Wave Geometry No. 50

Cosmology in Terms of Wave Geometry (XI). The Solar System as a Local Irregularity in the Universe.

By

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§ 1. Introduction and Outline of Theory.

On the motion of the planets in the solar system there exist two famous theories, the classical theory of Newton and the relativistic theory of Einstein. As is well-known, the former succeeded in explaining the three laws of Kepler, and the latter the secular advance of the perihelion of the Mercury orbit, which could not be explained by Newton's theory.

The purpose of this paper is to build a theory on the motion of the planets in the solar system in terms of Wave Geometry. The outline of the theory is as follows:

Prof. Iwatsuki and one of us have put forward a theory of spiral nebulae.(1) In that theory a nebula was considered as a physical system which might be taken as a local irregularity around a point in the universe. In such a physical system, the physical law is invariantly expressed by the rotation of coordinate system around a spacial point and by the translation with regard to time. In other words, the physical law in such a physical system is expressed in an invariant form by the following infinitesimal transformations:

$$
(1.1)\quad \begin{cases} -\cos\varphi\dfrac{\partial}{\partial\theta}+\cot\theta\sin\varphi\dfrac{\partial}{\partial\varphi}, \\[2ex] \sin\varphi\dfrac{\partial}{\partial\theta}+\cot\theta\cos\varphi\dfrac{\partial}{\partial\varphi}, \\[2ex] \dfrac{\partial}{\partial\varphi}, \qquad \dfrac{\partial}{\partial t}. \end{cases}
$$

As these transformations form a four-parameter group, we shall denote it by G_4. The theory of spiral nebulae in Wave Geometry was established as an invariant theory for G_4.

Since the solar system may also be considered as a local irregularity

(1) T. Iwatsuki and T. Sibata: This Journ., **81** (1941), 74 (W. G. No. 44), hereafter referred to as **I.**

in the universe, it is not unnatural to suppose the law controlling the solar system to be expressible in an invariant form under the transformation G_4. But the physical system in which the law is described by an expression invariant for G_4 is not only adequate for the solar system, but may also be adequate for spiral nebulae, nebular clusters, and other local irregularities in the universe. Therefore, the law which is proper to the solar system must be characterized by some additional conditions which are consistent with the physical nature of the solar system. In fact, in the theory of spiral nebulae, in order to characterize the spiral nebulae, the following three conditions were taken[(1)]:

$\mathrm{I_N}$ u^i (particle momentum-density vector to describe the motion of a particle in a spiral nebula) gives a plane motion in each plane (z=const.)

$\mathrm{II_N}$ A system of trajectories (3-dimensional) generated by u^i is axial-symmetric with respect to the z-axis,

$\mathrm{III_N}$ The trajectories generated by u^i are stationary.

In the same way, in the theory of the solar system, the following conditions will be taken in order to characterize the theory for the solar system:

$\mathrm{I_s}$ $k\rho \ll 1$ (k being the inverse of the radius of the universe and $\rho = r\sin\theta$).

$\mathrm{II_s}$ The particle in the solar system is governed by the central force of the sun.

Along the lines of the theoretical consideration above-mentioned, we shall develop a theory on the motion of planets and get the following results:

(i) *As the first approximation, a particle in the solar system (planet) describes a conic. This is consistent with Newton's theory.*

(ii) *The perihelion of the orbit of a planet advances in secular time. This is consistent with the result of the relativistic theory.*

(iii) *The eccentricity and latus rectum also change in secular time. This result has never been obtained in any previous theory.*

§ 2. The fundamental equation which is invariant for G_4 and the equation of motion.

In Wave Geometry, as the fundamental equation for physical phenomena we take

$$\frac{\partial \Psi}{\partial x^i} = (\Gamma_i + \textstyle\sum_i)\Psi, \tag{2.1}$$

where Γ_i is a 4–4 matrix given by $\dfrac{\partial \gamma_j}{\partial x^i} = \{^{\,k}_{ij}\}\gamma_k + \Gamma_i\gamma_j - \gamma_j\Gamma_i$, and $\sum_i$ is also

(1) **I.** p. 54. There were other two conditions $\mathrm{VI_N}$ and $\mathrm{V_N}$ in **I.** But these are not necessary when we restrict our discussion to the motion of a particle in a spiral nebula.

a 4–4 matrix whose elements are, at present, arbitrary functions of x^1, x^2, x^3, x^4; and γ_i is a matrix given by $\gamma_{(i}\gamma_{j)}=g_{ij}I$.

