}(dU)\mathfrak{f}_{\mathfrak{b}_i} = \boldsymbol{T}_f \sum_{i=1}^{\nu} \mathfrak{f}_{\mathfrak{b}_i}$ converges strongly when $\nu \to \infty$.(2) Then, since $\boldsymbol{T}_f$ is closed, $\sum_{i=1}^{\infty} \mathfrak{f}_{\mathfrak{b}_i}$ belongs to $\mathfrak{D}(f)$. Hence if we put

$$\mathfrak{f} = \sum_{i=1}^{\infty} \mathfrak{f}_{\mathfrak{b}_i} + \mathfrak{f}'',$$

then $\mathfrak{f}''$ belongs to $\mathfrak{D}(f)$, and $\mathfrak{f}''$ is orthogonal to all $\mathfrak{M}(\mathfrak{q}_{\mathfrak{b}})$ $(\mathfrak{b} \in \mathfrak{P})$, which is absurd unless $\mathfrak{f}'' = 0$. Consequently

$$\mathfrak{f} [=] \sum_{\mathfrak{b} \in \mathfrak{P}} \mathfrak{f}_{\mathfrak{b}} \tag{4.1}$$

for any $\mathfrak{f} \in \mathfrak{D}(f)$.

Now $\{\mathfrak{q}_{\mathfrak{b}}(U)\}$ is complete in $\mathfrak{H}$. If not, there exists a non-null element $\mathfrak{g}$ which is orthogonal to all $\mathfrak{M}(\mathfrak{q}_{\mathfrak{b}})$ $(\mathfrak{b} \in \mathfrak{P})$. From (4.1) $\mathfrak{g}$ must be orthogonal to all $\mathfrak{f} \in \mathfrak{D}(f)$, which is absurd since $\mathfrak{D}(f)$ is dense in $\mathfrak{H}$.

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(1) For $\boldsymbol{E}(U)\mathfrak{f}_{\mathfrak{b}} = \int_U \frac{\xi_{\mathfrak{b}}(U)\,\mathfrak{q}_{\mathfrak{b}}(U)}{\sigma_{\mathfrak{b}}(U)}$, where $\xi_{\mathfrak{b}}(U) = \big(\mathfrak{f}, \mathfrak{q}_{\mathfrak{b}}(U)\big)$. (Cf. F. Maeda, this journal, **4** (1934), 71).

(2) Since $\int_V f(\lambda)\boldsymbol{E}(dU)f_{\mathfrak{b}_i} \in \mathfrak{M}(\mathfrak{q}_{\mathfrak{b}_i})$ $(i = 1, 2, \dots)$.

Thus we have proved this theorem.

Now label all the elements of $\mathfrak{P}$ by (finite or transfinite) ordinal numbers α, so that any element of $\mathfrak{P}$ is expressed by $\mathfrak{f}_\alpha$ ($\alpha \in \mathfrak{A}$), where $\mathfrak{A}$ is the aggregate of the suffices. Since $\mathfrak{f}_\alpha$ belongs to $\mathfrak{D}(f)$, $\mathfrak{q}_\alpha(U) = \boldsymbol{E}(U)\mathfrak{f}_\alpha$ belongs to $\mathfrak{D}(f)$ for all U. For

$$\int_V |f(\lambda)|^2 \| \boldsymbol{E}(dU)\mathfrak{q}_\alpha(U)\|^2 = \int_U |f(\lambda)|^2 \| \boldsymbol{E}(dU)\mathfrak{f}_\alpha\|^2$$

is finite. Therefore $\mathfrak{D}(f)$ contains all $\mathfrak{M}'(\mathfrak{q}_\alpha)$ ($\alpha \in \mathfrak{A}$). Since, from (3.1) $\boldsymbol{T}_f^* = \boldsymbol{T}_{\bar{f}}$, and $\mathfrak{D}(\bar{f}) = \mathfrak{D}(f)$, the domain of $\boldsymbol{T}_f^*$ contains also all $\mathfrak{M}'(\mathfrak{q}_\alpha)$ ($\alpha \in \mathfrak{A}$). Hence *we can use* $\{\mathfrak{q}_\alpha(U)\}$ *as the basis of the representation of* $\boldsymbol{T}_f$.

