Wave Geometry No. 34

Spinor Calculus II.

By

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§ 1. Introduction.

In Spinor Calculus,(1) published in August 1938, we constructed from spinor $\psi \equiv \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$ the quantities (vectors and tensors) invariant by spin transformations, and investigated their properties. In this paper we shall study other properties of spinors.

In part I, corresponding to the fact that any 4–4 matrix can be expanded in terms of sedenion, we consider the expansion of any 1–4 matrix in terms of a certain definite set of bases, and by using this consideration, we shall obtain some relations between ψ and the spin-invariant quantities.

In part II, introducing the covariant differentiation of spinor, we shall find the relations among the spin-invariant quantities when ψ is the solution of the fundamental equation for ψ.

Part I.

§ 2. Preparatory statements.

As for the 4–4 matrices γ_i satisfying $\gamma_{(i}\gamma_{j)} = g_{ij}$ $(i, j = 1, \ldots, 4)$, we obtained the following :(2)

Theorem 1. *The matrix A is determined uniquely, except for a real factor, such that*

$$A\gamma_i = (A\gamma_i)^\dagger \equiv \gamma_i^\dagger A^\dagger , \tag{2.1}$$

$$A^\dagger = A , \qquad |A| \neq 0 , \tag{2.2}$$

(1) T. Sibata; This Journal, **8** (1938), (W. G. No. 26), 169.

(2) loc. cit., 170.

where $\gamma_i^\dagger$ *and* $A^\dagger$ *denote the conjugate transposed matrices of* γ_i *and* A *respectively.*

In addition to γ_i $(i=1,\dots,4)$ if we take the fifth matrix γ_5 so that

$$\gamma_i\gamma_5+\gamma_5\gamma_i=0 \qquad (i=1,\dots,4) \tag{2.3}$$

$$\gamma_5\gamma_5=\eta, \tag{2.4}$$

where $\eta=1$ or $=-1$ according as $|g_{ij}|>0$ or <0, we have

$$\gamma_{(\lambda}\gamma_{\mu)}=g_{\lambda\mu}, \qquad |g_{\lambda\mu}|>0 \tag{2.5}$$

and

$$A\gamma_\lambda=(A\gamma_\lambda)^\dagger \qquad (\lambda,\mu=1,\dots,5) \tag{2.6}$$

(In general relativity, since $|g_{ij}|<0$, $\eta=-1$).[1]

By any spin-transformation:

$$\psi'=S^{-1}\psi, \qquad \gamma_i'=S^{-1}\gamma_i S, \tag{2.7}$$

A is transformed as

$$A'=S^\dagger A S. \tag{2.8}$$

Because of (2.7) and (2.8), the real quantities

$$M\equiv\psi^\dagger A\psi, \quad N\equiv\psi^\dagger A\gamma_5\psi, \quad u^j\equiv\psi^\dagger A\gamma^j\psi$$

are invariant by the spin-transformations.

Theorem 2.[2] *Except for a factor, there exists a matrix* C *uniquely such that*

$$\tilde{\gamma}_i=-C\gamma_i C^{-1} \qquad (i=1,\dots,4) \tag{2.9}$$

$$\tilde{\gamma}_5=C\gamma_5 C^{-1} \tag{2.10}$$

where $\tilde{\gamma}_i$ *is the transposed matrix of* γ_i; *and the matrices*

$$C, \quad C\gamma_5, \quad C\gamma_i\gamma_5 \quad (i=1,\dots,4) \tag{2.11}$$

form the bases of any antisymmetric matrix, and

$$C\gamma_i, \quad C\gamma_{[i}\gamma_{j]}\equiv\frac{1}{2}C(\gamma_i\gamma_j-\gamma_j\gamma_i) \quad (i,j=1,\dots,4) \tag{2.12}$$

form the bases of any symmetric matrix.

Theorem 3.[3] *When 4–4 matrices* γ_λ $(\lambda=1,\dots,5)$ *satisfy the relations* $\gamma_{(\lambda}\gamma_{\mu)}=g_{\lambda\mu}$ $(\lambda,\mu=1,\dots,5)$, *there exists a matrix* D *such that*

(1) We take γ_5 so that $\eta=-1$ throughout this paper.
(2) loc. cit., 177.
(3) loc. cit., 181.

$$\bar{\gamma}_\lambda = \zeta D \gamma_\lambda D^{-1} \qquad (\lambda=1,\ldots\ldots,5),$$

where $\bar{\gamma}_\lambda$ *is the conjugate of* γ_λ *and* $\zeta=1$ *or* $=-1$ *according as* $|g_{\lambda\mu}|>0$ *or* <0.

By the spin-transformation (2.7), D is transformed as follows:

$$D' = \bar{S}^{-1} D S, \quad \text{or} \quad (D^{-1})' = S^{-1} D^{-1} \bar{S},$$

hence $D^{-1}\bar{\psi}$, say Ψ, undergoes the same transformation as ψ, namely

$$\Psi' \equiv D'^{-1} \bar{\psi}' = S^{-1} \Psi,$$

where bar means the conjugate of the corresponding quantities.

§ 3. Invariant bases of 1–4 matrix by spin-transformations.

From the theory of sedenion, we know that when 4–4 matrices γ_λ $(\lambda=1,\ldots\ldots,5)$ satisfy the relation $\gamma_{(\lambda}\gamma_{\mu)} = g_{\lambda\mu}$, and the determinant $|g_{\lambda\mu}|$ does not vanish identically, the 16 matrices

$$I, \quad \gamma_\lambda, \quad \gamma_{[\lambda}\gamma_{\mu]} \equiv \frac{1}{2}(\gamma_\lambda\gamma_\mu - \gamma_\mu\gamma_\lambda) \quad (\lambda, \mu=1,\ldots\ldots,5) \qquad (3.1)$$

form the bases of a hypercomplex number system, i. e. any 4–4 matrix is uniquely expressed by a linear combination of the matrices (3.1) in the form:

$$A + A^\lambda \gamma_\lambda + A^{\lambda\mu}\gamma_\lambda\gamma_\mu \qquad (A^{\lambda\mu} = -A^{\mu\lambda}), \qquad (3.2)$$

where A, A^λ, and $A^{\lambda\mu}$, are scalar, vector, and tensor, respectively. Corresponding to this, as for 1–4 matrix, we have

Theorem 4. *If* ψ *and* ϕ *are any 1–4 matrices* $\psi \equiv \begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}$, $\phi \equiv \begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\end{pmatrix}$, *satisfying*

$$(\tilde{\psi} C \phi)^2 + (\tilde{\psi} C \gamma_5 \phi)^2 \neq 0, \qquad (3.3)$$

where C *being a matrix defined by* (2.9), *then the following 4 matrices*

$$\psi, \quad \gamma_5\psi, \quad \phi, \quad \gamma_5\phi, \qquad (3.4)$$

form the bases of 1–4 matrix; i. e. any 1–4 matrix χ *is uniquely expressed by a linear combination of the matrices* (3.4) *in the form:*

$$\chi = p\psi + q\gamma_5\psi + r\phi + s\gamma_5\phi. \qquad (3.5)$$

Proof. To prove the theorem it is enough to show that the determinant of the matrix :[1]

$$P \equiv (\psi,\ \gamma_5\psi,\ \phi,\ \gamma_5\phi) \tag{3.6}$$

does not vanish identically.

The determinant of the matrix P is calculated as follows. From (3.6) we have

$$CP=(C\psi,\ C\gamma_5\psi,\ C\phi,\ C\gamma_5\phi) \quad \text{and} \quad \tilde{P}=\begin{pmatrix} \tilde{\psi} \\ \tilde{\psi}\tilde{\gamma}_5 \\ \tilde{\phi} \\ \tilde{\phi}\tilde{\gamma}_5 \end{pmatrix},$$

therefore

$$\tilde{P}CP=\begin{pmatrix} \tilde{\psi}C\psi & \tilde{\psi}C\gamma_5\psi & \tilde{\psi}C\phi & \tilde{\psi}C\gamma_5\phi \\ \tilde{\psi}\tilde{\gamma}_5C\psi & \tilde{\psi}\tilde{\gamma}_5C\gamma_5\psi & \tilde{\psi}\tilde{\gamma}_5C\phi & \tilde{\psi}\tilde{\gamma}_5C\gamma_5\phi \\ \tilde{\phi}C\psi & \tilde{\phi}C\gamma_5\psi & \tilde{\phi}C\phi & \tilde{\phi}C\gamma_5\phi \\ \tilde{\phi}\tilde{\gamma}_5C\psi & \tilde{\phi}\tilde{\gamma}_5C\gamma_5\psi & \tilde{\phi}\tilde{\gamma}_5C\phi & \tilde{\phi}\tilde{\gamma}_5C\gamma_5\phi \end{pmatrix}. \tag{3.7}$$

But, since C and $C\gamma_5$ are both antisymmetric matrices, we have

$$\tilde{\psi}C\psi=0, \qquad \tilde{\psi}C\gamma_5\psi=0,$$

$$\tilde{\phi}C\phi=0, \qquad \tilde{\phi}C\gamma_5\phi=0,$$

and, from (2.10), we have

$$\tilde{\psi}\tilde{\gamma}_5C\psi=\tilde{\psi}C\gamma_5\psi=0, \qquad \tilde{\psi}\tilde{\gamma}_5C\gamma_5\psi=-\tilde{\psi}C\psi=0;$$

similarly, $$\tilde{\phi}\tilde{\gamma}_5C\phi=0, \qquad \tilde{\phi}\tilde{\gamma}_5C\gamma_5\phi=0.$$

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(1) We define $(\psi,\ \gamma_5\psi,\ \phi,\ \gamma_5\phi)$ and $\begin{pmatrix} \tilde{\psi} \\ \tilde{\psi}\tilde{\gamma}_5 \\ \tilde{\phi} \\ \tilde{\phi}\tilde{\gamma}_5 \end{pmatrix}$ by

$$(\psi,\ \gamma_5\psi,\ \phi,\ \gamma_5\phi)=\begin{pmatrix} \psi_1 & (\gamma_5\psi)_1 & \phi_1 & (\gamma_5\phi)_1 \\ \psi_2 & (\gamma_5\psi)_2 & \phi_2 & (\gamma_5\phi)_2 \\ \psi_3 & (\gamma_5\psi)_3 & \phi_3 & (\gamma_5\phi)_3 \\ \psi_4 & (\gamma_5\psi)_4 & \phi_4 & (\gamma_5\phi)_4 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \tilde{\psi} \\ \tilde{\psi}\tilde{\gamma}_5 \\ \tilde{\phi} \\ \tilde{\phi}\tilde{\gamma}_5 \end{pmatrix}=\begin{pmatrix} \psi_1 & \psi_2 & \psi_3 & \psi_4 \\ (\tilde{\psi}\tilde{\gamma}_5)_1 & (\tilde{\psi}\tilde{\gamma}_5)_2 & (\tilde{\psi}\tilde{\gamma}_5)_3 & (\tilde{\psi}\tilde{\gamma}_5)_4 \\ \phi_1 & \phi_2 & \phi_3 & \phi_4 \\ (\tilde{\phi}\tilde{\gamma}_5)_1 & (\tilde{\phi}\tilde{\gamma}_5)_2 & (\tilde{\phi}\tilde{\gamma}_5)_3 & (\tilde{\phi}\tilde{\gamma}_5)_4 \end{pmatrix}.$$

Hence (3.7) becomes

$$\tilde{P}CP=\begin{pmatrix} 0 & 0 & \tilde{\psi}C\phi & \tilde{\psi}C\gamma_5\phi \\ 0 & 0 & \tilde{\psi}C\gamma_5\phi & -\tilde{\psi}C\phi \\ \tilde{\phi}C\psi & \tilde{\phi}C\gamma_5\psi & 0 & 0 \\ \tilde{\phi}C\gamma_5\psi & -\tilde{\phi}C\psi & 0 & 0 \end{pmatrix}.$$

Taking the determinant of the matrix above, we have

$$|P|\,|C|\,|P|=\{(\tilde{\phi}C\psi)^2+(\tilde{\phi}C\gamma_5\psi)^2\}\,\{(\tilde{\psi}C\phi)^2+(\tilde{\psi}C\gamma_5\phi)^2\}\,. \quad (3.8)$$

Further, since, by the antisymmetrical property of C and $C\gamma_5$,

$$\tilde{\phi}C\psi=-\tilde{\psi}C\phi\,, \qquad \tilde{\phi}C\gamma_5\psi=-\tilde{\psi}C\gamma_5\phi\,, \quad (3.9)$$

(3.8) becomes

$$|P|^2\,|C|=\{(\tilde{\psi}C\phi)^2+(\tilde{\psi}C\gamma_5\phi)^2\}^2.$$

But since $|C|\neq 0$, $|P|$ does not vanish identically providing that

$$(\tilde{\psi}C\phi)^2+(\tilde{\psi}C\gamma_5\phi)^2\neq 0\,, \quad \text{or} \quad (3.3). \qquad \text{Q. E. D.}$$

Next, we shall find the bases of 1–4 matrix constructed by only one given matrix, say ψ, instead of ψ and ϕ in theorem 4.

