Ring-Decomposition without Chain-Condition.

By

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In algebra, the decomposition of a ring in a direct sum of simple right ideals is discussed on the basis of "chain-condition" or "minimum-condition." Thus, a ring $\mathfrak{R}$ without radical, with minimum-condition for right ideals, is a direct sum of simple right ideals, i. e.

$$\mathfrak{R}=\mathfrak{a}_1+\mathfrak{a}_2+\cdots\cdots+\mathfrak{a}_n\,, \tag{1}$$

and there exist idempotents e_i $(i=1, 2, \ldots\ldots, n)$, such that

$$\mathfrak{a}_i=(e_i)_r\,, \qquad e_ie_j=0 \quad \text{when} \quad i\neq j\,,$$

and $$1=e_1+e_2+\cdots\cdots+e_n\,.$$ [1]

When the ring $\mathfrak{R}$ does not satisfy the minimum-condition, we cannot decompose $\mathfrak{R}$ in a direct sum of *simple* right ideals as in (1). Hence we must consider ring-decomposition from another point of view. Since the set $R_{\mathfrak{R}}$ of all right ideals is a lattice,[2] from the point of view of the lattice theory we can investigate the set of right ideals which are used for the decompositions of $\mathfrak{R}$.

For example, consider the case where $\mathfrak{R}$ without radical satisfies the minimum-condition. Then the decomposition (1) shows that $\mathfrak{R}$ is the join of right ideals $(\mathfrak{a}_1, \mathfrak{a}_2, \ldots\ldots, \mathfrak{a}_n)$. Let V be the set of n positive integers $1, 2, \ldots\ldots, n$; and let U be any subset of V, whose elements are $i_1, i_2, \ldots\ldots, i_\nu$. And write

$$\mathfrak{a}_U=\mathfrak{a}_{i_1}+\mathfrak{a}_{i_2}+\cdots\cdots+\mathfrak{a}_{i_\nu}\,.$$

Then $$\mathfrak{a}_{U_1}\frown\mathfrak{a}_{U_2}=(0) \qquad \text{when} \qquad U_1U_2=0\,.$$

And when V is a sum of mutually disjoint sets, i. e.

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(1) B. L. van der Waerden [1], 156–161. The numbers in square brackets refer to the list given at the end of this paper.

(2) J. v. Neumann [5], 4.

$$V = U_1 + U_2 + \cdots + U_m,$$

then we have a decomposition

$$\mathfrak{R} = \mathfrak{a}_{U_1} + \mathfrak{a}_{U_2} + \cdots + \mathfrak{a}_{U_m}.$$

Hence we have different decompositions of $\mathfrak{R}$, so far as we decompose V in different ways. Corresponding to these decompositions, there exist idempotents e_U, such that

$$\mathfrak{a}_U = (e_U)_r, \qquad e_{U_1} e_{U_2} = 0 \quad \text{when} \quad U_1 U_2 = 0,$$

and $\quad 1 = e_{U_1} + e_{U_2} + \cdots + e_{U_m} \quad \text{when} \quad V = U_1 + U_2 + \cdots + U_m.$

From the lattice-theory aspect the set of all $\mathfrak{a}_U$ forms a complemented distributive sublattice of $R_{\mathfrak{R}}$, which is lattice-isomorphic to the system $\{U\}$ of all subsets of V. We may call this system $\left\{\mathfrak{a}_U;\ U \in \{U\}\right\}$ a *decomposition system of right ideals.* Corresponding to this system $\left\{\mathfrak{a}_U;\ U \in \{U\}\right\}$, we find a system $\left\{e_U;\ U \in \{U\}\right\}$ of idempotents, such that $(e_U)_r = \mathfrak{a}_U$. When we define the inclusion of lattice-theory for idempotents as follows—we say $e_i > e_j$ when $e_i e_j = e_j e_i = e_j$, then $\left\{e_U;\ U \in \{U\}\right\}$ is a complemented distributive lattice which is lattice-isomorphic to $\left\{\mathfrak{a}_U;\ U \in \{U\}\right\}$. We call $\left\{e_U;\ U \in \{U\}\right\}$ a *decomposition system of idempotents.* e_U satisfies similar conditions to the resolution of identity $E(U)$ in Hilbert space.

When $\mathfrak{R}$ satisfies the chain-condition, we can easily obtain the correspondence between $\left\{\mathfrak{a}_U;\ U \in \{U\}\right\}$ and $\left\{e_U;\ U \in \{U\}\right\}$, since V is a finite set. But we may expect this correspondence to hold good also for rings without chain-condition. The object of the present paper is to show that this expectation is true.

I investigate this problem in two cases, i. e.

(i) To decompose $\mathfrak{R}$ in a direct sum of finite system of right ideals.

(ii) To decompose $\mathfrak{R}$ in a direct sum of enumerably infinite system of right ideals.

The last case is considered in the complete rank-ring introduced by J. v. Neumann.[1]

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(1) J. v. Neumann [3], 344; [4], 161.

Ring-Decomposition.

1. Let $\mathfrak{R}$ be a (not necessarily commutative) ring with unit 1, and denote by $R_{\mathfrak{R}}$ ($L_{\mathfrak{R}}$) the set of all right (left) ideals. Then $R_{\mathfrak{R}}$ ($L_{\mathfrak{R}}$) is a lattice where the inclusion $\subset$ of the lattice theory means the set-theoretical implication of ideals. The zero element of $R_{\mathfrak{R}}$ is $(0)_r=(0)$, and the unit element is $(1)_r=\mathfrak{R}$.(1) In what follows, all discussions about right ideals also hold good for left ideals.

Let $\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_n$ be right ideals, and

$$\mathfrak{a}=\mathfrak{a}_1\cup\mathfrak{a}_2\cup\cdots\cup\mathfrak{a}_n .$$

If every element a of $\mathfrak{a}$ is expressible uniquely in the form

$$a=a_1+a_2+\cdots+a_n , \qquad a_i \in \mathfrak{a}_i ,$$

then we say that $\mathfrak{a}$ is the *direct sum* of $\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_n$, and we write

$$\mathfrak{a}=\mathfrak{a}_1+\mathfrak{a}_2+\cdots+\mathfrak{a}_n .$$

Let $\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_n, \ldots$ be a system of right ideals. Then we say that this system is *independent* when

$$\sum(\mathfrak{a}_i\,;\, i\in I)\cap\sum(\mathfrak{a}_j\,;\, j\in J)=(0)$$

for every pair of non-intersecting subsets I, J of the set of integers $(1, 2, \ldots, n, \ldots)$, the notation $\sum(\mathfrak{a}_i\,;\, i\in I)$ denoting the least upper bound of the class of all right ideals $\mathfrak{a}_i\,;\, i\in I$.(2) Then we can easily prove that *when* $\mathfrak{a}=\mathfrak{a}_1\cup\mathfrak{a}_2\cup\cdots\cup\mathfrak{a}_n$, $\mathfrak{a}$ *is the direct sum of* $\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_n$ *when, and only when, the system of right ideals* $(\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_n)$ *is independent.*

2. Let B be a complemented distributive sublattice of $R_{\mathfrak{R}}$ with unit element $\mathfrak{R}$. To designate each right ideal in B, we attribute to them indices U. The system of indices $\{U\}$ is a complemented distributive lattice which is lattice-isomorphic to B, V and 0 being the unit and zero element of $\{U\}$ respectively. And we denote by $\mathfrak{a}_U$ the right ideal in B which corresponds to U. With respect to this system of indices $\{U\}$, we write, for the sake of simplicity,

(1) J. v. Neumann [5], 4.
(2) J. v. Neumann [4], 9.

$$U_1 \frown U_2 \equiv U_1 U_2$$

and when $U_1, U_2, \ldots, U_n$ are independent

$$U_1 \smile U_2 \smile \cdots \smile U_n \equiv U_1 + U_2 + \cdots + U_n .$$

Since $\{\mathfrak{a}_U ; U \in \{U\}\}$ is a complemented distributive lattice, there exist the following relations:

(α) $\mathfrak{a}_{U_1} \frown \mathfrak{a}_{U_2} = \mathfrak{a}_{U_1 U_2}$;

(β) $\mathfrak{a}_U = \mathfrak{a}_{U_1} + \mathfrak{a}_{U_2} + \cdots + \mathfrak{a}_{U_n}$ when $U = U_1 + U_2 + \cdots + U_n$;

(γ) $\mathfrak{a}_V = \mathfrak{R}$.