The physical law which governs a local irregularity such as spiral nebulae, the solar system, and so on, must also be considered as based on (2.1), and moreover must be invariant under the transformation given by (1.1).

Expanding $\sum_i$ in (2.1) in sedenion, we put, as in the theory of spiral nebulae,(1) (2.1) in the form:

$$\left(\frac{\partial}{\partial x^i}-\Gamma_i\right)\psi=(A_i+A_i^{\cdot 5}\gamma_5+A_i^{\cdot j}\gamma_j+A_i^{\cdot j5}\gamma_j\gamma_5)\,\psi\,.$$

In order that this be invariant by G_4; $A_i^{\cdot j}$, $A_i^{\cdot 5}$, A_i and $A_i^{\cdot j5}$ must be invariant tensors and vectors for the operators: $\cos\varphi\dfrac{\partial}{\partial\theta}-\cot\theta\sin\varphi\dfrac{\partial}{\partial\varphi}$, $\sin\varphi\dfrac{\partial}{\partial\theta}+\cot\theta\cos\varphi\dfrac{\partial}{\partial\varphi}$, and $\dfrac{\partial}{\partial\varphi}$ respectively, so that we have(1)

$$A_{11}(r,t)\,,\qquad A_2^2(r,t)=A_3^3(r,t)\,,\qquad A_{41}(r,t)\,,$$

$$A_{14}(r,t)\,,\qquad A_{23}=-A_{32}=\sin\theta\,R_{23}(r,t)\,,\qquad A_{44}(r,t)\,,$$

the other

$$A_{ij}=0\,,$$

and

$$A_1(r,t)\,,\qquad A_2=A_3=0\,,\qquad A_4(r,t)\,.$$

Here if we consider that the fundamental equation (2.1) is completely integrable, the form of $\sum_i$ is determined and the solution ψ is given as follows:(2)

$$\text{(2.2)}\qquad \psi=\left(\cos\frac{\delta}{2}-\sin\frac{\delta}{2}\gamma_5\right)\phi\,,$$

$$\text{(2.2a)}\quad\begin{cases}\phi_1=A\cos\dfrac{\theta}{2}+B\sin\dfrac{\theta}{2}\,,\\[2mm] \phi_2=-\left\{i\sqrt{1-k^2r^2}\,B+krB'\right\}\cos\dfrac{\theta}{2}+\left\{i\sqrt{1-k^2r^2}\,A+krA'\right\}\sin\dfrac{\theta}{2}\,,\\[2mm] \phi_3=A'\cos\dfrac{\theta}{2}+B'\sin\dfrac{\theta}{2}\,,\\[2mm] \phi_4=\left\{i\sqrt{1-k^2r^2}\,B'+krB\right\}\cos\dfrac{\theta}{2}-\left\{i\sqrt{1-k^2r^2}\,A'+krA\right\}\sin\dfrac{\theta}{2}\,,\end{cases}$$

$$\text{(2.2b)}\quad\begin{cases}A=\alpha e^{-\frac{i}{2}\varphi}\,,\qquad B=\beta e^{\frac{i}{2}\varphi}\,,\\[1mm] A'=\alpha' e^{-\frac{i}{2}\varphi}\,,\qquad B'=\beta' e^{\frac{i}{2}\varphi}\,,\end{cases}$$

(1) I. p. 61.
(2) I. Note III, p. 70.

$$
(2.2\text{c})\quad \begin{cases} \alpha = a\left\{\sqrt{1-k^2r^2}+ikr\right\}^{\frac{1}{2}} + b\left\{\sqrt{1-k^2r^2}+ikr\right\}^{-\frac{1}{2}}, \\ \alpha' = a\left\{\sqrt{1-k^2r^2}+ikr\right\}^{\frac{1}{2}} - b\left\{\sqrt{1-k^2r^2}+ikr\right\}^{-\frac{1}{2}}, \\ \beta = c\left\{\sqrt{1-k^2r^2}+ikr\right\}^{\frac{1}{2}} + d\left\{\sqrt{1-k^2r^2}+ikr\right\}^{-\frac{1}{2}}, \\ \beta' = c\left\{\sqrt{1-k^2r^2}+ikr\right\}^{\frac{1}{2}} - d\left\{\sqrt{1-k^2r^2}+ikr\right\}^{-\frac{1}{2}}, \end{cases}
$$

