Prime Ideals in Boolean Rings.

By

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(Received January 21, 1939.)

1. Introduction. Boolean rings may be regarded as abstract commutative rings in which every element is idempotent.[(1)] The ideal theory of these rings can be developed easily as a special instance of the ideal theory of general rings,[(2)] and the important results of the algebraic theory of Boolean rings developed by M. H. Stone[(3)] will follow from the general ideal theory. In the present paper we shall be concerned primarily with the properties of the prime ideals[(4)] in Boolean rings which involve the *well-ordering hypothesis*.

The following lemma has been found valid for idempotent principal ideals in general commutative rings,[(5)] and Stone[(6)] has proved it in Boolean rings. We shall prove it here merely for its importance for further investigation.

Lemma. *If* $\mathfrak{a}(\neq \mathfrak{o})$ *is a principal ideal in a Boolean ring* A, A *can be represented as the direct sum* $\mathfrak{a}+\mathfrak{a}'$, *where* $\mathfrak{a}'$ *is the orthocomplement of* $\mathfrak{a}$.

From the definition of a principal ideal, it is evident that $\mathfrak{a}$ is the subclass consisting of all the elements b such that, for a fixed non-zero element a, $ab=b$. If we denote the subclass consisting of all the elements c orthogonal to the element a by $\mathfrak{a}'$, then $\mathfrak{a}'$ is an ideal orthogonal to $\mathfrak{a}$. We let d be an arbitrary element in A and write

$$d=ad+d-ad\,.$$

(1) M. H. Stone, The Theory of Representations for Boolean Algebras, Trans. of Amer. Math. Soc. **40** (1936), 39.

(2) W. Krull, Idealtheorie in Ringen ohne Endlichkeitsbedingung, Math. Annalen **101** (1929), 729.

(3) Stone, loc. cit., 100–105.

(4) Ibid., 100. W. Krull, loc. cit., 735.

(5) S. Mori, Über allgemeine Multiplikationsringe, this Journal **4** (1934), I. *A* Boolean ring is a special multiplicative ring.

(6) Stone, loc. cit., 63.

Then we have $a.ad=ad$, $a(d-ad)=ad-a^2d=0$, and thus conclude that ad is an element in $\mathfrak{a}$ and $d-ad$ an element in $\mathfrak{a}.'$ Since d is arbitrary, we have $\mathfrak{e}=\mathfrak{a}+\mathfrak{a}'$.

2. Special ideals. The relations between special ideals and prime ideals are given in the following theorems.

Theorem 1. *If $\mathfrak{p}$ is a prime ideal in a Boolean ring, then the following properties are equivalent:*

(1) *$\mathfrak{p}$ is simple:*[7]

(2) *$\mathfrak{p}$ is normal:*[8]

(3) *$\mathfrak{p}$ is a zero-divisor:*[9]

(4) *the intersection of any infinite number of ideals, none of which is contained in $\mathfrak{p}$, is also not contained in $\mathfrak{p}$.*

The equivalence of properties (1) and (2) is evident from the property of prime ideal. Property (3) follows at once from (1), or (2). To prove (4), we observe first that if $\mathfrak{p}$ is simple, then the orthocomplement $\mathfrak{p}'$ must be a non-zero ideal; and by the property of prime ideal, we see that $\mathfrak{p}'$ is a two-element Boolean ring. We now let b be an element not contained in $\mathfrak{p}$, and p' the non-zero element in $\mathfrak{p}'$; then we have $bp' \not\equiv 0\ (\mathfrak{p})$, $bp'=p'$, and it is evident that every ideal not contained in $\mathfrak{p}$ must contain $\mathfrak{p}'$. From this result we can obtain property (4).

If $\mathfrak{p}$ is a zero-divisor, then we have $\mathfrak{p}\mathfrak{b}=\mathfrak{o}$, $\mathfrak{b}\neq\mathfrak{o}$, and $\mathfrak{b}$ is a two-element Boolean ring from the property of prime ideal. To establish $\mathfrak{p}+\mathfrak{b}=\mathfrak{e}$, we let b be an atomic element in $\mathfrak{b}$. Then we have $cb\neq o$, $cb=b$, for any element c not contained in $\mathfrak{p}$, since $\mathfrak{p}$ is a prime ideal and b an atomic element. Therefore we have $(c-b)b=0$, and hence we see that $c-b$ is an element in $\mathfrak{p}$. It follows from this result that c must be an element in $\mathfrak{p}+\mathfrak{b}$, and hence that $\mathfrak{p}$ is simple. We shall now show that the final property implies (1), (2), and (3). By the fundamental proposition[10] of ideal arithmetic, the intersection of all the prime ideals in A is $\mathfrak{o}$. But we see, by virtue of property (4), that the intersection $\mathfrak{d}$ of all the prime ideals outside $\mathfrak{p}$ is a non-zero ideal, and that $\mathfrak{p}\mathfrak{d}=\mathfrak{o}$.