Conversely, *when for every U in a complemented distributive lattice $\{U\}$ there corresponds one, and only one, right ideal $\mathfrak{a}_U$ such as to satisfy the conditions* (α), (β), (γ), *then* $\{\mathfrak{a}_U ; U \in \{U\}\}$ *is a sublattice of $R_{\mathfrak{R}}$ which is lattice-isomorphic to* $\{U\}$.

Let U_1 and U_2 be any elements in $\{U\}$; since $\{U\}$ is a Boolean algebra,(1) we can express U_1, U_2 in the following way:

$$U_1 = U_1 U_2 + U_3 , \qquad U_2 = U_1 U_2 + U_4 ,$$

and $U_3 U_4 = 0$.(2) Then $U_1 U_2, U_3, U_4$ are independent, and

$$U_1 \smile U_2 = U_1 U_2 + U_3 + U_4 .$$

Hence, by (β), $\quad \mathfrak{a}_{U_1} = \mathfrak{a}_{U_1 U_2} + \mathfrak{a}_{U_3} , \qquad \mathfrak{a}_{U_2} = \mathfrak{a}_{U_1 U_2} + \mathfrak{a}_{U_4} ,$

$$\mathfrak{a}_{U_1 \smile U_2} = \mathfrak{a}_{U_1 U_2} + \mathfrak{a}_{U_3} + \mathfrak{a}_{U_4} .$$

Therefore
$$\mathfrak{a}_{U_1 \smile U_2} = \mathfrak{a}_{U_1} \smile \mathfrak{a}_{U_2} . \tag{1}$$

From (α) and (1), we infer that B is a sublattice of $R_{\mathfrak{R}}$ which is lattice-isomorphic to $\{U\}$.

Since (α), (β), (γ) express the states of decompositions of $\mathfrak{R}$, we call $\{\mathfrak{a}_U ; U \in \{U\}\}$ (that is, a complemented distributive sublattice of $R_{\mathfrak{R}}$ with unit element $\mathfrak{R}$) a *decomposition system of right ideals.*

A decomposition system of right ideals $\{\mathfrak{a}_U ; U \in \{U\}\}$ is said to be *complete,* if there exists no decomposition system of right ideals of

(1) Boolean algebra means the complemented distributive lattice.

(2) For, put $W = U_3 U_4$, then since $W \subset U_1$, $W \subset U_2$, we have $W \subset U_1 U_2$. But, since $W \subset U_3$, $U_1 U_2 \frown U_3 = 0$, we have $W = 0$.

which $\left\{\mathfrak{a}_U;\, U\in\{U\}\right\}$ is a proper subset. As considered in the introduction, with respect to a ring $\mathfrak{R}$ without radical, with minimum-condition for right ideals, the complete decomposition system of right ideals is *atomistic*,(1) and it is lattice-isomorphic to the system of all subsets of a finite set.

3. *Let* $\left\{\mathfrak{a}_U;\, U\in\{U\}\right\}$ *be a decomposition system of right ideals. Then there exists a unique system of idempotents* $\left\{e_U;\, U\in\{U\}\right\}$ *such that*

(i) $(e_U)_r=\mathfrak{a}_U$,

(ii) $e_{U_1}e_{U_2}=0$ *when* $U_1U_2=0$,

(iii) $1=e_{U_1}+e_{U_2}+\cdots+e_{U_n}$ *when* $V=U_1+U_2+\cdots+U_n$.(2)

For any U in $\{U\}$, there exists in $\{U\}$ a unique inverse U' of U. Then, since

$$\mathfrak{a}_U\cup\mathfrak{a}_{U'}=\mathfrak{R}\,,\qquad \mathfrak{a}_U\cap\mathfrak{a}_{U'}=(0)\,,$$

we have a unique idempotent e such that

$$(e)_r=\mathfrak{a}_U\,,\qquad (1-e)_r=\mathfrak{a}_{U'}\,.^{(3)}$$

Denote this e by e_U. Thus we find a system of idempotents $\{e_U\}$ which is defined for all U in $\{U\}$. Of course, when

$$V=U+U'\,,$$

then $\quad 1=e_U+e_{U'}$ and $e_Ue_{U'}=e_{U'}e_U=0\,.$

Let U_1, U_2 be any two elements in $\{U\}$, such that $U_1U_2=0$. And put

$$V=U_1+U_1'\,,\qquad V=U_2+U_2'\,.$$

Then, since $\quad U_2=U_2V=U_2(U_1+U_1')=U_2U_1+U_2U_1'\,,$

we have $\quad U_2=U_2U_1'\,,$

that is $\quad U_1'\supset U_2\,.$

Hence $\quad \mathfrak{a}_{U_1'}\supset\mathfrak{a}_{U_2}\,.$

(1) For the meaning of atomistic see J. v. Neumann [6], 19.

(2) We may say that this theorem is a generalisation of the theorem proved in v. d. Waerden [1], 160.

(3) J. v. Neumann [2], 708; [5], 7.

Then, since $$e_{U_2}\in\mathfrak{a}_{U_2}\subset\mathfrak{a}_{U_1'}=(e_{U_1'})_r\,,$$
we have $$e_{U_2}=e_{U_1'}e_{U_2}\,. \tag{1}$$
Since $$1=e_{U_1}+e_{U_1'}\,,$$
we have $$e_{U_2}=(e_{U_1}+e_{U_1'})\,e_{U_2}=e_{U_1}e_{U_2}+e_{U_1'}e_{U_2}\,.$$
Hence, from (1), we have (ii):
$$e_{U_1}e_{U_2}=0\,.$$

Let $V=U_1+U_2+\cdots\cdots+U_n$. Then, since
$$\mathfrak{R}=\mathfrak{a}_{U_1}+\mathfrak{a}_{U_2}+\cdots\cdots+\mathfrak{a}_{U_n}\,,$$
we have $$\mathfrak{R}=(e_{U_1})_r+(e_{U_2})_r+\cdots\cdots+(e_{U_n})_r\,.$$
Hence 1 is expressible in the form
$$1=e_{U_1}x_1+e_{U_2}x_2+\cdots\cdots+e_{U_n}x_n\,,$$
where $e_{U_i}x_i$ is an element in $(e_{U_i})_r$. Therefore
$$e_{U_i}=e_{U_i}e_{U_1}x_1+e_{U_i}e_{U_2}x_2+\cdots\cdots+e_{U_i}e_{U_n}x_n\,.$$
From (ii) we have
$$e_{U_i}=e_{U_i}x_i \qquad \text{for all} \qquad i\,.$$
Consequently $$1=e_{U_1}+e_{U_2}+\cdots\cdots+e_{U_n}\,.$$
Thus we have (iii).