$$
(2.2\text{d})\quad \begin{cases} a = pe^{\frac{k}{2}\sqrt{1+L^2}\,t} + qe^{-\frac{k}{2}\sqrt{1+L^2}\,t}, \\ b = -i(\sqrt{1+L^2}+L)\,pe^{\frac{k}{2}\sqrt{1-L^2}\,t} + i(\sqrt{1+L^2}-L)\,qe^{-\frac{k}{2}\sqrt{1+L^2}\,t}, \\ c = le^{\frac{k}{2}\sqrt{1+L^2}\,t} + me^{-\frac{k}{2}\sqrt{1+L^2}\,t}, \\ d = -i(\sqrt{1+L^2}+L)\,le^{\frac{k}{2}\sqrt{1+L^2}\,t} + i(\sqrt{1+L^2}-L)\,me^{-\frac{k}{2}\sqrt{1+L2}\,t}. \end{cases}
$$

p, q, l, m are arbitrary integration constants, L is a constant involved in (2.1), and δ in (2.2) is a function of r occurring in (2.1).

But there are two sets of γ_λ's ($\lambda=1, 2, 3, 4, 5$) which satisfy

$$
\gamma_{(i}\gamma_{j)} = g_{ij}I, \qquad \gamma_i\gamma_5+\gamma_5\gamma_i=0, \qquad \gamma_5\gamma_5=-1. \qquad (i, j=1, \ldots, 4)
$$

Therefore, to find ψ after determining g_{ij} and A_{ij} from the field equation—the condition of integrability of (2.1),—we must consider two sets of fundamental equations[(1)]:

$$
(2.3)\qquad \left(\frac{\partial}{\partial x^i}-\Gamma_i\right)\overset{1}{\psi} = (A_i + A_i^{\cdot 5}\gamma_5 + A_i^{\cdot j}\gamma_j + A_i^{\cdot j5}\gamma_j\gamma_5)\overset{1}{\psi},
$$

and

$$
(2.4)\qquad \left(\frac{\partial}{\partial x^i}-\Gamma_i\right)\overset{2}{\psi} = (A_i - A_i^{\cdot 5}\gamma_5 + A_i^{\cdot j}\gamma_j - A_i^{\cdot j5}\gamma_j\gamma_5)\overset{2}{\psi}.
$$

Accordingly we have two sets of u^i made from the solutions of the above-given equation, as follows:

$$
\overset{1}{u}{}^i = \overset{1}{\psi}{}^\dagger A\gamma^i\overset{1}{\psi}, \qquad \overset{2}{u}{}^i = \overset{2}{\psi}{}^\dagger A\gamma^i\overset{2}{\psi}.
$$

The 4-vector u^i which expresses the actual motion of a particle is considered as a linear homogeneous function of $\overset{1}{u}_i$ and $\overset{2}{u}{}^i$, and is given as follows[(2)]:

$$
(2.5)\qquad u^i = \alpha\overset{1}{u}{}^i + \beta\overset{2}{u}{}^i.
$$

On the other hand, in (2.1) if we put

$$
(2.6)\qquad \delta = \xi + i\eta \qquad (\xi, \eta \text{ are real})
$$

(1) I. p. 64.

(2) I. p. 53, Assumption V.

then

$$\psi^\dagger A\gamma^i\psi=\cosh\eta\cdot\phi^\dagger A\gamma^i\phi-i\sinh\eta\cdot\phi^\dagger A\gamma^i\gamma_5\phi\,,$$

therefore we have

$$\overset{1}{u}{}^i=\cosh\eta\cdot v^i-i\sinh\eta\cdot v_5^i\,, \tag{2.7}$$

where

$$\begin{cases} v^i=\phi^\dagger A\gamma^i\phi\,, \\ v_5^i=\phi^\dagger A\gamma^i\gamma_5\phi\,. \end{cases} \tag{2.8}$$

In the same way we have $\overset{2}{u}{}^i$ by substituting $-\eta$, $-L$, p', q', l', m' for η, L, p, q, l, m in the expressions of u^i above. The explicit forms of v^i and v_5^i are given in cylindrical coordinates(1):

$$\begin{cases} v^\rho=-k\rho\sqrt{1-k^2r^2}\,T_0-k^2\rho zT_1-(1-k^2\rho^2)\cos\varphi T_2 \\ \qquad\qquad -(1-k^2\rho^2)\sin\varphi T_3\,, \\ v^z=-kz\sqrt{1-k^2r^2}\,T_0+(1-k^2z^2)\,T_1+k^2\rho z\cos\varphi T_2-k^2\rho z\sin\varphi T_3\,, \\ v^\varphi=\dfrac{\cos\varphi}{\rho}\,T_3+\dfrac{\sin\varphi}{\rho}\,T_2\,, \\ v^t=\dfrac{1}{\sqrt{1-k^2r^2}}\,T_4\,, \end{cases} \tag{2.9}$$