Immediate consequences of Lemma and Theorem I are the following propositions:

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(7), (8) Ibid., 63.

(9) The ideal $\mathfrak{a}$ is said to be a *zero-divisor* if $\mathfrak{a}\mathfrak{b}=\mathfrak{o}$ for a non-zero ideal $\mathfrak{b}$.

(10) Mori, loc. cit., 17. Stone, loc. cit., 105.

Theorem 2. *In order that a Boolean ring A shall contain an atomic element, it is necessary and sufficient that a prime ideal in A be normal.*

Theorem 2′. *In order that a Boolean ring A shall contain an atomic element, it is necessary and sufficient that there exist in A at least one element b such that the number of prime ideals not containing b is finite.*

Since the orthocomplement of a two-element ideal is a prime ideal, from Lemma, the necessity of the condition is evident. On the other hand, when an element b is not contained in only a finite number of prime ideals $\mathfrak{p}_1, \mathfrak{p}_2, \ldots . \mathfrak{p}_m$, it is evident that the orthocomplement $\mathfrak{a}'$ of the principal ideal $\mathfrak{a}=(b)$ must be contained in every $\mathfrak{p}_i (i=1, 2 \ldots . m)$, and that no other prime ideal contains $\mathfrak{a}'$, since by Lemma the ideal $\mathfrak{a}$ is simple. From the fundamental proposition of ideal arithmetic we have, therefore, that $\mathfrak{a}'$ is the intersection of all $\mathfrak{p}_i$. We now let $\mathfrak{a}_1'$ be the intersection of the prime ideals $\mathfrak{p}_2, \mathfrak{p}_3, \ldots . \mathfrak{p}_m$; then $\mathfrak{a}_1'$ is not contained in $\mathfrak{p}_1$ and hence we have $\mathfrak{a}\,\mathfrak{a}_1' \neq \mathfrak{o}$, $\mathfrak{a}\,\mathfrak{a}_1'\mathfrak{p}_1=\mathfrak{o}$. $\mathfrak{p}_1$ is thus a zero-divisor, and by virtue of Theorem 2 there exists an atomic element.

To obtain the condition for an ideal to be simple, we shall formulate the following theorem:

Theorem 3. *In order that an ideal $\mathfrak{a}$ in a Boolean ring A be simple, it is necessary and sufficient that there exist no prime ideal containing both the ideal $\mathfrak{a}$ and its orthocomplement $\mathfrak{a}'$.*

The necessity of the condition is evident. To establish the sufficiency, we assume that $\mathfrak{a}$ is not simple. Then we can find an element b such that $\mathfrak{a}+\mathfrak{a}'$ does not contain the element b. By the principle of transfinite induction, we can therefore prove the existence of a prime ideal $\mathfrak{p}$, which contains $\mathfrak{a}+\mathfrak{a}'$ but not the element b.[11]

3. Special Boolean rings. The chief task of the present section is to study the relations between the structure of special Boolean rings and the property of prime ideals.

Theorem 4. *In order that every ideal $\mathfrak{a}$ in a Boolean ring A be simple, it is necessary and sufficient that every prime ideal in A be a zero-divisor, or simple, or normal.*

If every ideal is simple, then every prime ideal is also simple, and Theorem 1 shows that every prime ideal is normal, or a zero-divisor.

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(11) Krull, loc. cit., 735. Stone, loc. cit., 101.

The condition is thus necessary. To establish the sufficiency, we now let every prime ideal in A be simple, $\mathfrak{a}$ be an arbitrary ideal in A and $\mathfrak{a}'$ the orthocomplement of the ideal $\mathfrak{a}$. If $\mathfrak{a}$ is not simple, then from Theorem 3 we can find a prime ideal $\mathfrak{p}$ such that $\mathfrak{p}$ is a divisor of $\mathfrak{a}+\mathfrak{a}'$. Since $\mathfrak{p}$ is simple, we have $\mathfrak{p}\mathfrak{b}=\mathfrak{o}$, $\mathfrak{b}\neq\mathfrak{o}$ from Theorem I. The relation $\mathfrak{p}\mathfrak{b}=\mathfrak{o}$ implies $\mathfrak{a}\mathfrak{b}=\mathfrak{o}$, and we have that $\mathfrak{a}'$ must contain the ideal $\mathfrak{b}$. Therefore $\mathfrak{b}$ is contained in $\mathfrak{p}$. On the other hand, $\mathfrak{p}$ does not contain $\mathfrak{b}$, by $\mathfrak{p}\mathfrak{b}=\mathfrak{o}$, which implies a contradiction. Hence an arbitrary ideal $\mathfrak{a}$ must be simple, and the condition also is sufficient.