Theorem 5. *If ψ is any 1–4 matrix satisfying*

$$(\psi^\dagger A\psi)^2+(\psi^\dagger A\gamma_5\psi)^2\neq 0\,, \quad (3.10)$$

then 4 matrices

$$\psi\,,\quad \gamma_5\psi\,,\quad D^{-1}\bar{\psi}\,,\quad \gamma_5 D^{-1}\bar{\psi}\,, \quad (3.11)$$

form the bases of 1–4 matrix.

Proof. In theorem 4, if we take ϕ as $\phi=D^{-1}\bar{\psi}$, (3.4) becomes (3.11) and the left-hand side of (3.3) becomes

$$(\tilde{\psi}CD^{-1}\bar{\psi})^2+(\tilde{\psi}C\gamma_5 D^{-1}\bar{\psi})^2. \quad (3.12)$$

Therefore, (3.11) form the bases of 1–4 matrix so far as (3.12) does not vanish identically. But we can show that (3.12) is equivalent to the left-hand side of (3.10).[1] So that the theorem is proved.

N. B. From theorem 5, we know that when at least one of the quantities $M\equiv\psi^\dagger A\psi$ and $N\equiv\psi^\dagger A\gamma_5\psi$ does not vanish identically, any

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(1) See Note 1.

1–4 matrix is expressible by a linear combination of the matrices (3.11). But, in the case when both M and N vanish identically, (3.10) vanishes identically, and the theorem cannot be well applied. This case will be treated in another way in note 7 at the end of this paper.

§ 4. The expansions of $\gamma_l \Psi$ and $\gamma_{[l}\gamma_{m]}\Psi$ in terms of the matrices (3.11).

Theorem 6. *When*

$$M^2+N^2 \neq 0, \tag{4.1}$$

$\gamma_l\Psi$ *and* $\gamma_{[l}\gamma_{m]}\Psi$ *are expressible by a linear combination of the matrices* (3.11) *as follows:*

$$\left.\begin{aligned}\gamma_l\Psi=\frac{1}{M^2+N^2}\Big[&\{(Mu_l-Nu_{l5})+(Nu_l+Mu_{l5})\gamma_5\}\Psi\\&+\rho_l(N-M\gamma_5)D^{-1}\bar{\Psi}\Big],\end{aligned}\right\} \tag{4.2}$$

$$\left.\begin{aligned}\gamma_{[l}\gamma_{m]}\Psi=\frac{1}{M^2+N^2}\Big[&\{(Mu_{lm}+Nu_{lm5})-(Mu_{lm5}-Nu_{lm})\gamma_5\}\Psi\\&-\{(M\rho_{lm5}-N\rho_{lm})+(M\rho_{lm}+N\rho_{lm5})\gamma_5\}D^{-1}\bar{\Psi}\Big]\end{aligned}\right\} \tag{4.3}$$

where

$$u_l=\Psi^\dagger A\gamma_l\Psi,\quad u_{l5}=\Psi^\dagger A\gamma_l\gamma_5\Psi,\quad u_{lm}=\Psi^\dagger A\gamma_{[l}\gamma_{m]}\Psi,\quad u_{lm5}=\Psi^\dagger A\gamma_{[l}\gamma_{m]}\gamma_5\Psi,$$

$$\rho_l=\tilde{\Psi}C\gamma_l\Psi,\quad \rho_{lm}=\tilde{\Psi}C\gamma_{[l}\gamma_{m]}\Psi,\quad \rho_{lm5}=\tilde{\Psi}C\gamma_{[l}\gamma_{m]}\gamma_5\Psi.$$

Proof. According to theorem 5, $\gamma_l\Psi$ may be expressed by a linear combination of the matrices (3.11) in the form:

$$\gamma_l\Psi=c_l\Psi+d_l\gamma_5\Psi+e_lD^{-1}\bar{\Psi}+f_l\gamma_5D^{-1}\bar{\Psi}. \tag{4.4}$$

In order to determine the actual values of c_l, d_l, e_l, and f_l, we use the following identities:[1]

$$\left.\begin{aligned}&u^l\gamma_l\Psi=(M+N\gamma_5)\Psi, && u^l_{5}\gamma_l\Psi=(-N+M\gamma_5)\Psi,\\&s^l\gamma_l\Psi=(-N+M\gamma_5)D^{-1}\bar{\Psi}, && t^l\gamma_l\Psi=i(-N+M\gamma_5)D^{-1}\bar{\Psi},\end{aligned}\right\} \tag{4.5}$$

where s^l and t^l are real and imaginary parts of ρ^l, i. e.

$$\rho^l=s^l+it^l. \tag{4.6}$$

(1) loc. cit., 173, 174, and 183.

If we multiply (4.4) by u^l, $u^l_{\cdot 5}$, s^l, and t^l, respectively, and contract them by the suffix l, we have

$$(M+N\gamma_5)\psi=u^lc_l\psi+u^ld_l\gamma_5\psi+u^le_lD^{-1}\overline{\psi}+u^lf_l\gamma_5D^{-1}\overline{\psi},$$
$$(-N+M\gamma_5)\psi=u^l_{\cdot 5}c_l\psi+u^l_{\cdot 5}d_l\gamma_5\psi+u^l_{\cdot 5}e_lD^{-1}\overline{\psi}+u^l_{\cdot 5}f_l\gamma_5D^{-1}\overline{\psi},$$
$$(-N+M\gamma_5)D^{-1}\overline{\psi}=s^lc_l\psi+s^ld_l\gamma_5\psi+s^le_lD^{-1}\overline{\psi}+s^lf_l\gamma_5D^{-1}\overline{\psi},$$
$$i(-N+M\gamma_5)D^{-1}\overline{\psi}=t^lc_l\psi+t^ld_l\gamma_5\psi+t^le_lD^{-1}\overline{\psi}+t^lf_l\gamma_5D^{-1}\overline{\psi}.$$

Comparing the coefficients of the bases in the equations above, we have

$$\left.\begin{array}{llll} M=u^lc_l, & N=u^ld_l, & 0=u^le_l, & 0=u^lf_l, \\ -N=u^l_{\cdot 5}c_l, & M=u^l_{\cdot 5}d_l, & 0=u^l_{\cdot 5}e_l, & 0=u^l_{\cdot 5}f_l, \\ 0=s^lc_l, & 0=s^ld_l, & -N=s^le_l, & M=s^lf_l, \\ 0=t^lc_l, & 0=t^ld_l, & -iN=t^le_l, & iM=t^lf_l, \end{array}\right\} \quad (4.7)$$

But, on the other hand, if we put

$$\overset{1}{\lambda}{}^l=\frac{u^l}{\sqrt{M^2+N^2}},\quad \overset{2}{\lambda}{}^l=\frac{u^l_{\cdot 5}}{\sqrt{M^2+N^2}},\quad \overset{3}{\lambda}{}^l=\frac{is^l}{\sqrt{M^2+N^2}},\quad \overset{4}{\lambda}{}^l=\frac{it^l}{\sqrt{M^2+N^2}}, \quad (4.8)$$

$\overset{a}{\lambda}{}^l(a=1,\dots\dots,4)$ satisfy the relations[1]

$$\overset{a}{\lambda}_l\overset{b}{\lambda}{}^l=\delta^{ab}\qquad (a,b=1,\dots\dots,4). \quad (4.9)$$

Using (4.9), and solving c_l, d_l, e_l, and f_l from (4.7), we have

$$\left.\begin{array}{ll} c_l=\dfrac{1}{\sqrt{M^2+N^2}}(M\overset{1}{\lambda}_l-N\overset{2}{\lambda}_l), & d_l=\dfrac{1}{\sqrt{M^2+N^2}}(N\overset{1}{\lambda}_l+M\overset{2}{\lambda}_l) \\ e_l=\dfrac{N}{\sqrt{M^2+N^2}}(-i\overset{3}{\lambda}_l+\overset{4}{\lambda}_l), & f_l=\dfrac{M}{\sqrt{M^2+N_2}}(i\overset{3}{\lambda}_l-\overset{4}{\lambda}_l). \end{array}\right\} \quad (4.10)$$

Substituting (4.10) into (4.4), we have

$$\left.\begin{array}{l} \gamma_l\psi=\dfrac{1}{\sqrt{M^2+N^2}}\Big[\{(M\overset{1}{\lambda}_l-N\overset{2}{\lambda}_l)+(N\overset{1}{\lambda}_l+M\overset{2}{\lambda}_l)\gamma_5\}\psi \\ \qquad\qquad +\{N(-i\overset{3}{\lambda}_l+\overset{4}{\lambda}_l)+M(i\overset{3}{\lambda}_l-\overset{4}{\lambda}_l)\gamma_5\}D^{-1}\overline{\psi}\Big], \end{array}\right\} \quad (4.11)$$

or, by (4.8) and (4.6),

(1) loc. cit., 185.

$$\left.\begin{aligned}\gamma_l\psi=\frac{1}{M^2+N^2}\Big[\{(Mu_l-Nu_{l5})+(Nu_l+Mu_{l5})\gamma_5\}\,\psi\\ +(N-M\gamma_5)\,\rho_l D^{-1}\bar{\psi}\Big].\end{aligned}\right\}\quad(4.12)$$

Thus the first part of the theorem is proved.