$$\begin{cases} iv^\rho_{\cdot5}=(\cos\varphi S_1+\sin\varphi S_2)\sqrt{1-k^2r^2}-kz\cos\varphi S_3+kz\sin\varphi S_4\,, \\ iv^z_{\cdot5}=\sqrt{1-k^2r^2}\,S_0+k\rho\cos\varphi S_3-k\rho\sin\varphi S_4\,, \\ iv^\varphi_{\cdot5}=\dfrac{\sqrt{1-k^2r^2}}{\rho}\,(\cos\varphi S_2-\sin\varphi S_1)+\dfrac{kz}{\rho}\,(\cos\varphi S_4+\sin\varphi S_3)+kS_5\,, \\ iv^t_{\cdot5}=-S_6-\dfrac{k}{\sqrt{1-k^2r^2}}\,(zS_7+\rho\cos\varphi S_8-\rho\sin\varphi S_9)\,, \end{cases} \tag{2.10}$$

where T_0, T_1, ...; S_0, S_1, ... are exponential functions of t, as shown in I. From (2.5), (2.7), (2.9), and (2.10) we have u^i, therefore the equation of motion is given by

$$\frac{d\rho}{u^\rho}=\frac{dz}{u^z}=\frac{d\varphi}{u^\varphi}=\frac{dt}{u^t}\,. \tag{2.11}$$

§ 3. The characteristic conditions for the solar system.

The form of u^i, determined by the considerations in § 2, is available for any local irregularity in the universe, and is not peculiar to the solar system. Accordingly, in order to obtain the equation of motion proper to the solar system, we must characterize u^i obtained in § 2 by some physical

(1) I. Note IV.

nature of the solar system. For this purpose we shall inquire into the existing theories.

First, we shall examine the classical theory of gravitation; in his theory Newton characterized the solar system by considering a planet as "a physical existence which is influenced by the central force of the sun." Under this consideration, the three laws of Kepler were deduced by means of his gravitational law (the inverse-square law) and the laws of motion.

On the other hand, in the general theory of relativity $K_{ij}=0$ (K_{ij} being the contracted Riemann-Christoffel tensor) was taken as the law of gravitation, and under the theoretical consideration that the gravitational field of the solar system is that of an isolated particle (the sun) continually at rest at the origin, the solution g_{ij} of the equation $K_{ij}=0$ was sought, and well-known Schwarzschild's line element was obtained.

Now, in our theory, as the quantity whose physical meaning has been clarified, we have only u^i for the present, so if we intend to give our theory the characteristics of the solar system, it is desirable to restrict u^i by some physical nature of the solar system. From this point of view, we prefer Newton's theory to Einstein's as our guide.

But in Wave Geometry, since the laws hitherto obtained are still kinematical, and not dynamical, a dynamical expression such as that "the solar system is a physical existence which is influenced by the central force of the sun" is not avairable as it stands. Therefore we must translate the terminology above into kinematical description. Now, in the Newtonian dynamics, the condition of a particle's being influenced by a central force is equivalent to the particle's moving in a plane and its areal velocity's being constant. Accordingly, instead of the condition "central force" we take "plane motion" and "areal law" as the condition to be imposed on u^i to characterize our theory as that for the solar system. These conditions, especially the areal law, may seem at first glance to be very stringent; they are not, however, but a quite weak restriction, at least not so stringent as to presume the property which is to be deduced as a result of the theory. On this point we shall give some detailed discussion in Note I.[1]

But the conditions; plane motion and areal law are those taken in Newtonian theory, therefore we must impose these conditions on our theory in its Newtonian approximation, but not in the general case. Then, the problem arises: In what form of approximation may our theory be taken as Newtonian? To answer this problem, we proceed with our consideration as follows.

The coefficients $T_0, T_1, \ldots ; S_0, S_1, \ldots$ in (2.8) and (2.9) are functions of t in the form of e^{lkt} (l is a constant) as was mentioned there, and, since $lkt \ll 1$, we have

(1) P. 246.

$$e^{lkt}=1+lkt+\frac{(lkt)^2}{2!}+\cdots\cdots. \tag{3.1}$$

Therefore, by putting $kt=0$ in (3.1), we have the first approximation of our theory, which may be expected to be Newtonian. Of course, this expectation is still a mere presumption. Whether this approximation is precisely Newtonian or not will be decided only when we have actually obtained the Newtonian theory as the first approximation ($kt=0$) of our general theory. To this problem we shall return later. At any rate, for the present we shall take our first approximation ($kt=0$) as Newtonian, and in that approximation we shall characterize the theory by the physical nature of the solar system.