The uniqueness of e_U is evident, since 1 expressible uniquely in the form (iii) where $e_{U_i}\in\mathfrak{a}_{U_i}$.

When $$U=U_1+U_2+\cdots\cdots+U_n\,,$$
put $$V=U+U'\,. \tag{2}$$
Then $$V=U'+U_1+U_2+\cdots\cdots+U_n\,.$$
Hence, by (iii), $$1=e_{U'}+e_{U_1}+e_{U_2}+\cdots\cdots+e_{U_n}\,.$$
But, from (2), we have
$$1=e_U+e_{U'}\,.$$

Consequently we have

$$e_U = e_{U_1} + e_{U_2} + \cdots + e_{U_n}. \tag{3}$$

Next, let U_1, U_2 be any two elements in $\{U\}$. And put

$$U_1 = U_1U_2 + U_3, \qquad U_2 = U_1U_2 + U_4.$$

Then $U_3U_4 = 0$. Hence, by (3)

$$e_{U_1} = e_{U_1U_2} + e_{U_3}, \qquad e_{U_2} = e_{U_1U_2} + e_{U_4}.$$

Consequently $\quad e_{U_1}e_{U_2} = e_{U_1U_2} + e_{U_3}e_{U_1U_2} + e_{U_1U_2}e_{U_4} + e_{U_3}e_{U_4}.$

Since $U_3(U_1U_2) = 0$, $(U_1U_2)U_4 = 0$, $U_3U_4 = 0$, from (ii) we have

$$e_{U_1}e_{U_2} = e_{U_1U_2}.$$

Thus, the *above-obtained* e_U *has the following properties:*

(α) $e_{U_1}e_{U_2} = e_{U_1U_2}$,

(β) $e_U = e_{U_1} + e_{U_2} + \cdots + e_{U_n}$ *when* $U = U_1 + U_2 + \cdots + U_n$,

(γ) $e_V = 1$.

In this way, e_U has similar properties to the resolution of identity $\boldsymbol{E}(U)$ in Hilbert space.[1] Now we define as follows: If, for each element U of a Boolean algebra $\{U\}$, there corresponds one, and only one, idempotent e_U which satisfies the following conditions,

(I) $e_{U_1}e_{U_2} = 0$ when $U_1U_2 = 0$,

(II) $1 = e_{U_1} + e_{U_2} + \cdots + e_{U_n}$ when $V = U_1 + U_2 + \cdots + U_n$,

then we call e_U a *decomposition of unit,* and $\{e_U;\, U \in \{U\}\}$ a *decomposition system of idempotents.* e_U satisfies the conditions (α), (β), (γ) cited above.

Decomposition of Unit in $\mathfrak{R}$.

4. Denote by $\mathfrak{E}$ the set of all idempotents in $\mathfrak{R}$. And let e_1, e_2

(1) The resolution of identity $\boldsymbol{E}(U)$ is a system of projections which is defined in a system of sets, and satisfies the following conditions:

(α) $\boldsymbol{E}(U_1)\boldsymbol{E}(U_2) = \boldsymbol{E}(U_1U_2)$,

(β) $\boldsymbol{E}(U) = \boldsymbol{E}(U_1) + \boldsymbol{E}(U_2) + \cdots + \boldsymbol{E}(U_n) + \cdots$ where $U = U_1 + U_2 + \cdots + U_n + \cdots$,

(γ) $\boldsymbol{E}(V) = \boldsymbol{1}$

where V is the total space. (Cf. F. Maeda [1], 78; [2], 198.) The logical structure of $\{\boldsymbol{E}(U)\}$ is the same as that of $\{\mathfrak{M}_U\}$, i.e. the orthogonal system of closed linear manifolds, which is investigated in F. Maeda [3], 18–21.

be two elements in $\mathfrak{E}$. When

$$e_1 e_2 = e_2 e_1 = e_2 ,$$

we say that $e_1 \succ e_2$.[1] Of course $e_1 \succ e_1$.

When $e_1 \succ e_2$, $e_2 \succ e_3$, we have $e_1 \succ e_3$. For, since $e_1 e_2 = e_2$, $e_2 e_3 = e_3$, we have $e_1 e_3 = e_1 e_2 e_3 = e_2 e_3 = e_3$. Similarly $e_3 e_1 = e_3$.

Hence, if we use " $\succ$ " as the inclusion in the lattice theory, $\mathfrak{E}$ is a partially ordered system. Consequently, we can define in $\mathfrak{E}$, the meet (greatest lower bound) $e_1 \frown e_2$, and the join (least upper bound) $e_1 \smile e_2$ of two idempotents. When a subset $\{e\}$ of $\mathfrak{E}$ is closed with respect to these two operations, I say that $\{e\}$ is a *sublattice of* $\mathfrak{E}$, even if $\mathfrak{E}$ is not a lattice.

When e_1, e_2 *are any two idempotents such that* $e_1 e_2 = e_2 e_1$, *then* $e_1 \smile e_2$ *and* $e_1 \frown e_2$ *always exist and* $e_1 \smile e_2 = e_1 + e_2 - e_1 e_2$, $e_1 \frown e_2 = e_1 e_2$.

Put
$$e_{12} \equiv e_1 + e_2 - e_1 e_2 .$$

Then we can easily see that e_{12} is an idempotent, and $e_{12} \succ e_1$, $e_{12} \succ e_2$. Let e' be any idempotent in $\mathfrak{E}$, such that $e' \succ e_1$, $e' \succ e_2$. Then, since $e_1 e' = e_1$, $e_2 e' = e_2$, we have

$$e_{12} e' = e_1 e' + e_2 e' - e_1 e_2 e' = e_1 + e_2 - e_1 e_2 = e_{12} .$$

Similarly, we have $e' e_{12} = e_{12}$. Therefore $e' \succ e_{12}$.

Consequently
$$e_1 \smile e_2 = e_{12} = e_1 + e_2 - e_1 e_2 .$$

Next, put
$$e^{12} \equiv e_1 e_2 .$$

Then we can easily see that e^{12} is an idempotent, and $e^{12} \prec e_1$, $e^{12} \prec e_2$. Let e'' be any idempotent in $\mathfrak{E}$, such that $e'' \prec e_1$, $e'' \prec e_2$. Then, since $e_1 e'' = e''$, $e_2 e'' = e''$, we have

$$e^{12} e'' = e_1 e_2 e'' = e_1 e'' = e'' .$$

Similarly we have $e'' e^{12} = e''$. Therefore $e'' \prec e^{12}$.

Consequently
$$e_1 \frown e_2 = e^{12} = e_1 e_2 .$$

5. *Let* e_U *be a decomposition of unit. Then* $\left\{e_U ;\, U \in \{U\}\right\}$ *is a*

(2) J. v. Neumann defined $e_1 > e_2$ by $e_1 e_2 = e_2$ for idempotents in a special commutative ring used in the quantum mechanical formalism, and investigated the lattice of idempotents. (J. v. Neumann [1], 443-447.)

complemented distributive sublattice of $\mathfrak{E}$, which is lattice-isomorphic to the index system $\{U\}$.