Next, in order to prove the second relation (4.3), we multiply (4.2) by the matrix γ_m and arrange it by the bases (3.11) and interchange the suffixes l and m; thus we have[1]

$$\left.\begin{aligned}\gamma_{[l}\gamma_{m]}\psi=2\{i\overset{[3\,4]}{\lambda_l\lambda_m}-\overset{[1\,2]}{\lambda_l\lambda_m}\gamma_5\}\,\psi+2\,\{(\overset{[2\,4]}{\lambda_l\lambda_m}-i\overset{[2\,3]}{\lambda_l\lambda_m})\\ -(i\overset{[1\,3]}{\lambda_l\lambda_m}-\overset{[1\,4]}{\lambda_l\lambda_m})\gamma_5\}\,D^{-1}\bar{\psi}.\end{aligned}\right\}\quad(4.13)$$

But, on the other hand, we know that[2]

$$u_{lm}=2(-N\overset{[1\,2]}{\lambda_l\lambda_m}+iM\overset{[3\,4]}{\lambda_l\lambda_m}) \qquad (4.14)$$

and[3]

$$\rho_{lm}=2M(i\overset{[1\,3]}{\lambda_l\lambda_m}-\overset{[1\,4]}{\lambda_l\lambda_m})+2N(\overset{[2\,4]}{\lambda_l\lambda_m}-i\overset{[2\,3]}{\lambda_l\lambda_m}). \qquad (4.15)$$

Also, from (4.14) and (4.15), we have

$$u_{lm5}=-\frac{i}{2}\epsilon_{lmpq}u^{pq}=2(iN\overset{[3\,4]}{\lambda_l\lambda_m}+M\overset{[1\,2]}{\lambda_l\lambda_m}) \qquad (4.16)$$

and $$\rho_{lm5}=-\frac{i}{2}\epsilon_{lmpq}\rho^{pq}=2M(\overset{[4\,2]}{\lambda_l\lambda_m}+i\overset{[2\,3]}{\lambda_l\lambda_m})-2N(i\overset{[3\,1]}{\lambda_l\lambda_m}+\overset{[1\,4]}{\lambda_l\lambda_m}). \qquad (4.17)$$

Therefore, using the relations (4.14)-(4.17), the equation (4.13) is written in the form (4.3). Q. E. D.

§ 5. The expansion of 1-4 matrix in terms of the matrix ψ only.

In § 3, we have seen that any 1-4 matrix is expressible by a linear combination of the matrices (3.11). But the matrices (3.11) are constructed by the two given matrices ψ and $\bar{\psi}$. Now we shall consider how to express a 1-4 matrix in terms of ψ only, avoiding the inconvenience of using $\bar{\psi}$.

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(1) See Note 2.
(2) loc. cit., 186.
(3) See Note 3.

Theorem 7. *When*

$$M^2+N^2 \neq 0\,, \tag{5.1}$$

any 1–4 matrix is expressible in a unique way by a linear combination of a set of the matrices

$$\psi\,,\quad \gamma_5\psi\,,\quad \gamma_{[l}\gamma_{m]}\psi\,,\quad (l,\ m=1,\dots,4)\,, \tag{5.2}$$

or

$$\gamma_l\psi\,,\quad \gamma_l\gamma_5\psi\,,\qquad (l=1,\dots,4)\,, \tag{5.3}$$

the coefficients being all real (or all purely imaginary).

Proof. Any 1–4 matrix is expressed in the form of

$$A\psi+A^5\gamma_5\psi+A^{lm}\gamma_l\gamma_m\psi \qquad (A^{lm}=-A^{ml})\,, \tag{5.4}$$

A, A^5, and A^{lm} being all real (or all purely imaginary). For, since any 1–4 matrix is expressible in the form (by theorem 5)

$$(c+d\gamma_5)\psi+(e+f\gamma_5)D^{-1}\bar{\psi}\,, \tag{5.5}$$

here it is enough to show that the real (or purely imaginary) quantities A, A^5, and A^{lm} can be uniquely determined so as to satisfy

$$\left.\begin{array}{c}(c+d\gamma_5)\psi+(e+f\gamma_5)D^{-1}\bar{\psi}=(A+A^5\gamma_5+A^{lm}\gamma_l\gamma_m)\psi \\ (A^{lm}=-A^{ml})\,.\end{array}\right\} \tag{5.6}$$

But, on the other hand, if we put

$$A^{lm}=\alpha_{ab}\overset{a}{\lambda}{}^l\overset{b}{\lambda}{}^m \qquad (\alpha_{ab}=-\alpha_{ba})\,, \tag{5.7}$$

from (4.13), we have

$$A^{lm}\gamma_l\gamma_m\psi=2\{i\alpha_{34}-\alpha_{12}\gamma_5\}\psi+2\{(\alpha_{24}-i\alpha_{23})-(i\alpha_{13}-\alpha_{14})\gamma_5\}D^{-1}\bar{\psi}\,. \tag{5.8}$$

Substituting (5.8) into (5.6), and comparing the coefficients of the bases of the resulting equation, we have

$$\left.\begin{array}{ll} c=2i\alpha_{34}+A\,, & d=-2\alpha_{12}+A^5\,, \\ e=2(\alpha_{24}-i\alpha_{23})\,, & f=-2(i\alpha_{13}-\alpha_{14})\,,\end{array}\right\} \tag{5.9}$$

But, when A^{lm} are real (or purely imaginary), since $\overset{1}{\lambda}{}^l$ are real and $\overset{\omega}{\lambda}{}^l$ $(\omega=2,3,4)$ are purely imaginary, it must follow that

$$\left\{\begin{array}{lll} \alpha_{1\omega} & (\omega=2,3,4) & \text{are purely imaginary (or real)} \\ \alpha_{\omega\delta} & (\omega,\delta=2,3,4) & \text{are real (or purely imaginary).}\end{array}\right.$$

Therefore, when c, d, e, and f, in (5.5), are given arbitrarily, from the relation (5.9), a_{ab}, A, and A^5 are uniquely determined such that A^{lm}, A, and A^5 are all real (or all purely imaginary). So the first part of the theorem is proved.

Next, we shall show that any 1–4 matrix is also expressed in the form:

$$(A^l\gamma_l + A^{l5}\gamma_l\gamma_5)\psi\,, \tag{5.10}$$

A^l and A^{l5} being real (or purely imaginary) vectors. Putting

$$A^l = \alpha_a \overset{a}{\lambda}{}^l, \qquad A^{l5} = \beta_a \overset{a}{\lambda}{}^l, \tag{5.11}$$

and using the relation (4.11), we have

$$\begin{aligned}(A^l\gamma_l + A^{l5}\gamma_l\gamma_5)\psi = \frac{1}{\sqrt{M^2+N^2}}\Big[&\{M(\alpha_1+\beta_2) - N(\alpha_2-\beta_1)\}\psi \\ &+ \{N(\alpha_1+\beta_2) + M(\alpha_2-\beta_1)\}\gamma_5\psi \\ &+ \{N(-i\alpha_3+\alpha_4) + M(i\beta_3-\beta_4)\}D^{-1}\overline{\psi} \\ &+ \{M(i\alpha_3-\alpha_4) + N(i\beta_3-\beta_4)\}\gamma_5 D^{-1}\overline{\psi}\Big].\end{aligned}$$

Hence, if we suppose that a matrix (5.5) is expressed in the form (5.10), i. e.

$$(c+d\gamma_5)\psi + (e+f\gamma_5)D^{-1}\overline{\psi} = (A^l\gamma_l + A^{l5}\gamma_l\gamma_5)\psi\,,$$

it must follow that

$$\left.\begin{aligned} c &= \frac{1}{\sqrt{M^2+N^2}}\{(M\alpha_1+N\beta_1) + (-N\alpha_2+M\beta_2)\}\,, \\ d &= \frac{1}{\sqrt{M^2+N^2}}\{(N\alpha_1-M\beta_1) + (M\alpha_2+N\beta_2)\}\,, \\ e &= \frac{1}{\sqrt{M^2+N^2}}\{i(-N\alpha_3+M\beta_3) + (N\alpha_4-M\beta_4)\}\,, \\ f &= \frac{1}{\sqrt{M^2+N^2}}\{i(M\alpha_3+N\beta_3) - (M\alpha_4+N\beta_4)\}\,. \end{aligned}\right\} \tag{5.12}$$

But, similarly as before, when A^l and A^{l5} are all real (or purely imaginary), we see that

$$\begin{cases} \alpha_1,\ \beta_1 & \text{are real (or purely imaginary),} \\ \alpha_\omega,\ \beta_\omega\ (\omega=2,3,4) & \text{are purely imaginary (or real).} \end{cases}$$

Hence, when c, d, e, and f, in (5.5), are given arbitrarily, because of the relation (5.12), α_a and β_a, and thus A^l and A^{l5}, are uniquely determined such that A^l and A^{l5} are all real (or purely imaginary). So the last part of the theorem is proved. Q. E. D.

N. B. If we use the relations (5.9) and (5.12), we can reduce a matrix of the form $\sum\psi$ to that of the form (5.5), $\sum$ being any 4-4 matrix. For, if we expand $\sum$ in sedenion:

$$\sum = A^{pq}\gamma_p\gamma_q + A^5\gamma_5 + A + A^l\gamma_l + A^{l5}\gamma_l\gamma_5 \quad (A^{pq} = -A^{qp}), \tag{5.13}$$

from the equation

$$\sum\psi = (c + d\gamma_5)\psi + (e + f\gamma_5) D^{-1}\bar{\psi}, \tag{5.14}$$

we can determine c, d, e, and f, as follows:

$$\left.\begin{aligned}
c &= A + 2iA^{pq}\overset{3}{\lambda}_p\overset{4}{\lambda}_q + \frac{1}{\sqrt{M^2+N^2}}\{(MA^p + NA^{p5})\overset{1}{\lambda}_p + (-NA^p + MA^{p5})\overset{2}{\lambda}_p\}, \\
d &= A^5 - 2A^{pq}\overset{1}{\lambda}_p\overset{2}{\lambda}_q + \frac{1}{\sqrt{M^2+N^2}}\{(NA^p - MA^{p5})\overset{1}{\lambda}_p + (MA^p + NA^{p5})\overset{2}{\lambda}_p\}, \\
e &= 2A^{pq}\overset{2}{\lambda}_p(\overset{4}{\lambda}_q - i\overset{3}{\lambda}_q) + \frac{1}{\sqrt{M^2+N^2}}\{i(NA^p + MA^{p5})\overset{3}{\lambda}_p + (NA^p - MA^{p5})\overset{4}{\lambda}_p\}, \\
f &= -2A^{pq}\overset{1}{\lambda}_p(i\overset{3}{\lambda}_q - \overset{4}{\lambda}_q) + \frac{1}{\sqrt{M^2+N^2}}\{i(MA^p + NA^{p5})\overset{3}{\lambda}_p - (MA^p + NA^{p5})\overset{4}{\lambda}_p\},
\end{aligned}\right\} \tag{5.15}$$

Lastly, we shall examine the relations between the sets of A^{lm}, A^5, A and A^l, A^{l5} when they are given as the sets of the coefficients of the two kinds of expansion of the same 1-4 matrix.

Theorem 8. *If*

$$(A^{lm}\gamma_l\gamma_m + A^5\gamma_5 + A)\psi \qquad (A^{lm} = -A^{ml}) \tag{5.16}$$

and

$$(A^l\gamma_l + A^{l5}\gamma_l\gamma_5)\psi \tag{5.17}$$

are the two kinds of expansion of the same 1-4 matrix, where A^{lm}, A^5, A, A^l, *and* A^{l5} *are all real (or all purely imaginary), there hold the relations:*

$$\left.\begin{aligned}
&\sqrt{M^2+N^2}\,A_{lm} = \{MA_{[l} + NA_{[l}^{\cdot 5}\}\overset{1}{\lambda}_{m]} - \frac{i}{2}\{-MA^{p5} + NA^p\}\epsilon_{lmpq}\overset{1}{\lambda}{}^q, \\
&\sqrt{M^2+N^2}\,A = (MA^l + NA^{l5})\overset{1}{\lambda}_l, \quad \sqrt{M^2+N^2}\,A^5 = (NA^l - MA^{l5})\overset{1}{\lambda}_l,
\end{aligned}\right\} \tag{5.18}$$

or

$$\left.\begin{aligned}
\sqrt{M^2+N^2}\,A_j &= -2(MA^l_{\cdot j} + iNA^{*l}_{\cdot j})\overset{1}{\lambda}_l + (MA + NA^5)\overset{1}{\lambda}_j, \\
\sqrt{M^2+N^2}\,A_j^{\cdot 5} &= -2(NA^l_{\cdot j} - iMA^{*l}_{\cdot j})\overset{1}{\lambda}_l + (NA - MA^5)\overset{1}{\lambda}_j.
\end{aligned}\right\} \tag{5.19}$$

Proof. Equating the right-hand side of (5.9) and (5.12), and solving A, A^5, and a_{ab} from the resulting equations, we have the relations (5.18). From (5.18), solving for A_j and $A_j^{\cdot 5}$, we have (5.19). Q. E. D.