The dimension of the solar system is less than 10^{15} cm., so $k\rho$ in (2.8) and (2.9) is less than 10^{-13}. Therefore we may neglect $k\rho$ compared with 1.

Thus we have the conditions for characterizing the solar system as follows:

$\mathrm{I_s}$ $k\rho \ll 1$.

$\mathrm{II_s}$ Particles in the solar system are governed by the central force of the sun,

or in kinematical terms:

$\mathrm{II_s}$a A particle in the solar system moves in a plane,

$\mathrm{II_s}$b The areal velocity of a particle in the system is constant.

§ 4. The theory of the solar system.

From (2.8) and (2.9), using conditions $\mathrm{I_s}$, $\mathrm{II_s}$a, and $\mathrm{II_s}$b in § 3, we have the following simple expressions for u^i (1):

$$\begin{cases} u^\rho = K_1 \cos\varphi + K_2 \sin\varphi\,, \\ u^z = 0\,, \\ u^\varphi = \dfrac{1}{\rho}(K_2\cos\varphi - K_1\sin\varphi + K_3)\,, \\ u^t = 1\,, \end{cases} \tag{4.1}$$

where K_1, K_2, and K_3 are given by (N. 13) and (N. 14a, b, c) in Note II, and depend only on t in the form of e^{lkt}.

As the first approximation, which is expected to be Newtonian, putting $kt=0$ in K_1, K_2, and K_3, we have, by solving (2.11),

$$\rho = \frac{c_1}{K_2\cos\varphi - K_1\sin\varphi - K_3} \tag{4.2}$$

where c_1 is an integration constant of (2.11), and K_1, K_2, K_3 are con-

(1) See Note II of this paper (p. 265).

stants involved in (4.1). (4.2) shows that the orbit of a particle in the solar system is a conic, coinciding with the result of the Newtonian theory. Thus our expectation that the first approximation of our theory would be Newtonian is justified.

Next, in the higher approximation, neglecting the terms higher than k^2t^2 in (3.1), we have

$$
(4.3)\quad \begin{cases} u^\rho = (K_1 + K_1' kt)\cos\varphi + (K_2 + K_2' kt)\sin\varphi\,, \\ u^z = 0\,, \\ u^\varphi = \dfrac{1}{\rho}\Big\{(K_2 + K_2' kt)\cos\varphi - (K_1 + K_1' kt)\sin\varphi + K_3 + K_3' kt\Big\}\,, \\ u^t = 1\,. \end{cases}
$$

In (4.3), the terms multiplied by kt are very small compared with 1, so taking these as secular terms, by solving (2.11), we have the path-curve of a particle as follows:

$$
(4.4)\quad \rho = \frac{c_1}{(K_2 + K_2' kt)\cos\varphi - (K_1 + K_1' kt)\sin\varphi + (K_3 + K_3' kt)}\,,
$$

or

$$
\rho = \frac{C + \Delta C}{1 + (\epsilon + \Delta\epsilon)\cos(\varphi - \omega - \Delta\omega)}\,,
$$

where

$$
\epsilon = \frac{\sqrt{(K_1)^2 + (K_2)^2}}{K_3}\,, \qquad \Delta\epsilon = \frac{\sqrt{(K_1)^2 + (K_2)^2}}{K_3}\left(\frac{K_1K_1' + K_2K_2'}{(K_1)^2 + (K_2)^2} - \frac{K_3'}{K_3}\right)kt\,,
$$

$$
\omega = \tan^{-1}\frac{K_1}{K_2}\,, \qquad \Delta\omega = \frac{K_1'K_2 - K_1K_2'}{(K_1)^2 + (K_2)^2}kt\,,
$$

$$
c = \frac{c_1}{K_3}\,, \qquad \Delta C = -\frac{c_1K_3'}{(K_3)^2}kt\,.
$$

This result shows that *the path-curve of a particle in the solar system is approximately a conic, but, in secular time, it varies as regards its eccentricity, its longitude of perihelion, and its latus rectum.*

§ 5. The third law of Kepler and integration constants.

In order to review the results obtained in the previous section, we shall take some constants in our theory into consideration, making comparison with the Newtonian theory.