Let e_{U_1}, e_{U_2} be any two elements in $\left\{e_U;\, U\in\{U\}\right\}$. Since, from sec. 3 ($\alpha$), e_{U_1} and e_{U_2} are commutative, by sec. 4 we have

$$e_{U_1}\cup e_{U_2}=e_{U_1}+e_{U_2}-e_{U_1}e_{U_2}\,, \tag{1}$$

and

$$e_{U_1}\cap e_{U_2}=e_{U_1}e_{U_2}\,. \tag{2}$$

Since $\{U\}$ is a Boolean algebra, we can decompose U_1 and U_2 as follows:

$$U_1=U_1U_2+U_3\,,\qquad U_2=U_1U_2+U_4\,,$$

where $U_3U_4=0$.(1) Then

$$U_1\cup U_2=U_1U_2+U_3+U_4\,.$$

Therefore, by (β), $e_{U_1}=e_{U_1U_2}+e_{U_3}\,,\qquad e_{U_2}=e_{U_1U_2}+e_{U_4}\,,$

and

$$e_{U_1\cup U_2}=e_{U_1U_2}+e_{U_3}+e_{U_4}\,.$$

Hence

$$e_{U_1\cup U_2}=e_{U_1}+e_{U_2}-e_{U_1U_2}\,.$$

Consequently, from (1),

$$e_{U_1}\cup e_{U_2}=e_{U_1\cup U_2}\,.$$

That is, $e_{U_1}\cup e_{U_2}$ belongs to $\left\{e_U;\, U\in\{U\}\right\}$.

From (2) and (α), we have

$$e_{U_1}\cap e_{U_2}=e_{U_1U_2}\,. \tag{4}$$

That is, $e_{U_1}\cap e_{U_2}$ belongs to $\left\{e_U;\, U\in\{U\}\right\}$.

From (3) and (4), $\left\{e_U;\, U\in\{U\}\right\}$ is a sublattice of $\mathfrak{E}$, which is lattice-isomorphic to $\{U\}$. Thus the theorem is proved.

6. We have the converse theorem:

Let $\{e\}$ be a subset of $\mathfrak{E}$ which satisfies the following conditions:

(i) *$\{e\}$ is a complemented distributive sublattice of $\mathfrak{E}$, with unit element* 1;

(ii) *when e belongs to $\{e\}$, then $1-e$ also belongs to $\{e\}$.*

Then $\{e\}$ is a decomposition system of idempotents.(2)

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(1) Cf. sec. 2, footnote.

(2) Analogous theorems are given in F. Maeda [3], 19 and 22.

Let $\{U\}$ be a Boolean algebra which is lattice-isomorphic to $\{e\}$. And denote by e_U the element of $\{e\}$ which corresponds to U. Then, we have

$$e_{U_1} \frown e_{U_2} = e_{U_1 U_2}, \qquad e_{U_1} \smile e_{U_2} = e_{U_1 \smile U_2}.$$

1°. Since $e(1-e)=(1-e)e$, we have, by sec. 4,

$$e \smile (1-e) = e + (1-e) - e(1-e) = 1, \qquad e \frown (1-e) = e(1-e) = 0.$$

Hence, $1-e$ is the complement of e.

2°. When $U = U_1 + U_2$, we have

$$e_U = e_{U_1} \smile e_{U_2}, \qquad e_{U_1} \frown e_{U_2} = 0. \tag{1}$$

But from property of the Boolean algebra[1] (1) has a unique solution

$$e_{U_2} = e_U \frown (1 - e_{U_1}),$$

where $1 - e_{U_1}$ is the complement of e_{U_1}.

Since $U \supset U_1$, $e_U > e_{U_1}$. Hence $e_U e_{U_1} = e_{U_1} e_U = e_{U_1}$.

And $$e_U(1 - e_{U_1}) = (1 - e_{U_1}) e_U.$$

Therefore we have, by sec. 4,

$$e_U \frown (1 - e_{U_1}) = e_U(1 - e_{U_1}) = e_U - e_{U_1}.$$

Consequently the unique solution of (1) is

$$e_{U_2} = e_U - e_{U_1}.$$

That is, $$e_U = e_{U_1} + e_{U_2}.$$

3°. When $$V = U_1 + U_2 + \cdots\cdots + U_n,$$

from 2°, we can easily obtain the relation

$$1 = e_{U_1} + e_{U_2} + \cdots\cdots + e_{U_n}.$$

4°. When $U_1 U_2 = 0$, put $U = U_1 + U_2$.

Then, from 2°, $$e_U = e_{U_1} + e_{U_2}.$$

Since $e_U > e_{U_1}$, we have $e_{U_1} e_U = e_{U_1}$.

(1) F. Maeda (3), 17.

Hence $$e_{U_1}=e_{U_1}e_U=e_{U_1}+e_{U_1}e_{U_2}.$$

That is, $$e_{U_1}e_{U_2}=0.$$

5°. Thus $\{e_U\}$ is a system of idempotents defined in a Boolean algebra, and satisfies (I) (II) in sec. 3. Hence e_U is a decomposition of unit.

7. In sec. 3 we obtained the decomposition of unit from the decomposition system of right ideals. Now we proceed to the converse problem.

Let e_U be a decomposition of unit. Then $\left\{(e_U)_r;\, U\in\{U\}\right\}$ is a decomposition system of right ideals which is lattice-isomorphic to $\left\{e_U;\, U\in\{U\}\right\}$.(1)

Since $\left\{e_U;\, U\in\{U\}\right\}$ is commutative, we have, by sec. 4,

$$e_{U_1}\cap e_{U_2}=e_{U_1}e_{U_2}=e_{U_1U_2}, \qquad e_{U_1}\cup e_{U_2}=e_{U_1}+e_{U_2}-e_{U_1}e_{U_2}. \tag{1}$$

Since $e_{U_1U_2}=e_{U_1}e_{U_2}\in(e_{U_1})_r$, $e_{U_1U_2}=e_{U_2}e_{U_1}\in(e_{U_2})_r$, we have

$$(e_{U_1U_2})_r\subset(e_{U_1})_r\cap(e_{U_2})_r. \tag{2}$$

Let x be any element in $\mathfrak{R}$ such that

$$x\in(e_{U_1})_r\cap(e_{U_2})_r.$$

Then $$x=e_{U_1}x=e_{U_2}x.$$

Hence $$x=e_{U_1}e_{U_2}x=e_{U_1U_2}x,$$

that is, $$x\in(e_{U_1U_2})_r.$$

Therefore $$(e_{U_1})_r\cap(e_{U_2})_r\subset(e_{U_1U_2})_r. \tag{3}$$

From (1), (2), and (3), we have

$$(e_{U_1}\cap e_{U_2})_r=(e_{U_1})_r\cap(e_{U_2})_r. \tag{4}$$

With respect to the relation of $\cup$, first consider the case where $U=U_1+U_2$. Then, since $U_1U_2=0$, by (1)

(1) We may say that this theorem is a generalization of the theorem given in van der Waerden [1], 161, Aufgabe 4.

$$e_{U_1} \cup e_{U_2} = e_{U_1} + e_{U_2} = e_U .$$

Since $e_U = e_{U_1} + e_{U_2} \in (e_{U_1})_r \cup (e_{U_2})_r$, we have

$$(e_U)_r \subset (e_{U_1})_r \cup (e_{U_2})_r . \tag{5}$$

And since $e_U e_{U_1} = e_{U_1}$, we have $(e_U)_r \supset (e_{U_1})_r$. Similarly $(e_U)_r \supset (e_{U_2})_r$.