Similarly as before, we have

Theorem 9. *If*

$$(A^{lm}\gamma_l\gamma_m+A^5\gamma_5+A)\psi \qquad (A^{lm}=-A^{ml}),$$

and

$$(B^{lm}\gamma_l\gamma_m+B^5\gamma_5+B)\psi \qquad (B^{lm}=-B^{ml})$$

are the two kinds of expansion of the same 1–4 matrix where A^{lm}, A^5, and A are all real and B^{lm}, B^5, and B are all purely imaginary, then there hold the relations:

$$\left.\begin{aligned} A&=iB^{lm}\overset{3}{\lambda}_l\overset{4}{\lambda}_m, \qquad A^5=-B^{lm}\overset{1}{\lambda}_l\overset{2}{\lambda}_m, \\ A_{lm}&=-B^5\overset{[1}{\lambda}_l\overset{2]}{\lambda}_m-iB\overset{[3}{\lambda}_l\overset{4]}{\lambda}_m+iB^{pq}\{(\overset{1}{\lambda}_p\overset{4}{\lambda}_q\overset{[1}{\lambda}_l\overset{3]}{\lambda}_m-\overset{1}{\lambda}_p\overset{3}{\lambda}_q\overset{[1}{\lambda}_l\overset{4]}{\lambda}_m) \\ &\qquad +(\overset{2}{\lambda}_p\overset{4}{\lambda}_q\overset{[2}{\lambda}_l\overset{3]}{\lambda}_m-\overset{2}{\lambda}_p\overset{3}{\lambda}_q\overset{[2}{\lambda}_l\overset{4]}{\lambda}_m)\}. \end{aligned}\right\} \tag{5.20}$$

Theorem 10. *If*

$$(A^l\gamma_l+A^{l5}\gamma_l\gamma_5)\psi \quad \text{and} \quad (B^l\gamma_l+B^{l5}\gamma_l\gamma_5)\psi$$

are the two kinds of expansion of the same 1–4 matrix where A^l, A^{l5} are all real and B^l, B^{l5} are all purely imaginary, then there hold the relations:

$$\left.\begin{aligned} A_j&=B^{l5}(\overset{2}{\lambda}_l\overset{1}{\lambda}_j-\overset{1}{\lambda}_l\overset{2}{\lambda}_j)+iB^l(\overset{4}{\lambda}_l\overset{3}{\lambda}_j-\overset{3}{\lambda}_l\overset{4}{\lambda}_j), \\ A_j^{\cdot 5}&=-B^l(\overset{2}{\lambda}_l\overset{1}{\lambda}_j-\overset{1}{\lambda}_l\overset{2}{\lambda}_j)+iB^{l5}(\overset{4}{\lambda}_l\overset{3}{\lambda}_j-\overset{3}{\lambda}_l\overset{4}{\lambda}_j). \end{aligned}\right\} \tag{5.21}$$

Combining theorems 8 and 10, we have

Theorem 11. *If*

$$(A^{pq}\gamma_p\gamma_q+A^5\gamma_5+A)\psi \quad (A^{pq}=-A^{qp}) \quad \text{and} \quad (B^p\gamma_p+B^{p5}\gamma_p\gamma_5)\psi$$

are the two kinds of expansion of the same 1–4 matrix where A^{pq}, A^5, and A are all real, and B^p and B^{p5} are all purely imaginary, then there hold the relations:

$$\left.\begin{aligned}
\sqrt{M^2+N^2}\,A_{lm} = \Big[& M\{-B^{p5}\overset{1}{\lambda}_p\overset{2}{\lambda}_{[l}+iB^p(\overset{4}{\lambda}_p\overset{3}{\lambda}_{[l}-\overset{3}{\lambda}_p\overset{4}{\lambda}_{[l})\} \\
& +N\{B^p\overset{1}{\lambda}_p\overset{2}{\lambda}_{[l}+iB^{p5}(\overset{4}{\lambda}_p\overset{3}{\lambda}_{[l}-\overset{3}{\lambda}_p\overset{4}{\lambda}_{[l})\}\Big]\overset{1}{\lambda}_{m]} \\
& -i\Big[-M\{B^p\overset{1}{\lambda}_p\overset{[4}{\lambda}_l\overset{3]}{\lambda}_m+iB^{p5}(\overset{4}{\lambda}_p\overset{[2}{\lambda}_l\overset{4]}{\lambda}_m-\overset{3}{\lambda}_p\overset{[2}{\lambda}_l\overset{3]}{\lambda}_m)\} \\
& +N\{B^{p5}\overset{1}{\lambda}_p\overset{[3}{\lambda}_l\overset{4]}{\lambda}_m+iB^{p5}(\overset{4}{\lambda}_p\overset{[2}{\lambda}_l\overset{4]}{\lambda}_m-\overset{3}{\lambda}_p\overset{[2}{\lambda}_l\overset{3]}{\lambda}_m)\}\Big], \\
\sqrt{M^2+N^2}\,A = MB^{p5}\overset{2}{\lambda}_p & -NB^p\overset{2}{\lambda}_p, \quad \sqrt{M^2+N^2}\,A^5=NB^{p5}\overset{2}{\lambda}_p+MB^p\overset{2}{\lambda}_p.
\end{aligned}\right\} \tag{5.22}$$

N. B. If we use theorems 7–11, we can reduce the matrix of the form $\sum\psi$, where

$$\sum = A^{pq}\gamma_p\gamma_q+A^5+A^p\gamma_p+A^{p5}\gamma_p\gamma_5 \qquad (A^{pq}=-A^{qp}), \tag{5.23}$$

to the matrix of the form:

$$(E^{pq}\gamma_p\gamma_q+E^5\gamma_5+E)\psi \qquad (E^{pq}=-E^{qp}) \tag{5.24}$$

where E^{pq}, E^5, and E are real and are determined in a unique way. The calculation, being easy but tedious, is omitted here.

§ 6. The vector which satisfies the relation $v^\lambda\gamma_\lambda\psi=v\psi$.

As an application of the statements made above (§ 4–§ 5), when ψ is given we shall find the general form of the vector v^λ satisfying the relation

$$v^\lambda\gamma_\lambda\psi=v\psi \qquad (\lambda=1,\ldots\ldots,5). \tag{6.1}$$

In the previous paper,[1] we saw that the vectors $u^\lambda\equiv\psi^\dagger A\gamma^\lambda\psi$ and $\rho^l\equiv\tilde{\psi}C\gamma^l\psi$ ($\lambda=1,\ldots\ldots,5$; $l=1,\ldots\ldots,4$) satisfy the relations:[2]

$$u^\lambda\gamma_\lambda\psi=M\psi, \qquad \rho^l\gamma_l\psi=0. \tag{6.2}$$

Therefore, $v^\lambda=u^\lambda$, $v=M$ and $v^l=\rho^l$, $v^5=v=0$ are two special solutions of (6.1).

To obtain the general solution, using the relation (4.11), rewriting (6.1) as

(1) loc. cit., 174 and 180.
(2) See Note 4.

$$\left.\begin{array}{r}\dfrac{v^l}{\sqrt{M^2+N^2}}\Big[\{(M\overset{1}{\lambda}_l-N\overset{2}{\lambda}_l)+(N\overset{1}{\lambda}_l+M\overset{2}{\lambda}_l)\gamma_5\}\psi \\ +(-i\overset{3}{\lambda}_l+\overset{4}{\lambda}_l)(N-M\gamma_5)D^{-1}\overline{\psi}\Big]+v^5\gamma_5\psi=v\psi\,, \\ (l=1,2,3,4)\,,\end{array}\right\} \quad (6.3)$$

and comparing the coefficients of the bases of (6.3), we have

$$\left.\begin{array}{c}\dfrac{v^l}{\sqrt{M^2+N^2}}(M\overset{1}{\lambda}_l-N\overset{2}{\lambda}_l)=v\,, \quad \dfrac{v^l}{\sqrt{M^2+N^2}}(N\overset{1}{\lambda}_l+M\overset{2}{\lambda}_l)+v^5=0, \\ v^l(-i\overset{3}{\lambda}_l+\overset{4}{\lambda}_l)=0\,,\end{array}\right\} \quad (6.4)$$

which is reduced to

$$\left.\begin{array}{c}\sqrt{M^2+N^2}\,v^l\overset{1}{\lambda}_l=vM-v^5N\,, \quad \sqrt{M^2+N^2}\,v^l\overset{2}{\lambda}_l=-v^5M-vN\,, \\ v^l(-i\overset{3}{\lambda}_l+\overset{4}{\lambda}_l)=0\,.\end{array}\right\} \quad (6.5)$$

Putting

$$v^l=w_a\overset{a}{\lambda}{}^l \qquad (a=1,\ldots\ldots,4)\,, \qquad (6.6)$$

from (6.5), we have

$$\left.\begin{array}{c}\sqrt{M^2+N^2}\,w_1=vM-v^5N\,, \qquad \sqrt{M^2+N^2}\,w_2=-v^5M-vN\,, \\ -iw_3+w_4=0\,;\end{array}\right\} \quad (6.7)$$

hence, (6.6) becomes

$$v^l=\frac{1}{\sqrt{M^2+N^2}}\{(vM-v^5N)\overset{1}{\lambda}{}^l-(v^5M+vN)\overset{2}{\lambda}{}^l\}-iw_4\overset{3}{\lambda}_l+w_4\overset{4}{\lambda}_l\,, \qquad (6.8)$$

or, since $-i\overset{3}{\lambda}_l+\overset{4}{\lambda}_l=\dfrac{\rho_l}{\sqrt{M^2+N^2}}$, if we put $w_4\equiv w$, we have

$$v^l=\frac{1}{\sqrt{M^2+N^2}}\{(vM-v^5N)\overset{1}{\lambda}{}^l-(v^5M+vN)\overset{2}{\lambda}{}^l+w\rho^l\}\,, \qquad (6.9)$$

which is the general solution of (6.1), where v, v^5, and w are regarded as arbitrary.

In the special case in which v^λ and v are all real, since w_1 is real and w_ω $(\omega=2,3,4)$ are purely imaginary, from (6.7) we have

$$\sqrt{M^2+N^2}\,w_1=vM-v^5N, \quad 0=v^5M+vN, \quad w_2=w_3=w_4=0. \quad (6.10)$$

Therefore, it follows that

$$v=\tau M\,, \quad v^5=-\tau N\,, \quad v^l=\tau u^l \quad (l=1,\ldots\ldots,4)\,, \qquad (6.11)$$

where τ is real but arbitrary, so that (6.11) is the general real solution of (6.1).

Putting together the results obtained above, we have

Theorem 12. *When*

$$M^2+N^2\neq 0\,, \tag{6.12}$$

the general solution v^l of the equation

$$v^\lambda\gamma_\lambda\psi=v\psi \qquad (\lambda=1,\ldots.,5)$$

is given by

$$v^l=\frac{1}{\sqrt{M^2+N^2}}\{(vM-v^5N)\overset{1}{\lambda}{}^l-(v^5M+vN)\overset{2}{\lambda}{}^l+w\rho^l\}\,, \quad (l=1,\ldots.,4)$$

where v^5, v, and w are arbitrary. And the general real solution is given by

$$v^\lambda=\tau\psi^\dagger A\gamma^\lambda\psi\,, \quad v=\tau\psi^\dagger A\psi \quad (\lambda=1,\ldots.,5) \tag{6.13}$$

where τ is real but arbitrary.