From (4.2), we have

$$
\rho^2\frac{u^\varphi}{u^t} = \rho^2\dot{\varphi} = c_1\,, \qquad \left(\begin{array}{l}\text{dot represents differenta-}\\ \text{tion with respect to } t\end{array}\right)
$$

therefore

$$
(5.1)\qquad c_1 = h = \text{twice the areal velocity.}
$$

Next, if a be half the major axis of the elliptic orbit, it is given by

$$a=\frac{C}{1-\epsilon^2}=\frac{K_3c_1}{(K_3)^2-(K_1)^2-(K_2)^2}, \tag{5.2}$$

and if b be half the minor axis,

$$b=\frac{C}{\sqrt{1-\epsilon^2}}=\frac{c_1}{\sqrt{(K_3)^2-(K_1)^2-(K_2)^2}},$$

therefore the period of revolution along the elliptic orbit is given by

$$T=\frac{2\pi ab}{h}=\frac{2\pi K_3c_1}{(K_3)^2-(K_1)^2-(K_2)^2)^{\frac{3}{2}}}. \tag{5.3}$$

Accordingly, we have, from (5.2) and (5.3)

$$\frac{T^2}{a^3}=\frac{4\pi^2}{K_3c_1}. \tag{5.4}$$

On the other hand, according to Newton's theory

$$\frac{T^2}{a^3}=\frac{4\pi^2}{fM}$$

where f is the universal constant of gravitation, and M the mass of the sun.

Therefore, if we put

$$K_3c_1=fM, \tag{5.5}$$

the right-hand side of (5.4) becomes a constant common to all planets in the solar system, and Kepler's third law holds good in our theory.

It must be noticed that K_3 and c_1 are integration constants of fundamental equation (2.1) and of differential equation (2.11) respectively, and that these cannot be independent of each other, but are connected by (5.5).

Lastly, we shall inquire into the Law of force in our theory in terms of Newtonian dynamics.

In cylindrical coordinates, the component of acceleration in ρ-direction is given from (4.1) and (4.2) as follows:

$$a_\rho=\ddot{\rho}-\rho^2\dot{\varphi}=-\frac{K_3c_1}{\rho^2},$$

therefore, from (5.5),

$$a_\rho=-\frac{fM}{\rho^2}.$$

This is nothing but the Newtonian law of gravitation.

§ 6. Conclusions.

Considering the solar system as a local irregularity in the universe we obtained the following results:

(i) *The orbit of a particle (a planet) in the solar system coincides with the result of the Newtonian Law of Gravitation,*

in the higher approximation,

(ii) *The longitude of perihelion of the elliptic orbit of a planet advances in secular time,*

(iii) *the eccentricity of the orbit varies in secular time, and*

(iv) *the latus rectum of the orbit varies in secular time.*

Result (ii) has already been obtained in the relativistic theory of gravitation, whereas (iii) and (iv) are new theoretical results that have never been deduced in any previous theory. As for the secular change of eccentricity, we have the experimental data[1]:

Mercury	$-0.''88 \pm 0.''33$
Venus	$+0.21 \pm 0.21$
Earth	$+0.02 \pm 0.07$
Mars	$+0.29 \pm 0.18$.

As the experimental error is quite large, it is hard to say anything definite on the secular change of eccentricity; but of the data above, two exceed probable error, and one is about 2.7 times in excess of probable error, so that the existence of secular change of eccentricity may perhaps be true. This is what our theory expects.

As to the secular change of latus rectum (iv), we have nothing to say experimentally.

In our theory, the number of arbitrary constants to be determined exceeds that of expected experimental data, therefore we can always identify our theoretical values with any observational data. Accordingly, we can give no numerical value to anticipate theoretically the secular changes of eccentricity or latus rectum. Consequently as regards determination of their numerical value we must deduce some more physical relations from our theory, the solution of this problems being left for the future.

Note I.

Let P denote the acceleration directed to the centre of force, i. e., the sun, (P may be any function of $\rho, \varphi, t, \dot{\rho}, \dot{\varphi}$ etc.); then we have the following equations in cylindrical coordinates:

(N. 1) $$\ddot{\rho} - \rho\dot{\theta}^2 = -P,$$

(N. 2) $$\frac{d}{dt}(\rho^2\dot{\theta}) = 0.$$

The latter equation gives on integration

$$\rho^2\dot{\theta} = h = \text{constant},$$

and eliminating t from (N. 1),

(1) Cf. Eddington: The Mathematical Theory of Relativity, p. 89.

$$\text{(N. 3)} \qquad \frac{h}{\rho^2}\frac{d}{d\theta}\left(\frac{h}{\rho^2}\frac{d\rho}{d\theta}\right)-\frac{h^2}{\rho^3}=-P \qquad \left(\frac{d}{dt}=\frac{h}{\rho^2}\frac{d}{d\theta}\right)$$

This gives the orbit of motion of a particle.