Hence
$$e(_U)_r \supset (e_{U_1})_r \cup (e_{U_2})_r . \tag{6}$$

From (5) and (6) we have

$$(e_U)_r = (e_{U_1})_r \cup (e_{U_2})_r . \tag{7}$$

Next, let U_1, U_2 be any two elements in $\{U\}$. Then

$$U_1 = U_1U_2 + U_3 , \qquad U_2 = U_1U_2 + U_4 .$$

and $U_3U_4 = 0$.[1] Hence

$$U_1 \cup U_2 = U_1U_2 + U_3 + U_4 .$$

Then, by (7), $(e_{U_1})_r = (e_{U_1U_2})_r \cup (e_{U_3})_r$, $(e_{U_2})_r = (e_{U_1U_2})_r \cup (e_{U_4})_r$,

$$(e_{U_1 \cup U_2})_r = (e_{U_1U_2})_r \cup (e_{U_3})_r \cup (e_{U_4})_r .$$

Hence

$$(e_{U_1})_r \cup (e_{U_2})_r = (e_{U_1U_2})_r \cup (e_{U_3})_r \cup (e_{U_4})_r = (e_{U_1 \cup U_2})_r = (e_{U_1} \cup e_{U_2})_r . \tag{8}$$

Since $e_V = 1$, $\{(e_U)_r ; U \in \{U\}\}$ contains $\mathfrak{R}$.

Hence, from (3) and (8), $\{(e_U)_r ; U \in \{U\}\}$ and $\{e_U ; U \in \{U\}\}$ are lattice-isomorphic, and $\{(e_U)_r ; U \in \{U\}\}$ is a decomposition system of right ideals.

8. Let $\mathfrak{Z}$ be the centre of $\mathfrak{R}$, that is, the set of those $a \in \mathfrak{R}$ which commute with every $x \in \mathfrak{R}$: $ax = xa$. And denote by $\mathfrak{Z}_e$ the set of all idempotents contained in $\mathfrak{Z}$. Then $\mathfrak{Z}_e$ is a commutative system, and when $e_1, e_2 \in \mathfrak{Z}_e$ then, by sec. 4,

$$e_1 \cup e_2 = e_1 + e_2 - e_1e_2 , \qquad e_1 \cap e_2 = e_1e_2 ,$$

and they belong to $\mathfrak{Z}_e$.

Let e_1, e_2, e_3 be any idempotents in $\mathfrak{Z}_e$. Then

$$e_1 \cap (e_2 \cup e_3) = e_1 (e_2 + e_3 - e_2e_3) = e_1e_2 + e_1e_3 - e_1e_2e_3 ,$$

(1) Cf. sec. 2, footnote.

$$(e_1 \frown e_2) \smile (e_1 \frown e_3) = e_1e_2 \smile e_1e_3 = e_1e_2 + e_1e_3 - e_1e_2e_1e_3 = e_1e_2 + e_1e_3 - e_1e_2e_3 .$$

Hence in $\mathfrak{Z}_e$, the distributive law

$$e_1 \frown (e_2 \smile e_3) = (e_1 \frown e_2) \smile (e_1 \frown e_3)$$

holds good.

When e belongs to $\mathfrak{Z}_e$, then $1-e$ belongs also to $\mathfrak{Z}_e$. And since

$$e \smile (1-e) = e + (1-e) - e(1-e) = 1 , \qquad e \frown (1-e) = e(1-e) = 0 ,$$

$1-e$ is the inverse of e.

Hence $\mathfrak{Z}_e$ is a complemented distributive sublattice of $\mathfrak{E}$, and, by sec. 6, *$\mathfrak{Z}_e$ is a decomposition system of idempotents.* When $e \in \mathfrak{Z}_e$, $(e)_r = (e)_l$. That is, it is a two-sided ideal, which we denote by $(e)_*$. Then by sec. 7, *$\{(e)_*, e \in \mathfrak{Z}_e\}$ is a decomposition system of two-sided ideals, which is lattice-isomorphic to $\mathfrak{Z}_e$.*

Now denote by $R'_{\mathfrak{R}}$ $(L'_{\mathfrak{R}})$ the set of all principal right (left) ideals of the form $(e)_r$ $\big((e)_l\big)$, e being any idempotent in $\mathfrak{R}$. And let $Z'_{\mathfrak{R}}$ be the intersection of $R'_{\mathfrak{R}}$ and $L'_{\mathfrak{R}}$. When $(e)_r$ is an ideal in $Z'_{\mathfrak{R}}$, there exists an idempotent f such that

$$(e)_r = (f)_l .$$

Then, since $f \in (e)_r$, we have $f = ef$; and, since $e \in (f)_l$, we have $e = ef$. Hence $e = f$. Consequently any ideal in $Z'_{\mathfrak{R}}$ is a two-sided ideal $(e)_*$, and this e is uniquely determined.

Let x be any element in $\mathfrak{R}$. Since $ex \in (e)_*$, we have $ex = ex \cdot e$, and since $xe \in (e)_*$, we have $xe = e \cdot xe$. Hence $ex = xe$. Therefore $e \in \mathfrak{Z}_e$.

Consequently, *the elements of $Z'_{\mathfrak{R}}$ are precisely the ideals $(e)_*$ with $e \in \mathfrak{Z}_e$, and the correspondence between ideals in $Z'_{\mathfrak{R}}$ and idempotents in $\mathfrak{Z}_e$ thus defined is one-to-one.* And, by sec. 7, *$Z'_{\mathfrak{R}}$ is a decomposition system of two-sided ideals $\{(e)_* ; e \in \mathfrak{Z}_e\}$.* Hence *$\{(e)_* ; e \in \mathfrak{Z}_e\}$ is the unique complete decomposition system of two-sided ideals.*[(1)]

When $\mathfrak{R}$ is a *regular* ring, since every principal right (left) ideal is expressed in the form $(e)_r$ $\big((f)_l\big)$, e (f) being idempotents, $R'_{\mathfrak{R}}$ $(L'_{\mathfrak{R}})$ is the set of all principal right (left) ideals. Hence the above-obtained results coincide with J. von Neumann's.[(2)]

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(1) The completeness is defined in a similar way to that in sec. 2.

(2) J. v. Neumann [2], 713, Theorem 7; [5], 14, Theorem 2.10.

Decomposition of Complete Rank-Ring.

9. Next assume that $\mathfrak{R}$ is a *rank-ring*, that is, that there exists a real function $R(a)$, $a \in \mathfrak{R}$, such that

(α) $0 \leqq R(a) \leqq 1$ for every $a \in \mathfrak{R}$;

(β) $R(a)=0$ if, and only if, $a=0$;

(γ) $R(1)=1$;

(δ) $R(ab) \leqq R(a), R(b)$;

(ϵ) For $e^2=e, f^2=f, ef=fe=0$ we have $R(e+f)=R(e)+R(f)$.

If we define the rank-distance by $R(a-b)$, then $\mathfrak{R}$ is a metric space.(1) A sequence $\{a_i; i=1, 2, \ldots\}$ is convergent to the limit a if $\lim_{i\to\infty} R(a_i-a)=0$, and we write $a=\lim_{i\to\infty} a_i$. Assume that $\mathfrak{R}$ is complete in the topology of the rank-distance $R(a-b)$.