From theorem 12, we have

Theorem 13. *When*

$$M^2+N^2\neq 0\,,$$

(i) *in order that v^λ in the equation*

$$v^\lambda\gamma_\lambda\psi=0 \qquad (\lambda=1,\ldots.,5) \tag{6.14}$$

may be real, it must be true that

$$M\equiv\psi^\dagger A\psi=0\,, \qquad v^\lambda=\tau\psi^\dagger A\gamma^\lambda\psi\,,$$

where τ is real but arbitrary.

(ii) *in order that v^l and v in the equation*

$$v^l\gamma_l\psi=v\psi \qquad (l=1,\ldots.,4) \tag{6.15}$$

may be real, it must be true that

$$v^l=\tau\psi^\dagger A\gamma^l\psi\,, \quad v=\tau\psi^\dagger A\psi\,, \quad N\equiv\psi^\dagger A\gamma_5\psi=0$$

where τ is real but arbitrary.

(iii) *a real solution v^l of the equation*

$$v^l\gamma_l\psi=0 \qquad (l=1,\ldots.,4)\,, \tag{6.16}$$

does not exist. But, if we put aside the condition that v^l be real, the general solution of (6.16) *is given by*

$$v^l = w\rho^l,$$

where w is an arbitrary scalar.

Lastly, we shall consider the converse problem: *for given v^λ and v to find the solution ψ of the equation $v^\lambda\gamma_\lambda\psi = v\psi$* ($\lambda = 1, \ldots, 5$). Rewriting $v^\lambda\gamma_\lambda\psi = v\psi$ as

$$P\psi = 0, \quad \text{where} \quad P = v^\lambda\gamma_\lambda - v, \tag{6.17}$$

the necessary and sufficient condition for the existence of ψ satisfying (6.17) becomes that the determinant $|P|$ vanishes identically. $|P|$ can be calculated as follows: Take the matrix

$$Q = v^\lambda\gamma_\lambda + v.$$

Then we have

$$QP = (v^\lambda v_\lambda - v^2)I. \tag{6.18}$$

But since the determinants of the matrices P and Q are equal,(1) from (6.18) we have

$$|P|^2 = (v^\lambda v_\lambda - v^2)^4.$$

So we have

Theorem 14. *The necessary and sufficient condition for the existence of ψ satisfying the relation*

$$v^\lambda\gamma_\lambda\psi = v\psi,$$

is that

$$v^\lambda v_\lambda = v^2.$$

When $|P| = 0$, the number of independent solutions ψ of $v^\lambda\gamma_\lambda\psi = v\psi$ is classified by the rank of P. That is to say, when the rank of P is equal to 3 or 2, the number of independent solutions ψ of $v^\lambda\gamma_\lambda\psi = v\psi$

(1) If we take γ_l and γ_5 as in the previous paper (cf. loc. cit., 172)

$$\gamma_l = \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{0}} & \times & \times \\ & & \times & \times \\ \hline \times & \times & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ \times & \times & & \end{array}\right), \qquad \gamma_5 = i\left(\begin{array}{cc|cc} 1 & & \multicolumn{2}{c}{0} \\ & 1 & & \\ \hline \multicolumn{2}{c|}{0} & -1 & \\ & & & -1 \end{array}\right),$$

P and Q have the following forms respectively:

$$P = \left(\begin{array}{cc|cc} a & & \times & \times \\ & a & \times & \times \\ \hline \times & \times & b & \\ \times & \times & & b \end{array}\right), \qquad Q = \left(\begin{array}{cc|cc} -b & & \times & \times \\ & -b & \times & \times \\ \hline \times & \times & -a & \\ \times & \times & & -a \end{array}\right)$$

where $a = -v + iv^5$, $b = -v - iv^5$, from which we see that

$$|P| = |Q|.$$

is 1 or 2. But we see that when $|P|=0$, the rank of P is, at most, 2. For, since

$$(v^\lambda\gamma_\lambda+v)(v^\lambda\gamma_\lambda-v)=(v^\lambda v_\lambda-v^2)I$$

and the determinant of $P\equiv v^\lambda\gamma_\lambda-v$ is equal to $\pm(v^\lambda v_\lambda-v^2)^2$, every cofactor of the matrix $v^\lambda\gamma_\lambda-v$ is equal to every element of the matrix:

$$\pm(v^\lambda v_\lambda-v^2)(v^\lambda\gamma_\lambda+v),$$

which vanishes on the assumption $|P|=0$, or $v^\lambda v_\lambda-v^2=0$; so that every cofactor of P must vanish, i.e. the rank of P is 2 at most. So we have

Theorem 15. *When*

$$v^\lambda v_\lambda=v^2$$

the number of independent solutions ψ *of*

$$v^\lambda\gamma_\lambda\psi=v\psi$$

is, at least, 2.

Part II.

In part I, we considered ψ as arbitrary. Now we shall investigate what relations exist among u_l, u_{l5} etc. when ψ is a solution of the fundamental equation for ψ. For this purpose, first we shall find the covariant derivative of the matrices A, B, C, and D, defined in part I.

§ 7. The covariant derivative of A, B, C, and D.

We know that γ_j's defined by $\gamma_{(j}\gamma_{k)}=g_{jk}$ satisfy the following relation:

$$\frac{\partial\gamma_j}{\partial x^k}-\left\{{h \atop jk}\right\}\gamma_h-\Gamma_k\gamma_j+\gamma_j\Gamma_k=0\,. \tag{7.1}$$

Following Schrödinger and Pauli, we define the left-hand side of this equation as the *covariant derivative* of γ_j and denote it by $\nabla_k\gamma_j$ i.e.

$$\nabla_k\gamma_j=\frac{\partial\gamma_j}{\partial x^k}-\left\{{h \atop jk}\right\}\gamma_h-\Gamma_k\gamma_j+\gamma_j\Gamma_k\,. \tag{7.2}$$

Corresponding to (7.2), we define the *covariant derivative* of 1–4 matrix ψ as follows

$$\nabla_k\psi=\frac{\partial\psi}{\partial x^k}-\Gamma_k\psi\,. \tag{7.3}$$

As to the matrix A, Pauli has proved[1] that

$$\frac{\partial A}{\partial x^k}+A\Gamma_k+\Gamma_k{}^\dagger A=a_kA$$

where a_k is a gradient vector. In a similar manner, we can prove that

$$\frac{\partial B}{\partial x^k}+B\Gamma_k+\tilde{\Gamma}_kB=b_kB$$

$$\frac{\partial C}{\partial x^k}+C\Gamma_k+\tilde{\Gamma}_kC=c_kC$$

$$\frac{\partial D}{\partial x^k}+D\Gamma_k-\bar{\Gamma}_kD=d_kD\,,$$

where b_k, c_k and d_k are certain gradient vectors. We call the expression of the left-hand side of the equations given above the *covariant derivative* of $A, B, C,$ and D, respectively, and denote them by ∇_kA, ∇_kB, ∇_kC, and ∇_kD, i. e.,

$$\nabla_kA=\frac{\partial A}{\partial x^k}+A\Gamma_k+\Gamma_k{}^\dagger A\,, \tag{7.4}$$

$$\nabla_kB=\frac{\partial B}{\partial x^k}+B\Gamma_k+\tilde{\Gamma}_kB\,, \tag{7.5}$$

$$\nabla_kC=\frac{\partial C}{\partial x^k}+C\Gamma_k+\tilde{\Gamma}_kC\,, \tag{7.6}$$

$$\nabla_kD=\frac{\partial D}{\partial x^k}+D\Gamma_k-\bar{\Gamma}_kD\,. \tag{7.7}$$

Since we can normalize $A, B, C,$ and D, by taking suitable multipliers, so that a_k, b_k, c_k, and d_k vanish, we have

$$\nabla_kA=0\,,\quad \nabla_kB=0\,,\quad \nabla_kC=0\,,\quad \nabla_kD=0\,. \tag{7.8}$$

Under such definitions of covariant derivatives of $A, B, C, D,$ and ψ, we see[2] that the covariant differentiation of spinor, vector, and tensor, obeys the ordinary rule of differentiation; for instance,

$$\left.\begin{aligned}\nabla_k(\psi^\dagger A\gamma_j\psi)&=(\nabla_k\psi)^\dagger A\gamma_j\psi+\psi^\dagger A\gamma_j\nabla_k\psi\,,\\ \nabla_k(A^p\gamma_p\psi)&=(\nabla_kA^p)\gamma_p\psi+A^p\gamma_p\nabla_k\psi\,,\end{aligned}\right\} \tag{7.9}$$

where A^p is any vector and ∇_kA^p denotes the covariant derivative of A^p with respect of $\{{}^{\,l}_{jk}\}$.

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(1) Pauli, Ann. d. Physik, **18** (1933), 358, 359.

(2) Cf. loc. cit., 358.

§ 8. The conditions for integrability of the fundamental equation for ψ.

In wave geometry, using the notation (7.3), we know that[1] the general fundamental equation for ψ has the form

$$\nabla_k \psi = \Sigma_k \psi\,, \tag{8.1}$$

where

$$\Sigma_k = A_k^{\cdot pq}\gamma_p\gamma_q + A_k + A_k^{\cdot 5}\gamma_5 + A_k^{\cdot p}\gamma_p + A_k^{\cdot p5}\gamma_p\gamma_5 \quad (A_k^{\cdot pq} = -A_k^{\cdot qp})\,. \tag{8.2}$$

Now, making use of the fact that the covariant differentiation of spinor, vector, and tensor, obeys the ordinary rule of differentiation, we can obtain the condition for integrability of (8.1). Operating ∇_j on both sides of (8.1), we have

$$\nabla_{[j}\nabla_{k]}\psi = (\nabla_{[j}\Sigma_{k]} + \Sigma_{[k}\Sigma_{j]})\,\psi\,. \tag{8.3}$$