As is seen from (N. 1) and (N. 2), whatever the form of P may be, the areal law always holds good, as long as the central force is assumed, while this assumption gives no definite knowledge as to the orbit of motion, because P is arbitrary and (N. 3) gives any form of the orbit.

Note II.

From the assumption ($\mathrm{II_s}$a) we choose the plane of motion as xy-plane, i. e. $z=0$. Then, from (2.8) and (2.9), putting $z=0$ and taking account of the assumption $\mathrm{I_s}$, we have

$$\text{(N. 4)} \quad \begin{cases} v^\rho=-k\rho T_0-T_2\cos\varphi+T_3\sin\varphi\,, \\ v^z=T_1\,, \\ v^\varphi=\dfrac{1}{\rho}(T_3\cos\varphi+T_2\sin\varphi)\,, \\ v^t=T_4\,, \end{cases}$$

$$\text{(N. 5)} \quad \begin{cases} iv^\rho_{\cdot 5}=S_1\cos\varphi+S_2\sin\varphi\,, \\ iv^z_{\cdot 5}=S_0\,, \\ iv^\varphi_{\cdot 5}=\dfrac{1}{\rho}(S_2\cos\varphi-S_1\sin\varphi)+kS_5\,, \\ iv^t_{\cdot 5}=-S_6\,. \end{cases}$$

Substituting (N. 4) and (N. 5) into (2.7) and (2.5), and taking account of the assumption ($\mathrm{II_s}$a), u^i are given, disregarding the common factor, as follows:

$$\text{(N. 6)} \quad \begin{cases} u^\rho=(a_1+a_2\mu)k\rho+(a_3+a_4\mu+a_5\mu^2)\cos\varphi+(a_6+a_7\mu+a_8\mu^2)\sin\varphi\,, \\ u^z=0\,, \\ u^\varphi=\dfrac{1}{\rho}\Big\{(a_6+a_7\mu+a_8\mu^2)\cos\varphi-(a_3+a_4\mu+a_5\mu^2)\sin\varphi \\ \qquad\qquad\qquad +(a_9+a_{10}\mu)\mu k\rho\Big\}\,, \\ y^t=a_{11}+a_{12}\mu+a_{13}\mu^2 \end{cases}$$

where

$$\text{(N. 7)} \quad \begin{cases} a_1\equiv T_1T_0'-T_1'T_0\,, & a_8\equiv S_0S_2'-S_0'S_2\,, \\ a_2\equiv S_0'T_0+S_0T_0'\,, & a_0\equiv T_1'S_5+T_1S_5'\,, \\ a_3\equiv T_1T_2'-T_1'T_2\,, & a_{10}\equiv S_0'S_5-S_0S_6'\,, \\ a_4\equiv S_0'T_2+S_0T_2'+T_1'S_1+T_1S_1'\,, & a_{11}\equiv T_1'T_4-T_1T_4'\,, \\ a_5\equiv S_0S_1'-S_0'S_1\,, & a_{12}\equiv-(S_0'T_4+S_0T_4'+T_1'S_6+T_1S_6')\,, \\ a_6\equiv T_1'T_3-T_1T_3'\,, & a_{13}\equiv S_0'S_6-S_0S_6'\,, \\ a_7\equiv-(S_0'T_3+S_0T_3')\,, & \mu\equiv\tanh\eta\,. \end{cases}$$

Putting

$$A_1 \equiv (a_1 + a_2\mu)k\rho$$
$$A_2 \equiv a_3 + a_4\mu + a_5\mu^2$$
$$A_3 \equiv a_6 + a_7\mu + a_8\mu^2$$
$$A_4 \equiv (a_9 + a_{10}\mu)\mu k\rho$$
$$A_5 \equiv a_{11} + a_{12}\mu + a_{12}\mu^2 ,$$

(N. 6) is expressed as

$$\text{(N. 8)} \quad \begin{cases} u^\rho = A_1 + A_2 \cos\varphi + A_3 \sin\varphi \\ u^z = 0 \\ u^\varphi = \dfrac{1}{\rho}(A_3 \cos\varphi - A_2 \sin\varphi + A_4) \\ u^t = A_5 \end{cases}$$