Let $a_1+a_2+\cdots+a_n+\cdots$ be a series of elements in R. And put

$$s_n=a_1+a_2+\cdots+a_n.$$

If there exists an element a, in $\mathfrak{R}$, such that

$$\lim_{n\to\infty} R(s_n-a)=0,$$

then we say that the series converges to a, and we write

$$a=a_1+a_2+\cdots+a_n+\cdots.$$

Let $\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_n, \ldots$ be a sequence of right ideals, and

$$\mathfrak{a}=\mathfrak{a}_1 \cup \mathfrak{a}_2 \cup \cdots \cup \mathfrak{a}_n \cup \cdots.$$

If every element of a of $\mathfrak{a}$ is expressible uniquely in the form

$$a=a_1+a_2+\cdots+a_n+\cdots \qquad (a_n \in \mathfrak{a}_n),$$

then we say that $\mathfrak{a}$ is a *direct sum* of $\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_n, \ldots$, and write

$$\mathfrak{a}=\mathfrak{a}_1+\mathfrak{a}_2+\cdots+\mathfrak{a}_n+\cdots.$$

Then we can easily prove the following theorem:

When $\mathfrak{a}=\mathfrak{a}_1 \cup \mathfrak{a}_2 \cup \cdots \cup \mathfrak{a}_n \cup \cdots,$

(1) J. v. Neumann [3], 344; [5], 161. J. v. Neumann writes $\bar{R}(a)$ instead of $R(a)$. Here it is assumed that $\mathfrak{R}$ is regular. But if we add (ζ) $R(a+b) \leqq R(a)+R(b)$, then this assumption is superfluous in the present paper.

$\mathfrak{a}$ *is the direct sum of* $\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_n, \ldots$ *when, and only when, the system of ideals* $(\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_n, \ldots)$ *is independent.*

10. *Any sequence of idempotents such that*

$$e_1 > e_2 > \cdots > e_i > \cdots \tag{1}$$

$$(e_2 < e_2 < \cdots < e_i < \cdots) \tag{2}$$

converges to an idempotent e such that

$$e_i > e \qquad (e_i < e) \qquad \text{for all} \qquad i.$$

And $$\Pi(e_i\,;\, i=1,2,\ldots)=e\,, \quad \left(\sum(e_i\,;\, i=1,2,\ldots)=e\right).^{(1)} \tag{3}$$

$$\Pi\Big((e_i)_r\,;\, i=1,2,\ldots\Big)=(e)_r\,, \quad \Big(\sum\Big((e_i)_r\,;\, i=1,2,\ldots\Big)=(e)_r\Big). \tag{4}$$

First consider case (1). When $j<i$, since $e_ie_j=e_je_i=e_i$, we can easily see that e_j-e_i is an idempotent, and

$$(e_j-e_i)e_i=e_i(e_j-e_i)=0\,.$$

Hence, by sec. 9 (ϵ), we have

$$R(e_j)=R(e_j-e_i)+R(e_i)\,. \tag{5}$$

Hence $$R(e_1)\geqq R(e_2)\geqq \cdots \geqq R(e_i)\geqq \cdots,$$

and, $R(e_i)$ being convergent,

$$\lim_{\substack{i\to\infty\\ j\to\infty}} \{R(e_j)-R(e_i)\}=0\,.$$

Therefore, from (5),

$$\lim_{\substack{i\to\infty\\ j\to\infty}} R(e_j-e_i)=0\,.$$

Since $\mathfrak{R}$ is complete, there exists an element e in $\mathfrak{R}$, such that

$$\lim_{i\to\infty} e_i=e\,.$$

In the equality $e_ie_j=e_je_i=e_i$ $(j<i)$, let $i\to\infty$; then, since, by sec. 9 (δ), $\lim\limits_{i\to\infty} e_ie_j=ee_j$, $\lim\limits_{i\to\infty} e_je_i=e_je$, we have

(1) In this paper the symbol $\sum(e_i\,;\, i=1,2,\ldots)$ means the least upper bound in the lattice theory, and not the sum in the ring theory. Similarly for Π.

$$ee_j = e_j e = e. \tag{6}$$

Next let $j \to \infty$; then we have $e^2 = e$; that is, e is an idempotent. And, from (6), $e_i \succ e$ for all i.

Next, let f be any idempotent in $\mathfrak{E}$, such that

$$e_i \succ f \quad \text{for all} \quad i.$$

Then
$$e_i f = f e_i = f.$$
Let $i \to \infty$; we have
$$ef = fe = f; \quad \text{that is,} \quad e \succ f.$$

Hence, from the definition of the greatest lower bound of $\{e_i; i=1, 2, \ldots.\}$, we have
$$\Pi(e_i; i=1, 2, \ldots.) = e.$$
Thus we have (3).

Since $\quad e_i \succ e, \quad$ we have $\quad e_i e = e.$

Therefore $\quad e \in (e_i)_r \quad$ for all $\quad i.$

Hence $$e \in \Pi\big((e_i)_r; i=1, 2, \ldots.\big),$$

that is, $$(e)_r \subset \Pi\big((e_i)_r; i=1, 2, \ldots.\big).$$

Next, let x be any element in $\mathfrak{R}$, such that

$$x \in \Pi\big((e_i)_r; i=1, 2, \ldots.\big).$$

Then, since $x \in (e_i)_r$, we have

$$x = e_i x \quad \text{for all} \quad i.$$
Let $i \to \infty$; we have
$$x = ex; \quad \text{that is,} \quad x \in (e)_r.$$

Hence $$\Pi\big((e_i)_r; i=1, 2, \ldots.\big) \subset (e)_r.$$

Consequently $$\Pi\big((e_i)_r; i=1, 2, \ldots.\big) = (e)_r.$$
Thus we have (4).

In a similar manner we can prove for case (2).

11. Let B be a complemented distributive $\aleph_1$-sublattice[1] of $R_{\mathfrak{R}}$ whose unit element is $\mathfrak{R}$. Let $\{U\}$ be a complemented distributive

(1) For the definition of $\aleph_1$-lattice see J. v. Neumann [6], 5.

$\aleph_1$-lattice which is lattice-isomorphic to B. And denote by $\mathfrak{a}_U$ the right ideal in B which corresponds to U. Then there exist the following relations:

(α) $\mathfrak{a}_{U_1} \frown \mathfrak{a}_{U_2} = \mathfrak{a}_{U_1 U_2}$;

(β) $\mathfrak{a}_U = \mathfrak{a}_{U_1} + \mathfrak{a}_{U_2} + \cdots\cdots + \mathfrak{a}_{U_n} + \cdots\cdots$
when $U = U_1 + U_2 + \cdots\cdots + U_n + \cdots\cdots$;

(γ) $\mathfrak{a}_V = \mathfrak{R}$, V being the unit element in $\{U\}$.

Since (α), (β), (γ) express the states of decomposition of $\mathfrak{R}$, we term $\left\{\mathfrak{a}_U;\, U \in \{U\}\right\}$ a *decomposition system of right ideals in the generalized sense.*

Let $B = \left\{\mathfrak{a}_U;\, U \in \{U\}\right\}$ *be a decomposition system of right ideals in the generalized sense. Then there exists a unique system of idempotents* $\left\{e_U;\, U \in \{U\}\right\}$ *such that*

(i) $(e_U)_r = \mathfrak{a}_U$,

(ii) $e_{U_1} e_{U_2} = 0$ *when* $U_1 U_2 = 0$,

(iii) $1 = e_{U_1} + e_{U_2} + \cdots\cdots + e_{U_n} + \cdots\cdots$
when $V = U_1 + U_2 + \cdots\cdots + U_n + \cdots\cdots$.