But, because of (8.2), we have

$$\nabla_{[j}\Sigma_{k]} = \nabla_{[j}A_{k]}^{\cdot pq}\gamma_p\gamma_q + \nabla_{[j}A_{k]} + \nabla_{[j}A_{k]}^{\cdot 5}\gamma_5 + \nabla_{[j}A_{k]}^{\cdot p}\gamma_p + \nabla_{[j}A_{k]}^{\cdot p5}\gamma_p\gamma_5\,, \tag{8.4}$$

$$\left.\begin{aligned}\Sigma_{[k}\Sigma_{j]} = {} & A_{[k}^{\cdot pq}A_{j]}^{\cdot rs}\gamma_p\gamma_q\gamma_r\gamma_s + A_{[k}^{\cdot pq}A_{j]}^{\cdot r}\gamma_p\gamma_q\gamma_r + A_{[k}^{\cdot pq}A_{j]}^{\cdot r5}\gamma_p\gamma_q\gamma_r\gamma_5 \\ & + A_{[k}^{\cdot 5}A_{j]}^{\cdot p}\gamma_5\gamma_p + A_{[k}^{\cdot 5}A_{j]}^{\cdot p5}\gamma_p \\ & + A_{[k}^{\cdot p}A_{j]}^{\cdot qr}\gamma_p\gamma_q\gamma_r + A_{[k}^{\cdot p}A_{j]}^{\cdot 5}\gamma_p\gamma_5 + A_{[k}^{\cdot p}A_{j]}^{\cdot q}\gamma_p\gamma_q + A_{[k}^{\cdot p}A_{j]}^{\cdot q5}\gamma_p\gamma_q\gamma_5 \\ & + A_{[k}^{\cdot p5}A_{j]}^{\cdot qr}\gamma_p\gamma_5\gamma_q\gamma_r - A_{[k}^{\cdot p5}A_{j]}^{\cdot 5}\gamma_p + A_{[k}^{\cdot p5}A_{j]}^{\cdot q}\gamma_p\gamma_5\gamma_q + A_{[k}^{\cdot p5}A_{j]}^{\cdot q5}\gamma_p\gamma_q \\ = {} & 4A_{[k}^{\cdot pl}A_{j]l}^{\cdot\cdot q}\gamma_p\gamma_q + 4A_{[k}^{\cdot pq}A_{j]q}\gamma_p + 4A_{[k}^{\cdot pq}A_{j]q}^{\cdot\cdot 5}\gamma_p\gamma_5 + A_{[k}^{\cdot p5}A_{j]}^{\cdot q5}\gamma_p\gamma_q \\ & - 2A_{[k}^{\cdot 5}A_{j]}^{\cdot p}\gamma_p\gamma_5 + 2A_{[k}^{\cdot 5}A_{j]}^{\cdot p5}\gamma_p + A_{[k}^{\cdot p}A_{j]}^{\cdot q}\gamma_p\gamma_q + 2A_{[k}^{\cdot p}A_{j]}^{\cdot q5}g_{pq}\gamma_5\,,\end{aligned}\right\} \tag{8.5}$$

and[2]

$$\nabla_{[j}\nabla_{k]}\psi = \frac{1}{8}K_{jk}^{\cdot\cdot pq}\gamma_p\gamma_q\psi\,. \tag{8.6}$$

Substituting (8.4)–(8.6) into (8.3), (8.3) is reduced to the form

$$(P^{pq}\gamma_p\gamma_q + P + P^5\gamma_5 + P^p\gamma_p + P^{p5}\gamma_p\gamma_5)\,\psi = 0\,, \tag{8.7}$$

where

$$\left.\begin{aligned} P^{pq} &= \frac{1}{8}K_{kj}^{\cdot\cdot pq} + A_{[k}^{\cdot p}A_{j]}^{\cdot q} + A_{[k}^{\cdot p5}A_{j]}^{\cdot q5} + \nabla_{[j}A_{k]}^{\cdot pq} + 4A_{[k}^{\cdot pl}A_{j]l}^{\cdot\cdot q}\,, \\ P &= \nabla_{[j}A_{k]}\,, \qquad P^5 = \nabla_{[j}A_{k]}^{\cdot 5} + 2A_{[k}^{\cdot p}A_{j]}^{\cdot q5}g_{pq}\,, \\ P^p &= \nabla_{[j}A_{k]}^{\cdot p} + 2A_{[k}^{\cdot 5}A_{j]}^{\cdot p5} + 4A_{[k}^{\cdot pq}A_{j]q}\,, \\ P^{p5} &= \nabla_{[j}A_{k]}^{\cdot p5} - 2A_{[k}^{\cdot 5}A_{j]}^{\cdot p} + 4A_{[k}^{\cdot pq}A_{j]q}^{\cdot\cdot 5}.\end{aligned}\right\} \tag{8.8}$$

---

(1) When we consider the most general transformation and parallel displacement which make $dx^i\gamma_i\psi = 0$ invariant, the fundamental equation for ψ becomes (8.1).

(2) See Note 5.

So that the condition for integrability of (8.1) is given by (8.7) together with (8.8).

§ 9. The differential equations for u_j, u_{j5}, etc., when ψ is a solution of the fundamental equation.

We shall obtain the differential equations for u_j, u_{j5}, etc., when ψ is the solution of (8.1). From (8.1), we have

$$\nabla_k u_j = \psi^\dagger (\Sigma_k^\dagger A\gamma_j + A\gamma_j \Sigma_k)\psi . \tag{9.1}$$

But since

$$\Sigma_k^\dagger = \bar{A}_k^{\cdot pq}\gamma_q^\dagger\gamma_p^\dagger + \bar{A}_k^{\cdot 5}\gamma_5^\dagger + \bar{A}_k + \bar{A}_k^{\cdot l}\gamma_l^\dagger + \bar{A}_k^{\cdot l5}\gamma_5^\dagger\gamma_l^\dagger , \tag{9.2}$$

it follows that

$$\Sigma_k^\dagger A = A(\bar{A}_k^{\cdot pq}\gamma_q\gamma_p + \bar{A}_k^{\cdot 5}\gamma_5 + \bar{A}_k + \bar{A}_k^{\cdot l}\gamma_l + \bar{A}_k^{\cdot l5}\gamma_5\gamma_l) . \tag{9.3}$$

Substituting this equation into (9.1), we have

$$\left.\begin{aligned} \nabla_k u_j = \psi^\dagger A \{ & (\bar{A}_k^{\cdot pq}\gamma_q\gamma_p + \bar{A}_k^{\cdot 5}\gamma_5 + \bar{A}_k + \bar{A}_k^{\cdot l}\gamma_l + \bar{A}_k^{\cdot l5}\gamma_5\gamma_l)\gamma_j \\ & + \gamma_j(A_k^{\cdot pq}\gamma_p\gamma_q + A_k^{\cdot 5}\gamma_5 + A_k + A_k^{\cdot l}\gamma_l + A_k^{\cdot l5}\gamma_l\gamma_5)\}\psi . \end{aligned}\right\} \tag{9.4}$$

Separating $A_k^{\cdot pq}$ etc. into real and imaginary parts, i. e.

$$A_k^{\cdot pq} = a_k^{\cdot pq} + ib_k^{\cdot pq} , \quad A_k^{\cdot 5} = a_k^{\cdot 5} + ib_k^{\cdot 5} , \quad A_k = a_k + ib_k ,$$
$$A_k^{\cdot l} = a_k^{\cdot l} + ib_k^{\cdot l} , \quad A_k^{\cdot l5} = a_k^{\cdot l5} + ib_k^{\cdot l5} ,$$

(9.4) becomes as follows:

$$\left.\begin{aligned} \nabla_k u_j = & 4a_{kj}^{\cdot\cdot q}u_q + 2a_k u_j + 2(a_{kj}M + a_{kj}^{\cdot\cdot 5}N) \\ & + 2i(b_k^{\cdot pq}u_{jpq} + b_k^{\cdot 5}u_{j5} + b_k^{\cdot l}u_{jl} + b_k^{\cdot l5}u_{jl5}) , \end{aligned}\right\} \tag{9.5}$$

which is the differential equation for $u_j \equiv \psi^\dagger A\gamma_j\psi$ when ψ is the solution of (8.1). Similarly, we have

$$\left.\begin{aligned} \nabla_k u_{j5} = & 4a_{kj}^{\cdot\cdot q}u_{q5} + 2a_k u_{j5} - 2a_k^{\cdot l}u_{jl5} + 2a_k^{\cdot l5}u_{jl} \\ & + 2i(b_k^{\cdot pq}u_{jpq5} - b_k^{\cdot 5}u_j - b_{kj}N + b_{kj}^{\cdot\cdot 5}M) , \end{aligned}\right\} \tag{9.6}$$

$$\nabla_k \rho_j = 4A_{kj}^{\cdot\cdot q}\rho_q + 2A_k\rho_j + 2A_k^{\cdot p}\rho_{jp} + 2A_k^{\cdot p5}\rho_{jp5} , \tag{9.7}$$

$$\nabla_k M = 2(ib_k^{\cdot pq}u_{pq} + a_k M + a_k^{\cdot 5}N + a_k^{\cdot l}u_l + ib_k^{\cdot l5}u_{l5}) , \tag{9.8}$$

$$\nabla_k N = 2(ib_k^{\cdot pq}u_{pq5} + a_k N - a_k^{\cdot 5}M - ib_k^{\cdot l}u_{l5} + a_k^{\cdot l5}u_l) . \tag{9.9}$$

But, on the other hand, we saw that(1) the determination of ψ in

(1) T. Sibata; This Journal, **8** (1938), 186.

spin space is equivalent to that of u_j, u_{j5}, ρ_j, M, and N, in vector space. So we have the result: *the fundamental equation for* ψ (9.1) *can be replaced by the equations* (9.5)–(9.9).

§ 10. Simplification of the form of the fundamental equation for ψ.

Theorem. *Assuming that* $M^2+N^2\neq 0$ *the general fundamental equation for* ψ

$$\nabla_k\psi=\sum_k\psi\,, \tag{10.1}$$

where $\sum_k$ *is any 4–4 matrix, can be reduced to the form:*

$$\nabla_k\psi=A_k^{\cdot pq}\gamma_p\gamma_q\psi \qquad (A_k^{\cdot qp}=-A_k^{\cdot pq}) \tag{10.2}$$

where $A_k^{\cdot pq}$ *are all real, in which case* M *and* N *are constant.*

Proof. By theorem 7, when $M^2+N^2\neq 0$, the equation (10.1) is reduced to the form:

$$\nabla_k\psi=(A_k^{\cdot pq}\gamma_p\gamma_q+A_k^{\cdot 5}\gamma_5+A_k)\psi \qquad (A_k^{\cdot pq}=-A_k^{\cdot qp})\,, \tag{10.3}$$

where $A_k^{\cdot pq}$, $A_k^{\cdot 5}$, and A_k are real and are determined in a unique way. Then, by (8.7) and (8.8), the condition for integrability of (10.3) becomes

$$\left\{\left(\frac{1}{8}K_{kj}^{\cdot\cdot pq}+\nabla_{[j}A_{k]}^{\cdot pq}+4A_{[k}^{\cdot pl}A_{j]l}^{\cdot\cdot q}\right)\gamma_p\gamma_q+\nabla_{[j}A_{k]}^{\cdot 5}\gamma_5+\nabla_{[j}A_{k]}\right\}\psi=0\,.$$

But as the coefficients of $\gamma_{[p}\gamma_{q]}\psi$, $\gamma_5\psi$, and ψ, of the equation above are real, by theorem 7 it must follow that

$$\frac{1}{8}K_{kj}^{\cdot\cdot pq}+\nabla_{[j}A_{k]}^{\cdot pq}+4A_{[k}^{\cdot pl}A_{j]l}^{\cdot\cdot q}=0\,. \tag{10.4}$$

$$\nabla_{[j}A_{k]}=0\,, \qquad \nabla_{[j}A_{k]}^{\cdot 5}=0\,. \tag{10.5}$$

From (10.5), A_k and $A_k^{\cdot 5}$ must be gradient vectors, i. e.

$$A_k=\frac{\partial a}{\partial x^k}\,, \qquad A_k^{\cdot 5}=\frac{\partial b}{\partial x^k}\,,$$

where a and b are any scalars. If we put[1]

$$\psi=\frac{1}{2}e^a\{e^{ib}(1+\mathring{\gamma}_5)+e^{-ib}(1-\mathring{\gamma}_5)\}\psi'\equiv e^a\{\cos b+i\mathring{\gamma}_5\sin b\}\psi', \tag{10.6}$$

(1) See Note 6.

(10.3) becomes

$$\nabla_k \psi' = A_k^{\cdot pq} \gamma_p \gamma_q \psi'.$$

Replacing ψ' by ψ, we have

$$\nabla_k \psi = A_k^{\cdot pq} \gamma_p \gamma_q \psi ,$$

the condition for integrability being given by (10.4).

Further, from (9.8) and (9.9) we have

$$\nabla_k M = 0 , \qquad \nabla_k N = 0 .$$

So the theorem is proved.