5. *In the diagonal representative of* $\boldsymbol{T}_f$, *we have*

$$\boldsymbol{E}(U)\boldsymbol{T}_{\mathfrak{K}} \subseteqq \boldsymbol{T}_{\mathfrak{K}}\boldsymbol{E}(U).$$

First consider the case where $\{\mathfrak{q}_\alpha(U)\}$ obtained in the preceding section, is composed of only one completely additive vector valued differential set function $\mathfrak{q}(U)$. And use $\mathfrak{q}(U)$ as the basis of the representation. Then by sec. 1, $\mathfrak{H}$ is represented by $\mathfrak{L}_2(\sigma)$, and since the domains of $\boldsymbol{T}_f$ and $\boldsymbol{T}_f^*$ contain $\mathfrak{M}'(\mathfrak{q})$, the representatives of $\boldsymbol{T}_f$ and $\boldsymbol{T}_f^*$ are

$$\mathfrak{K}(E, E') = \big(\boldsymbol{T}_f\mathfrak{q}(E'), \mathfrak{q}(E)\big) = \int_{EE'} f(\lambda)\,\sigma(dU), \tag{5.1}$$

$$\mathfrak{K}^*(E, E') = \big(\boldsymbol{T}_f^*\mathfrak{q}(E'), \mathfrak{q}(E)\big) = \big(\boldsymbol{T}_{\bar{f}}\,\mathfrak{q}(E'), \mathfrak{q}(E)\big) = \int_{EE'} \overline{f(\lambda)}\,\sigma(dU)$$

respectively. Thus we have diagonal representatives.

Let $\mathfrak{S}_U(E, E')$ be the representative of $\boldsymbol{E}(U)$. That is

$$\mathfrak{S}_U(E, E') = \big(\boldsymbol{E}(U)\mathfrak{q}(E'), \mathfrak{q}(E)\big) = \big(\mathfrak{q}(E'U), \mathfrak{q}(E)\big) = \sigma(EE'U). \tag{5.2}$$

Of course
$$\boldsymbol{E}(U) = \boldsymbol{T}_{\mathfrak{S}_U}.$$

When $\mathfrak{x}(E) \in \mathfrak{D}_{\mathfrak{K}}$, we have by (5.2)

$$\boldsymbol{T}_{\mathfrak{S}_U}\boldsymbol{T}_{\mathfrak{K}}\mathfrak{x}(E) = \boldsymbol{T}_{\mathfrak{S}_U}\mathfrak{K}\mathfrak{x}(E) = \mathfrak{K}\mathfrak{x}(EU). \tag{5.3}$$

And $$T_{\mathfrak{K}} T_{\mathfrak{S}_U} \xi(E) = T_{\mathfrak{K}} \xi(EU) = \int_V \frac{\mathfrak{K}(E, dE')\xi(dE' \cdot U)}{\sigma(dE')}$$

$$= \int_V \frac{\mathfrak{K}(E, dE' \cdot U)\xi(dE')}{\sigma(dE')}.$$

But by (5.1) $$\mathfrak{K}(E, E'U) = \mathfrak{K}(EU, E').$$

Hence $$T_{\mathfrak{K}} T_{\mathfrak{S}_U} \xi(E) = \int_V \frac{\mathfrak{K}(EU, dE')\xi(dE')}{\sigma(dE')} = \mathfrak{K}\xi(EU).$$

Therefore, we have by (5.3)

$$E(U) T_{\mathfrak{K}} \xi(E) = T_{\mathfrak{K}} E(U) \xi(E)$$

for all $\xi(E) \in \mathfrak{D}_{\mathfrak{K}}$. Hence

$$E(U) T_{\mathfrak{K}} \subseteqq T_{\mathfrak{K}} E(U).$$

Next consider the case where $\{\mathfrak{q}_\alpha(U)\}$ is composed of more than one completely additive vector valued differential set functions. Then by sec. 2, any element $\mathfrak{f}$ in $\mathfrak{H}$ is represented by $\xi = \{\xi_\alpha(U)\}$ in $\mathfrak{L}_2(\sigma)$, and the representatives of T_f and T_f^* are $\{\mathfrak{K}_{\alpha\beta}(E, E')\}$ and $\{\mathfrak{K}^*_{\alpha\beta}(E, E')\}$ respectively, where

$$\mathfrak{K}_{\alpha\beta}(E, E') = \big(T_f \mathfrak{q}_\beta(E'), \mathfrak{q}_\alpha(E)\big) = \int_V f(\lambda)\big(\mathfrak{q}_\beta(E' dU), \mathfrak{q}_\alpha(E)\big)$$