Since $B = \left\{\mathfrak{a}_U;\, U \in \{U\}\right\}$ is a complemented distributive sublattice of $R_{\mathfrak{R}}$, by sec. 3, there exists a unique system of idempotents $\left\{e_U;\, U \in \{U\}\right\}$ such that (i) (ii) and

(iii) $1 = e_{U_1} + e_{U_2} + \cdots\cdots + e_{U_n}$ when $V = U_1 + U_2 + \cdots\cdots + U_n$,

hold good. Hence we must prove (iii). When

$$V = U_1 + U_2 + \cdots\cdots + U_n + \cdots\cdots,$$

put

$$U^{(i)} = U_1 + U_2 + \cdots\cdots + U_i,$$

then $e_{U^{(i)}} = e_{U_1} + e_{U_2} + \cdots\cdots + e_{U_i}$ is an idempotent, and

$$e_{U^{(1)}} \prec e_{U^{(2)}} \prec \cdots\cdots \prec e_{U^{(i)}} \prec \cdots\cdots.$$

Hence, by sec. 10, there exists an idempotent e such that

$$\lim_{i \to \infty} e_{U^{(i)}} = e,$$

and

$$\sum\left((e_{U^{(i)}})_r;\, i = 1, 2, \ldots.\right) = (e)_r.$$

But, since $\sum\left((e_{U^{(i)}})_r;\, i = 1, 2, \ldots.\right) = \sum\left((e_{U_i})_r;\, i = 1, 2, \ldots.\right) = \mathfrak{R}$,

we have $e = 1$. Thus (iii) holds good.

As in sec. 3, the above-obtained e_U has the following properties:

(α) $e_{U_1}e_{U_2}=e_{U_1U_2}$,

(β) $e_U=e_{U_1}+e_{U_2}+\cdots\cdots+e_{U_n}+\cdots\cdots$

when $U=U_1+U_2+\cdots\cdots+U_n+\cdots\cdots$,

(γ) $e_V=1$.

Thus e_U has the same properties as the resolution of identity $\boldsymbol{E}(U)$ in Hilbert space.(1) Now we define as follows: If, for each element U of a $\aleph_1$-Boolean algebra $\{U\}$, there corresponds one, and only one, idempotent e_U which satisfies the following conditions,

(i) $e_{U_1}e_{U_2}=0$ when $U_1U_2=0$,

(ii) $1=e_{U_1}+e_{U_2}+\cdots\cdots e_{U_n}+\cdots\cdots$

when $V=U_1+U_2+\cdots\cdots+U_n+\cdots\cdots$,

then we call e_U a *decomposition of unit in the generalized sense*, and $\left\{e_U;\ U\in\{U\}\right\}$ *a decomposition system of idempotents in the generalized sense.* e_U satisfies the conditions (α), (β), (γ) cited above.

Decomposition of Unit in Complete Rank-Ring.

12. *Let* e_U *be a decomposition of unit in the generalized sense. Then* $\left\{e_U;\ U\in\{U\}\right\}$ *is a complemented distributive* $\aleph_1$*-sublattice of* $\mathfrak{E}$, *which is lattice-isomorphic to the index system* $\{U\}$.

By sec. 5, $\left\{e_U;\ U\in\{U\}\right\}$ is a complemented distributive sublattice of $\mathfrak{E}$, which is lattice-isomorphic to $\{U\}$. From sec. 11 (α), $e_{U_1}<e_{U_2}$ when, and only when, $U_1\subset U_2$. Hence, when

$$e_{U_1}<e_{U_2}<\cdots\cdots<e_{U_n}<\cdots\cdots,$$

then

$$U_1\subset U_2\subset\cdots\cdots\subset U_n\subset\cdots\cdots.$$

Now let $U_n=U_{n-1}+U''_{n-1}$ and $\sum(U_n;\ n=1,2,\ldots.)=U$.

Then $(U_1, U''_1, U''_2, \ldots., U''_n, \ldots.)$ is independent, and

$$U=U_1+U''_1+U''_2+\cdots\cdots+U''_n+\cdots\cdots.$$

Hence, by sec. 11 (β), we heve

$$e_U=e_{U_1}+e_{U''_1}+e_{U''_2}+\cdots\cdots+e_{U''_n}+\cdots\cdots.$$

(1) Cf. sec. 3, footnote.

Since $$e_{U_n}=e_{U_1}+e_{U_1''}+e_{U_2''}+\cdots\cdots+e_{U_{n-1}''},$$

we have $$\lim_{n\to\infty} e_{U_n}=e_U\,,$$

and by sec. 10 $$\sum(e_{U_n}\,;\,n=1,2,\ldots.)=e_U\,. \tag{1}$$

Next, let $\{e_{U^{(n)}}\,;\,n=1,2,\ldots.\}$ be any sequence of idempotents in $\big\{e_U;\,U\in\{U\}\big\}$. Put

$$U_n=U^{(1)}\cup U^{(2)}\cup\cdots\cdots\cup U^{(n)}\,,$$

then $$e_{U_n}=e_{U^{(1)}}\cup e_{U^{(2)}}\cup\cdots\cdots\cup e_{U^{(n)}}\,,$$

and $$e_{U_1}<e_{U_2}<\cdots\cdots<e_{U_n}<\cdots\cdots.$$

Hence, by (1),

$$\sum(e_{U^{(n)}}\,;\,n=1,2,\ldots.)=\sum(e_{U_n}\,;\,n=1,2,\ldots.)=e_U\,, \tag{2}$$

where $U=\sum(U_n\,;\,n=1,2,\ldots.)=\sum(U^{(n)}\,;\,n=1,2,\ldots.)$.

When $$e_{U_1}>e_{U_2}>\cdots\cdots>e_{U_n}>\cdots\cdots,$$

then $$U_1\supset U_2\supset\cdots\cdots\supset U_n\supset\cdots\cdots.$$

Put $\Pi(e_{U_n}\,;\,n=1,2,\ldots.)=e\,,\qquad \Pi(U_n\,;\,n=1,2,\ldots.)=U\,.$

Let U_n' be the inverse of U_n in the Boolean algebra $\{U\}$. Then we have

$$e_{U_1'}<e_{U_2'}<\cdots\cdots<e_{U_n'}<\cdots\cdots$$

and $$U_1'\subset U_2'\subset\cdots\cdots\subset U_n'\subset\cdots\cdots.$$

Then, by the preceding case,

$$\sum(e_{U_n'}\,;\,n=1,2,\ldots.)=e_{U'}$$

where $$U'=\sum(U_n'\,;\,n=1,2,\ldots.)\,.$$

Since $1=e_{U_n}+e_{U_n'}\,,\qquad \lim_{n\to\infty} e_{U_n}=e\,,\qquad \lim_{n\to\infty} e_{U_n'}=e_{U'}\,,$

we have $$1=e+e_{U'}\,.$$

But, since $1=U+U'$, we have $1=e_U+e_{U'}$. Hence $e=e_U$.

Consequently $$\Pi(e_{U_n}\,;\,n=1,2,\ldots.)=e_U\,, \tag{3}$$

where $$U=\Pi(U_n\,;\,n=1,2,\ldots.)\,.$$

Next let $\{e_{U^{(n)}}\,;\, n=1, 2, \ldots.\}$ be any sequence of idempotents in $\left\{e_U\,;\, U\in\{U\}\right\}$. Put

$$U_n = U^{(1)} \frown U^{(2)} \frown \cdots\cdot \frown U^{(n)},$$

then
$$e_{U_n} = e_{U^{(1)}} \frown e_{U^{(2)}} \frown \cdots\cdot \frown e_{U^{(n)}},$$

and
$$e_{U_1} > e_{U_2} > \cdots\cdot > e_{U_n} > \cdots\cdot .$$

Hence, by (3),

$$\Pi(e_{U^{(n)}}\,;\, n=1, 2, \ldots.) = \Pi\,(e_{U_n}\,;\, n=1, 2, \ldots.) = e_U, \tag{4}$$

where
$$U = \Pi(U_n\,;\, n=1, 2, \ldots.) = \Pi\,(U^{(n)}\,;\, n=1, 2, \ldots.).$$

By (2) and (4), we see that $\left\{e_U;\, U\in\{U\}\right\}$ is lattice-isomorphic to $\{U\}$, and it is a complemented distributive $\aleph_1$-sublattice of $\mathfrak{E}$.