When the fundamental equation (10.1) is reduced to (10.2), because of $M^2+N^2=$constant, (9.6)–(9.9) become

$$\nabla_k \overset{a}{\lambda}_j = 4A_{kjl} \overset{a}{\lambda}{}^l \quad (a=1, \ldots, 4), \quad \nabla_k M = 0., \quad \nabla_k N = 0 , \quad (10.7)$$

where $\overset{a}{\lambda}_j$'s are defined by the equation (4.8). So, taking account of the result in § 9, we are led to the conclusion that *the wave geometry characterized by* ψ *and the differential equation for* ψ *can be replaced by the "geometry of vectors" satisfying the relation* (10.7).

Notes.

Note 1.

Since

$$D = \widetilde{A}^{-1} B \quad \text{(loc. cit., 182), where} \quad B = -C\gamma_5 ,$$

we have

$$D^{-1} = B^{-1}\widetilde{A} = -\gamma_5^{-1} C^{-1} \widetilde{A}$$

$$= -\eta \gamma_5 C^{-1} \widetilde{A} \quad \text{(because of } \gamma_5 \gamma_5 = \eta) ,$$

hence

$$CD^{-1} = -\eta \widetilde{\gamma}_5 \widetilde{A}. \qquad (※)$$

Therefore, the first term of (3.12) is rewritten as

$$\widetilde{\psi} C D^{-1} \overline{\psi} = -\eta \widetilde{\psi} \widetilde{\gamma}_5 \widetilde{A} \overline{\psi}$$

$$= -\eta \psi^{\dagger} A \gamma_5 \psi \quad \text{(by transposing)} ,$$

and the second term is rewritten as

$$\widetilde{\psi} C \gamma_5 D^{-1} \overline{\psi} = \widetilde{\psi} \widetilde{\gamma}_5 C D^{-1} \overline{\psi} \quad (\text{since } \widetilde{\gamma}_5 = C \gamma_5 C^{-1})$$

$$= -\eta \widetilde{\psi} \widetilde{\gamma}_5 \widetilde{\gamma}_5 \widetilde{A} \overline{\psi} \quad \text{(by (※))}$$

$$= -\widetilde{\psi} \widetilde{A} \overline{\psi} \quad (\text{since } \widetilde{\gamma}_5 \widetilde{\gamma}_5 = \eta)$$

$$= -\psi^{\dagger} A \psi , \quad \text{(by transposing)}$$

so that $$(\widetilde{\psi}CD^{-1}\bar{\psi})^2+(\widetilde{\psi}C\gamma_5 D^{-1}\bar{\psi})^2=(\psi^\dagger A\psi)^2+(\psi^\dagger A\gamma_5\psi)^2$$

Note 2.

In the calculation we need the expansion of $\gamma_l D^{-1}\bar{\psi}$. For this, since $\bar{\gamma}_l=D\gamma_l D^{-1}$, we have

$$\gamma_l D^{-1}\bar{\psi}=D^{-1}\bar{\gamma}_l\bar{\psi}\,,$$

or, substituting the conjugate of (4.2) into the above and using the relation:

$$\bar{D}D=-I,\quad\text{or}\quad D=-\bar{D}^{-1},\quad(\text{loc. cit., 183}),$$

we have

$$\gamma_l D^{-1}\bar{\psi}=\frac{1}{M^2+N^2}\Big[\{(Mu_l+Nu_{l5})+(Nu_l-Mu_{l5})\gamma_5\}D^{-1}\bar{\psi}\\ -\bar{\rho}_l(N-M\gamma_5)\psi\Big].$$

Then, (4.13) follows from (4.2).

Note 3.

The relation (4.15) can be obtained in the same way as (4.14);

putting $$\rho_{lm}=f_{ab}\overset{a}{\lambda}_l\overset{b}{\lambda}_m\qquad(f_{ab}=-f_{ba}),$$

from the relations

$$\begin{aligned}u^k\rho_{kj}&=u^k\widetilde{\psi}C\gamma_{[k}\gamma_{j]}\psi=u^k\widetilde{\psi}C(g_{kj}-\gamma_j\gamma_k)\psi\\&=u_j\widetilde{\psi}C\psi-\widetilde{\psi}C\gamma_j(M+N\gamma_5)\psi\quad(\text{because } u^k\gamma_k\psi=(M+N\gamma_5)\psi)\\&=-M\rho_j,\qquad\qquad(\text{because } \widetilde{\psi}C\psi=0,\ \widetilde{\psi}C\gamma_j\gamma_5\psi=0),\end{aligned}$$

and $$\begin{aligned}u^k\rho_{kj}{}^*&=\frac{1}{2}u^k\widetilde{\psi}C\epsilon_{kjpq}\gamma^p\gamma^q\psi=iu^k\widetilde{\psi}C\gamma_5\gamma_{[k}\gamma_{j]}\psi\\&=iu^k\widetilde{\psi}C\gamma_5(g_{kj}-\gamma_j\gamma_k)\psi=-i\widetilde{\psi}C\gamma_5\gamma_j(M+N\gamma_5)\psi\\&=-iN\rho_j\,,\end{aligned}$$

it must follow that

$$f_{1b}\overset{b}{\lambda}_j=-M(-i\overset{3}{\lambda}_j+\overset{4}{\lambda}_j)\,,$$

and $$\frac{1}{2}\overset{1}{\lambda}{}^k\epsilon_{kjpq}f_{ab}\overset{a}{\lambda}{}^p\overset{b}{\lambda}{}^q=-iN(-i\overset{3}{\lambda}_j+\overset{4}{\lambda}_j)$$

or $$\frac{1}{2}\epsilon_{ab1d}\overset{d}{\lambda}_j f_{ab}=-iN(-i\overset{3}{\lambda}_j+\overset{4}{\lambda}_j)$$

i. e. $$f_{12}=0\,,\quad f_{13}=iM\,,\quad f_{14}=-M$$

and $$f_{34}=0\,,\quad f_{42}=-N\,,\quad f_{23}=-iN\,.$$

So that, substituting the values of f_{ab} into the expansion $\rho_{lm}=f_{ab}\overset{a}{\lambda}_l\overset{b}{\lambda}_m$,

we have $$\rho_{lm}=2M(i\overset{[1}{\lambda}_l\overset{3]}{\lambda}_m-\overset{[1}{\lambda}_l\overset{4]}{\lambda}_m)+2N(\overset{[2}{\lambda}_l\overset{4]}{\lambda}_m-i\overset{[2}{\lambda}_l\overset{3]}{\lambda}_m)\,.$$

Note 4.

As the generalized equation of (6.2), we have obtained the equation

$$(\phi\gamma^\lambda\psi)\gamma_\lambda\psi=(\phi\psi)\psi\,, \qquad (※)$$

where ϕ is any 4–1 matrix $\phi\equiv(\phi_1, \phi_2, \phi_3, \phi_4)$ (cf. loc. cit., 174). We can see that, if we take ϕ suitably, $v^\lambda=\phi\gamma^\lambda\psi$ and $v=\phi\psi$ may give the general solution of (6.1). However, in this section, we are looking for the solution of (6.1) in terms of ψ only, avoiding the complication of using ψ and ϕ.

If we put $\phi=\tilde{\chi}C$, where χ is a certain 1–4 matrix $\chi\equiv\begin{pmatrix}\chi_1\\ \chi_2\\ \chi_3\\ \chi_4\end{pmatrix}$, the relation (※) becomes

$$(\tilde{\chi}C\gamma^\lambda\psi)\gamma_\lambda\psi=(\tilde{\chi}C\psi)\psi\,, \qquad (\lambda=1,\ldots\ldots,5)$$

which is a spin-invariant. Corresponding to the relation above, we can also prove the following identity:

$$(\tilde{\chi}C\gamma^l\chi)\gamma_l\psi=2\{(\tilde{\chi}C\psi)+(\tilde{\chi}C\gamma_5\psi)\gamma_5\}\chi\,. \qquad (l=1,\ldots\ldots,4)$$

Note 5.

The relation (8.6) is obtained from the identity (7.2). For, from (7.3), we have

$$\nabla_{[j}\nabla_{k]}\psi=\left(-\frac{\partial\Gamma_{[k}}{\partial x^{j]}}+\Gamma_{[j}\Gamma_{k]}\right)\psi\,,$$

but, on the other hand, from the condition for integrability of the relation:

$$\frac{\partial\gamma_h}{\partial x^k}=\{{}^{\,l}_{hk}\}\gamma_l+\Gamma_k\gamma_h-\gamma_h\Gamma_k\,,$$

we have $$-\frac{\partial\Gamma_{[k}}{\partial x^{j]}}+\Gamma_{[j}\Gamma_{k]}=\frac{1}{8}K_{jk}^{\cdot\cdot pq}\gamma_p\gamma_q$$

where $K_{jk}^{\cdot\cdot pq}$ is the curvature tensor: $K_{jkp}{}^{q}=2\left(\frac{\partial\left\{{q \atop p[j}\right\}}{\partial x^{k]}}-\left\{{q \atop h[j}\right\}\left\{{h \atop |p|k]}\right\}\right)$, so that (8.6) is obtained. (Cf. E. Schrödinger; Berl. Ber. (1932), 105).

Note 6.

When ψ and $\bar{\psi}$ are related by (10.6), among the spin-invariant quantities constructed from ψ and ψ' the following relations hold good:

$$M=e^{2a}(\cos 2bM'+\sin 2bN') \qquad u^{l}=e^{2a}u'^{l}$$

$$N=e^{2a}(\cos 2bN'-\sin 2bM').$$

Note 7.

In the case when

$$M\equiv\psi^{\dagger}A\psi=0\,,\qquad N\equiv\psi^{\dagger}A\gamma_5\psi=0\,, \tag{N. 1}$$

we shall find the bases of any 1–4 matrix in terms of ψ. For this, we have

Theorem:—*When* $M=0$, *the matrices*

$$\psi,\quad D^{-1}\bar{\psi},\quad \phi,\quad D^{-1}\bar{\phi}, \tag{N. 2}$$

where

$$\phi\equiv w^{l}\gamma_{l}\psi\,, \tag{N. 3}$$

form the bases of 1–4 matrix so far as

$$w^{l}u_{l}\neq 0\,, \tag{N. 4}$$

where

$$u_{l}\equiv\psi^{\dagger}A\gamma_{l}\psi\,.$$

N. B. If $\psi\neq 0$, there exists a vector w^{l} satisfying $w^{l}u_{l}\neq 0$. For u_{l} cannot be zero when, and only when, $\psi\neq 0$, so we can take w^{l} such that $w^{l}u_{l}\neq 0$.