From Theorem I and Theorem 4, we have at once

Theorem 4′. *In order that every ideal in a Boolean ring A be normal, it is necessary and sufficient that every prime ideal in A be a zero-divisor, or simple, or normal.*

Theorem 5. *In order that every ideal of a Boolean ring A be principal, it is necessary and sufficient that A have unit e and that every prime ideal in A be simple.*

If every ideal in A is principal, and if $\mathfrak{a}$ is an arbitrary ideal in A, then $\mathfrak{a}$ is simple from Lemma, and the orthocomplement $\mathfrak{a}'$ of $\mathfrak{a}$ is also principal. We now let a and a' be the units of $\mathfrak{a}$ and $\mathfrak{a}'$; then the element $e=a+a'$ is also the unit of the Boolean ring A. By Lemma, every ideal in A is simple, and hence every prime ideal in A must also be simple. If every prime ideal is simple, then every ideal in A is also simple, by virtue of Theorem 4. Hence we have $\mathfrak{a}+\mathfrak{a}'=\mathfrak{e}$ for an arbitrary ideal $\mathfrak{a}$, and we have $a+a'=e$ for the unit e, where a and a' are the elements of $\mathfrak{a}$ and $\mathfrak{a}'$. Thus it is evident that $\mathfrak{a}=(a)$, $\mathfrak{a}'=(a')$.

4. Atomic basis. We shall now turn to the study of the existence of the atomic basis.

Theorem 6. *If the Boolean ring A has an atomic basis,*[12] *then every prime ideal in A is normal.*

If $\mathfrak{p}$ is an arbitrary prime ideal in A, then an element b is not contained in $\mathfrak{p}$, and b is represented as the sum of atomic elements. In this case, only one of those atomic elements in the representation is not contained in $\mathfrak{p}$. From Lemma we see thus that $\mathfrak{p}$ is a zero-divisor, and by Theorem I $\mathfrak{p}$ is normal.

Theorem 7. *In order that a Boolean ring A shall contain an atomic basis, it is necessary and sufficient that the orthocomplement of* *every principal ideal in A be represented as a finite product of prime ideals.*

For an arbitrary element a, we see from Lemma that the principal ideal $\mathfrak{a}=(a)$ is simple. By virtue of the existence of an atomic basis, the principal ideal $\mathfrak{a}$ is the direct sum of the two-element Boolean rings $\mathfrak{a}_i(i=1, 2, \ldots, m)$, and thus the orthocomplement $\mathfrak{a}'$ of $\mathfrak{a}$ is the intersection of the prime ideals $\mathfrak{p}_i(i=1, 2, \ldots, m)$, which is the orthocomplement of $\mathfrak{a}_i$.

If the orthocomplement $\mathfrak{a}'$ of an arbitrary principal ideal $\mathfrak{a}=(a)$ is the product of prime ideals $\mathfrak{p}_i(i=1, 2 \ldots, m)$, then, by the process described in the proof of Theorem 2′, $\mathfrak{p}_i(i=1, 2, \ldots, m)$ is simple, and the orthocomplement of $\mathfrak{p}_i$ is the two-element Boolean ring $\mathfrak{a}_i=(a_i)$. Therefore we have

$$\mathfrak{p}_i=\mathfrak{a}'+\mathfrak{a}_1+\cdots+\mathfrak{a}_{i-1}+\mathfrak{a}_{i+1}+\cdots+\mathfrak{a}_m, \quad \mathfrak{p}_i+\mathfrak{a}_i=\mathfrak{e},$$

$$\mathfrak{a}=\mathfrak{a}_1+\mathfrak{a}_2+\cdots+\mathfrak{a}_i+\cdots+\mathfrak{a}_m.$$

Hence it is true that the element a is the sum of the atomic elements $a_i(i=1, 2, \ldots, m)$. Class $\mathfrak{k}$, consisting of all the atomic elements in A, is therefore an atomic basis.

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(12) Stone, loc. cit., 50. A class $\mathfrak{k}$ of atomic elements is said to be an *atomic basis* if every non-zero element is the sum of elements in $\mathfrak{k}$.