13. *Let $\{e\}$ be a subset of $\mathfrak{E}$ which satisfies the following conditions:*

(i) *$\{e\}$ is a complemented distributive $\aleph_1$-sublattice of $\mathfrak{E}$ with unit element* 1.

(ii) *When e belongs to $\{e\}$, then $1-e$ also belongs to $\{e\}$.*

Then $\{e\}$ is a decomposition system of idempotents in the generalized sense.

Let $\{U\}$ be a $\aleph_1$-Boolean algebra which is lattice-isomorphic to $\{e\}$. And denote by e_U the element of $\{e\}$ which corresponds to U. Then, by sec. 6, e_U is a decomposition of unit in the restricted sence. Hence we must prove the following property: when

$$V = U_1 + U_2 + \cdots\cdot + U_n + \cdots\cdot,$$

then
$$1 = e_{U_1} + e_{U_2} + \cdots\cdot + e_{U_n} + \cdots\cdot . \tag{1}$$

Put
$$U^{(n)} = U_1 + U_2 + \cdots\cdot + U_n,$$

then
$$U^{(1)} \subset U^{(2)} \subset \cdots\cdot \subset U^{(n)} \subset \cdots\cdot,$$

and
$$V = \sum(U^{(n)}\,;\, n=1, 2, \ldots.).$$

Hence, by the lattice-isomorphism, we have

$$e_{U^{(1)}} < e_{U^{(2)}} < \cdots\cdot < e_{U^{(n)}} < \cdots\cdot,$$

and
$$1 = \sum(e_{U^{(n)}}\,;\, n=1, 2, \ldots.),$$

where
$$e_{U^{(n)}} = e_{U_1} + e_{U_2} + \cdots\cdot + e_{U_n}.$$

Hence, by sec. 10, $\lim_{n\to\infty} e_{U^{(n)}}=1$.

Consequently we have relation (1).

14. *Let e_U be a decomposition of unit in the generalized sense. Then $\{(e_U)_r\,;\ U\in\{U\}\}$ is a decomposition system of right ideals in the generalized sense, which is lattice-isomorphic to $\{e_U;\ U\in\{U\}\}$.*

In sec. 7, we have seen that $\{(e_U)_r;\ U\in\{U\}\}$ is a complemented distributive sublattice of $R_{\mathfrak{R}}$ which is lattice-isomorphic to $\{U\}$. Now we must prove that it is a $\aleph_1$-sublattice of $R_{\mathfrak{R}}$, which is lattice-isomorphic to $\{U\}$.

When $$e_{U_1}>e_{U_2}>\cdots\cdot>e_{U_n}>\cdots\cdot,$$
from sec. 10, we have
$$\Pi\left((e_{U_n})_r\,;\ n=1,2,\ldots.\right)=(e_U)_r\,,$$
where $$e_U=\Pi(e_{U_n}\,;\ n=1,2,\ldots.)\,.$$

If $\{e_{U^{(i)}}\,;\ i=1,2,\ldots.\}$ be any sequence, put
$$e_{U_i}=e_{U^{(1)}}\frown e_{U^{(2)}}\frown\cdots\cdot\frown e_{U^{(i)}}\,,$$
then $$(e_{U_i})_r=(e_{U^{(1)}})_r\frown(e_{U^{(2)}})_r\frown\cdots\cdot\frown(e_{U^{(i)}})_r\,,$$
and $$e_{U_1}>e_{U_2}>\cdots\cdot>e_{U_i}>\cdots\cdot.$$

Hence, by the above result, we have
$$\Pi\left((e_{U^{(i)}})_r\,;\ i=1,2,\cdots\cdot\right)=\Pi\left((e_{U_i})_r\,;\ i=1,2,\ldots.\right)=(e_U)_r\,,$$
where $$e_U=\Pi(e_{U_i}\,;\ i=1,2,\ldots.)=\Pi\,(e_{U^{(i)}}\,;\ i=1,2,\ldots.)\,.$$

Similarly, we can prove that
$$\sum\left((e_{U^{(i)}})_r\,;\ i=1,2,\ldots.\right)=(e_U)_r\,,$$
where $$e_U=\sum(e_{U^{(i)}}\,;\ i=1,2,\ldots.)\,.$$

Thus $\{(e_U)_r;\ U\in\{U\}\}$ is a $\aleph_1$-sublattice of $R_{\mathfrak{R}}$, which is lattice-isomorphic to $\{e_U;\ U\in\{U\}\}$, that is, to $\{U\}$.

15. Let $\mathfrak{Z}$ be the centre of a complete rank-ring $\mathfrak{R}$. And denote by $\mathfrak{Z}_e$ the set of all idempotents contained in $\mathfrak{Z}$. We have seen, in sec. 8, that $\mathfrak{Z}_e$ is a complemented distributive sublattice of $\mathfrak{E}$. Now I shall show that *it is a* $\aleph_1$-*sublattice of* $\mathfrak{E}$.

Let $\{e_i; i=1,2,\ldots.\}$ be a sequence such that

$$e_i \in \mathfrak{Z}_e, \qquad e_1 > e_2 > \cdots\cdot > e_i > \cdots\cdot.$$

Then, from sec. 10, there exists an idempotent e such that

$$\lim_{i\to\infty} e_i = e \qquad \text{and} \qquad \Pi(e_i; i=1,2,\ldots.) = e.$$

Since $e_i \in \mathfrak{Z}_e$,

$$e_i x = x e_i \qquad \text{for all} \qquad x \in \mathfrak{R}.$$

Let $i \to \infty$; then we have

$$ex = xe.$$

That is, e belongs to $\mathfrak{Z}_e$.

Next, let $\{e^{(i)}; i=1,2,\ldots.\}$ be an arbitrary sequence such that $e^{(i)} \in \mathfrak{Z}_e$. And put

$$e_i = e^{(1)} \frown e^{(2)} \frown \cdots\cdot \frown e^{(i)}.$$

Then $e_i \in \mathfrak{Z}_e$, and

$$e_1 > e_2 > \cdots\cdot > e_i > \cdots\cdot.$$

Hence, from the above discussion, there exists an idempotent e in $\mathfrak{Z}_e$, such that

$$\Pi(e^{(i)}; i=1,2,\ldots.) = \Pi(e_i; i=1,2,\ldots.) = e.$$

In a similar manner, we can prove that there exists an idempotent e in $\mathfrak{Z}_e$ such that

$$\sum(e^{(i)}; i=1,2,\ldots.) = e.$$

Thus $\mathfrak{Z}_e$ is a $\aleph_1$-sublattice of $\mathfrak{E}$.

Consequently, by sec. 13, $\mathfrak{Z}_e$ *is a decomposition system of idempotents in the generalized sense,* and, as in sec. 8, $\{(e)_*, e \in \mathfrak{Z}_e\}$ *is the unique complete decomposition system of two-sided ideals in the generalized sense.*

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