Proof. To prove the theorem it is sufficient to show that the determinant of the matrix

$$P\equiv(\psi,\ D^{-1}\bar{\psi},\ \phi,\ D^{-1}\bar{\phi}) \tag{N. 5}$$

does not vanish identically, the determinant $|P|$ being calculated as follows. Take a matrix B satisfying $\tilde{\gamma}_{\lambda}=B\gamma_{\lambda}B^{-1}$. Then, in consequence of $\tilde{P}=\begin{pmatrix}\tilde{\psi}\\ \psi^{\dagger}\tilde{D}^{-1}\\ \tilde{\phi}\\ \phi^{\dagger}\tilde{D}^{-1}\end{pmatrix}$, we have

$$\tilde{P}BP=\begin{pmatrix} \tilde{\Psi}B\Psi & \tilde{\Psi}BD^{-1}\bar{\Psi} & \tilde{\Psi}B\phi & \tilde{\Psi}BD^{-1}\bar{\phi} \\ \Psi^{\dagger}\tilde{D}^{-1}B\Psi & \Psi^{\dagger}\tilde{D}^{-1}B\tilde{D}^{-1}\bar{\Psi} & \Psi^{\dagger}\tilde{D}^{-1}B\phi & \Psi^{\dagger}\tilde{D}^{-1}BD^{-1}\bar{\phi} \\ \tilde{\phi}B\Psi & \tilde{\phi}BD^{-1}\bar{\Psi} & \tilde{\phi}B\phi & \tilde{\phi}BD^{-1}\bar{\phi} \\ \phi^{\dagger}\tilde{D}^{-1}B\Psi & \phi^{\dagger}\tilde{D}^{-1}BD^{-1}\bar{\Psi} & \phi^{\dagger}\tilde{D}^{-1}B\phi & \phi^{\dagger}\tilde{D}^{-1}BD^{-1}\bar{\phi} \end{pmatrix}. \tag{N. 6}$$

But B and D have the following properties[1]

$$\tilde{B}=-B, \tag{N. 7}$$

$$D=\tilde{A}^{-1}B, \quad \text{or} \quad D^{-1}=B^{-1}\tilde{A}, \tag{N. 8}$$

$$\tilde{D}^{-1}B=-A, \tag{N. 9}$$

and

$$\tilde{D}^{-1}BD^{-1}=-AD^{-1}=-AB^{-1}\tilde{A}=-B^{\dagger}, \tag{N. 10}$$

because of

$$AB^{-1}\tilde{A}=B^{\dagger}. \tag{N. 11}$$

Therefore, from (N. 7)-(N. 11), it follows that

$$\tilde{\Psi}B\Psi=0, \quad \tilde{\phi}B\phi=0,$$

$$\Psi^{\dagger}\tilde{D}^{-1}BD^{-1}\bar{\Psi}=-\Psi^{\dagger}B^{\dagger}\bar{\Psi}=-\overline{(\tilde{\Psi}\tilde{B}\Psi)}=0, \quad \phi^{\dagger}\tilde{D}^{-1}BD^{-1}\phi=0,$$

$$\tilde{\Psi}BD^{-1}\bar{\Psi}=\tilde{\Psi}\tilde{A}\bar{\Psi}=\Psi^{\dagger}A\Psi, \quad \Psi^{\dagger}\tilde{D}^{-1}B\Psi=-\Psi^{\dagger}A\Psi,$$

$$\tilde{\Psi}BD^{-1}\bar{\phi}=\tilde{\Psi}\tilde{A}\bar{\phi}=\Phi^{\dagger}A\Psi, \quad \Psi^{\dagger}\tilde{D}^{-1}B\phi=-\Psi^{\dagger}A\phi,$$

$$\Psi^{\dagger}\tilde{D}^{-1}BD^{-1}\bar{\phi}=-\Psi^{\dagger}B^{\dagger}\bar{\phi}=-\overline{(\tilde{\Psi}\tilde{B}\phi)}=\overline{(\tilde{\Psi}B\phi)}.$$

Hence, using the relation stated above, (N. 6) is rewritten in the form

$$\tilde{P}BP=\begin{pmatrix} 0 & \Psi^{\dagger}A\Psi & \tilde{\Psi}B\phi & \phi^{\dagger}A\Psi \\ -\Psi^{\dagger}A\Psi & 0 & -\Psi^{\dagger}A\phi & \overline{(\tilde{\Psi}B\phi)} \\ -\tilde{\Psi}B\phi & \Psi^{\dagger}A\phi & 0 & \phi^{\dagger}A\phi \\ -\phi^{\dagger}A\Psi & -\overline{(\tilde{\Psi}B\phi)} & -\phi^{\dagger}A\phi & 0 \end{pmatrix},$$

i. e. $\tilde{P}BP$ is an antisymmetric matrix, and the determinant is

$$|P|\,|B|\,|P|=\{(\Psi^{\dagger}A\Psi)(\phi^{\dagger}A\phi)-(\tilde{\Psi}B\phi)\overline{(\tilde{\Psi}B\phi)}-(\Psi^{\dagger}A\phi)(\phi^{\dagger}A\Psi)\}^{2}. \tag{N. 12}$$

If we normalize B such that $|B|=\pm 1$, from (N. 12), we have

$$|P|=(\Psi^{\dagger}A\Psi)(\phi^{\dagger}A\phi)-(\tilde{\Psi}B\phi)\overline{(\tilde{\Psi}B\phi)}-(\Psi^{\dagger}A\phi)(\phi^{\dagger}A\Psi). \tag{N. 13}$$

(1) (N. 8) and (N. 11) were given in the previous paper (loc. cit., 182) and in Pauli's paper; Annales de l'Institut H. Poincaré **6** (1936), 121. And (N. 9) is obtained by taking the transposed matrix of the latter of (N.8), because of $\tilde{B}=-B$.

Substituting the relations (N. 1) and (N. 3), i. e.

$$M\equiv\psi^\dagger A\psi=0\,,\qquad \phi\equiv w^l\gamma_l\psi\,,$$

into (N. 13), we have

$$|P|=-(\psi^\dagger Aw^l\gamma_l\psi)(\psi^\dagger A\bar{w}^m\gamma_m\psi)\,,$$

or
$$|P|=-(w^lu_l)(\bar{w}^mu_m)\,.$$

So that $|P|$ cannot vanish indentically so far as

$$w^lu_l\neq 0\,. \qquad \text{Q. E. D.}$$

Next we shall show that actually when $M=N=0$ the matrices (3.11) cannot be the bases of 1–4 matrix; in this case the matrices (3.11) are linearly dependent. For this, we have

Theorem :—*When*

$$M=0\,,\qquad N=0\,,$$

among the matrices ψ, $\gamma_5\psi$, *and* $D^{-1}\bar{\psi}$, *there exists the following relation :*

$$\gamma_5\psi=a\psi+bD^{-1}\bar{\psi}\,, \tag{N. 14}$$

where a *and* b *are certain scalars satisfying the relations*

$$u_{l5}=au_l\,,\qquad \rho_l=-bu_l\,. \tag{N. 15}$$

Proof. Expand the matrix $\gamma_5\psi$ by the bases (N.2), i. e.

$$\gamma_5\psi=a\psi+bD^{-1}\bar{\psi}+cw^l\gamma_l\psi+d\bar{w}^l\gamma_lD^{-1}\bar{\psi}\,. \tag{N. 16}$$

To determine a, b, c, and d, multiply (N. 16) by the matrix $\psi^\dagger A$, then we have

$$N=aM+b\psi^\dagger AD^{-1}\bar{\psi}+cw^lu_l+d\bar{w}^l\psi^\dagger A\gamma_lD^{-1}\bar{\psi}\,; \tag{N. 17}$$

but, because of $AD^{-1}=B^\dagger$ (by (N. 10)), we have

$$\psi^\dagger AD^{-1}\bar{\psi}=(\overline{\tilde{\psi}\tilde{B}\psi})=0\,,\quad \psi^\dagger A\gamma_lD^{-1}\bar{\psi}=\psi^\dagger\gamma_l{}^\dagger AD^{-1}\bar{\psi}=(\overline{\tilde{\psi}\tilde{\gamma}_l\tilde{B}\psi})=0\,,$$

hence, from (N. 17), using (N. 1) and (N. 4), we have

$$0=c\,; \tag{N. 18}$$

further, multiplying (N. 16) by $\tilde{\psi}B$ and using (N. 1) and (N, 18), we have

$$0=b\tilde{\psi}BD^{-1}\overline{\psi}+d\overline{w}^l\tilde{\psi}B\gamma_l D^{-1}\overline{\psi}\,, \tag{N. 19}$$

but, because of $BD^{-1}=\tilde{A}$ (by (N. 8)), we have

$$\tilde{\psi}BD^{-1}\overline{\psi}=\tilde{\psi}\tilde{A}\overline{\psi}=\psi^\dagger A\psi=0\,,$$

$$\tilde{\psi}B\gamma_l D^{-1}\overline{\psi}=\tilde{\psi}\tilde{\gamma}_l BD^{-1}\overline{\psi}=\tilde{\psi}\tilde{\gamma}_l\tilde{A}\overline{\psi}=\psi^\dagger A\gamma_l\psi=u_l\,,$$

hence, from (N. 19), using (N. 4), it follows that

$$0=d\,, \tag{N. 20}$$

so that we have the form of (N. 14), i. e.

$$\gamma_5\psi=a\psi+bD^{-1}\overline{\psi}\,. \tag{N. 21}$$

Multiplying (N. 21) by $\psi^\dagger A\gamma_l$, we have

$$u_{l5}=au_l+b\psi^\dagger A\gamma_l D^{-1}\overline{\psi}\,,$$

from which, because of the relation $\psi^\dagger A\gamma_l D^{-1}\overline{\psi}=\psi^\dagger\gamma_l{}^\dagger AD^{-1}\overline{\psi}=\psi^\dagger\gamma_l{}^\dagger B^\dagger\overline{\psi}=0$, it follows that

$$u_{l5}=au_l\,,$$

from which a is determined. Similarly, multiplying (N. 21) by $\tilde{\psi}B\gamma_l$, we have

$$\tilde{\psi}B\gamma_l\gamma_5\psi=b\tilde{\psi}B\gamma_l D^{-1}\overline{\psi}\,,$$

from which, in consequence of the relations

$$\tilde{\psi}B\gamma_l\gamma_5\psi=-\tilde{\psi}C\gamma_5\gamma_l\gamma_5\psi=-\rho_l\,,$$

and $$\tilde{\psi}B\gamma_l D^{-1}\overline{\psi}=\tilde{\psi}\tilde{\gamma}_l BD^{-1}\overline{\psi}=\tilde{\psi}\tilde{\gamma}_l\tilde{A}\overline{\psi}=u_l\,,$$

it follows that

$$\rho_l=-bu_l\,,$$

from which b is determined. Q. E. D.

From the theorem above, we have

Theorem :—*When* $M=N=0$, *there exist the relations*

$$u_{l5}=au_l\,,\qquad \rho_l=-bu_l\,,$$

where a and b satisfy the following equation

$$-a^2+b\bar{b}=1\,. \tag{N. 22}$$

Proof. Proof of the first part of the theorem being the same as in the preceding theorem, we shall only show that (N. 22) holds good. From (N. 14) and (N. 15), we have

$$u_l\gamma_5\psi = u_{l5}\psi - \rho_l D^{-1}\bar{\psi} . \qquad \text{(N. 23)}$$

Multiplying (N. 23) by $\psi^\dagger A\gamma_p\gamma_5$, we have

$$-u_l u_p = u_{l5}u_{p5} - \rho_l \psi^\dagger A\gamma_p\gamma_5 D^{-1}\bar{\psi} ,$$

from which, because of

$$\psi^\dagger A\gamma_p\gamma_5 D^{-1}\bar{\psi} = \psi^\dagger\gamma_p{}^\dagger\gamma_5{}^\dagger A D^{-1}\bar{\psi} = \psi^\dagger\gamma_p{}^\dagger\gamma_5{}^\dagger B^\dagger\bar{\psi} = \psi^\dagger\gamma_p{}^\dagger C^\dagger\bar{\psi} = \bar{\rho}_p ,$$

we have

$$-u_l u_p = u_{l5}u_{p5} - \rho_l\bar{\rho}_p \qquad \text{(N. 24)}$$

Substituting (N. 15) into (N. 24), we have

$$-u_l u_p = a^2 u_l u_p - b\bar{b} u_l u_p ,$$

or, in consequence of $u_l u_p \neq 0$,

$$1 = b\bar{b} - a^2.$$

So that the theorem is proved. (Q. E. D.)

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