$$= \begin{cases} \int_{EE'} f(\lambda)\sigma_\alpha(dU) & \text{when } \alpha = \beta \\ 0 & \text{when } \alpha \neq \beta, \end{cases} \tag{5.4}$$

$$\mathfrak{K}^*_{\alpha\beta}(E, E') = \big(T_f^* \mathfrak{q}_\beta(E'), \mathfrak{q}_\alpha(E)\big) = \big(T_{\bar{f}} \mathfrak{q}_\beta(E'), \mathfrak{q}_\alpha(E)\big)$$

$$= \begin{cases} \int_{EE'} \overline{f(\lambda)}\sigma_\alpha(dU) & \text{when } \alpha = \beta \\ 0 & \text{when } \alpha \neq \beta. \end{cases}$$

Thus we have diagonal representatives.

Let $\{\mathfrak{S}_{U,\alpha\beta}(E, E')\}$ be the representative of $E(U)$. That is

$$\mathfrak{S}_{U,\alpha\beta}(E, E') = \big(E(U)\,\mathfrak{q}_\beta(E'), \mathfrak{q}_\alpha(E)\big) = \big(\mathfrak{q}_\beta(E'U), \mathfrak{q}_\alpha(E)\big)$$

$$= \begin{cases} \sigma_\alpha(EE'U) & \text{when } \alpha = \beta \\ 0 & \text{when } \alpha \neq \beta. \end{cases} \tag{5.5}$$

Of course
$$E(U) = T_{\mathfrak{S}_U}.$$

When $\xi \in \mathfrak{D}_{\mathfrak{K}}$, since by (5.4)

$$\sum_{\gamma \in \mathfrak{A}} \mathfrak{K}_{\beta\gamma}\xi_\gamma(E) = \mathfrak{K}_{\beta\beta}\xi_\beta(E),$$

we have by (5.5)

$$\sum_{\beta \in \mathfrak{A}} \mathfrak{S}_{U,\alpha\beta}\,\{\sum_{\gamma \in \mathfrak{A}} \mathfrak{K}_{\beta\gamma}\xi_\gamma(E)\} = \sum_{\beta \in \mathfrak{A}} \mathfrak{S}_{U,\alpha\beta}\mathfrak{K}_{\beta\beta}\xi_\beta(E) = \mathfrak{K}_{\alpha\alpha}\xi_\alpha(EU). \tag{5.6}$$

And since by (5.5)

$$\sum_{\gamma \in \mathfrak{A}} \mathfrak{S}_{U,\beta\gamma}\xi_\gamma(E) = \xi_\beta(EU),$$

we have by (5.4)

$$\sum_{\beta \in \mathfrak{A}} \mathfrak{K}_{\alpha\beta}\,\{\sum_{\gamma \in \mathfrak{A}} \mathfrak{S}_{U,\beta\gamma}\xi_\gamma(E)\} = \int_U \frac{\mathfrak{K}_{\alpha\alpha}(E, dE')\,\xi_\alpha(dE')}{\sigma_\alpha(dE')}$$

$$= \int_V \frac{\mathfrak{K}_{\alpha\alpha}(EU, dE')\,\xi_\alpha(dE')}{\sigma_\alpha(dE')}^{(1)} = \mathfrak{K}_{\alpha\alpha}\xi_\alpha(EU). \tag{5.7}$$

Hence we have from (5.6) and (5.7)

$$E(U)T_{\mathfrak{K}}\xi = T_{\mathfrak{K}}E(U)\xi$$

for all $\xi \in \mathfrak{D}_{\mathfrak{K}}$. That is

$$E(U)\,T_{\mathfrak{K}} \subseteqq T_{\mathfrak{K}}E(U).$$

6. Now we have the conclusion:
In the diagonal representative of T_f, *we have*

$$T = T_{\mathfrak{K}} \qquad \textit{and} \qquad T_f^* = T_{\mathfrak{K}^*}.$$

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(1) For from (5.4) $\mathfrak{K}_{\alpha\alpha}(EU, E') = \mathfrak{K}_{\alpha\alpha}(E, EU')$.

From (1.2) and (2.2) we have $T_f \subseteqq T_{\mathfrak{R}}$. Hence from the theorems of sec. 3 and 5, we have

$$T_f = T_{\mathfrak{R}} .$$

Similarly we have $T_{\bar{f}} = T_{\mathfrak{R}^*}$. But $T_f^* = T_{\bar{f}}$. Hence we have

$$T_f^* = T_{\mathfrak{R}^*} .$$