In order that u^i shall satisfy the assumption (II_sb) in Newtonian approximation, it must be true that, because of $dx^i/dt = u^i/u^t$

$$\frac{d}{dt}\left(\rho^2 \frac{u^\varphi}{u^t}\right) = 0 ,$$

or

$$\text{(N. 9)} \qquad \frac{u^i}{u^t} \frac{\partial}{\partial x^i}\left(\rho^2 \frac{u^\varphi}{u^t}\right) = 0 .$$

Substituting (N. 8) into (N. 9) and putting $A_i/A_5 \equiv B_i$ ($i = 1, 2, 3, 4$), we have

$$(B_1 + B_2 \cos\varphi + B_3 \sin\varphi)\frac{\partial}{\partial\rho}\left\{\rho(B_3 \cos\varphi - B_2 \sin\varphi + B_4)\right\}$$
$$+ \frac{1}{\rho}(B_3 \cos\varphi - B_2 \sin\varphi + B_4)\frac{\partial}{\partial\varphi}\left\{\rho(B_3 \cos\varphi - B_2 \sin\varphi + B_4)\right\} = 0 ,$$

because B_1, B_2, B_3, B_4 are functions of ρ and do not involve t in Newtonian approximation. Evaluating the equation above, and taking the condition that the resulting equation must hold good for all values of φ, we have

$$\text{(N. 10)} \quad \begin{cases} B_2 B_3' + B_3 B_2' = 0 & \text{(a)} \\ B_3 B_3' - B_2 B_2' = 0 & \text{(b)} \\ (B_1 B_3' + B_2 B_4')\rho + B_3 B_1 = 0 & \text{(c)} \\ (-B_1 B_2' + B_3 B_4')\rho - B_2 B_1 = 0 & \text{(d)} \\ (-B_3 B_2' + B_1 B_4')\rho + B_4 B_1 = 0 , & \text{(e)} \end{cases}$$

where dash denotes differentiation with respect to ρ.

From (N. 10a) and (N. 10b), it follows that

$$\text{(N. 11)} \qquad B_2' = B_3' = 0 ,$$

or $\qquad (B_2)^2 + (B_3)^2 = 0 , \quad \text{i. e.,} \quad B_2 = B_3 = 0 ,$

because B_2 and B_3 are real. As the latter is contained in the former case, we need only consider the former case (N. 11). Then the remaining three equations of (N. 10) become

$$\text{(N. 12)}\quad \begin{cases} B_2B'_4\rho+B_3B_1=0 \\ B_3B'_4\rho-B_2B_1=0 \\ B_1B'_4\rho+B_4B_1=0 \end{cases}$$

respectively. From the last equation of these three it must be true that

Case I. $\qquad B_1=0\,,$

or Case II. $\quad B'_4\rho+B_4=0\,,\quad$ i. e., $\quad B_4=\dfrac{K_4}{\rho}\quad$ (K_4=constant)

In case II, from the first two equations of (N. 12), it must be true that

(A) $\qquad B_1=B_4=0\,,$

or (B) $\qquad B_2=B_3=0\,.$

Similarly, in case I we have

(C) $\qquad B_2=B_3=0\,,$

or (D) $\qquad B'_4=0\,.$

Thus, putting together the equations above, we get the result: in order that u^i given by (N. 8) shall satisfy the assumption (II_sb); A_1, A_2, A_3, A_4, A_5 must be one of the following four forms:

(A) $\quad \left(\dfrac{A_2}{A_5}\right)'=\left(\dfrac{A_3}{A_5}\right)'=0\,,\qquad A_1=A_4=0\,,$

(B) $\quad A_2=A_3=0\,,\qquad \dfrac{A_4}{A_5}=\dfrac{K_4}{\rho}\qquad$ (K_4=constant)

(C) $\quad A_1=A_2=A_3=0\,,\qquad \dfrac{A_4}{A_5}$ = arbitrary function of ρ,

(D) $\quad \left(\dfrac{A_2}{A_5}\right)'=\left(\dfrac{A_3}{A_5}\right)'=\left(\dfrac{A_4}{A_5}\right)'=0\,,\qquad A_1=0\,.$

Now we shall investigate the cases above in detail.

Case (A). $\qquad \dfrac{A_2}{A_5}=K_1\,,\quad \dfrac{A_2}{A_5}=K_2\,,\quad A_1=A_4=0\,,$

i. e.,

$$\frac{a_3+a_4\mu+a_5\mu^2}{a_{11}+a_{12}\mu+a_{12}\mu^2}=K_1\,,\qquad \frac{a_6+a_7\mu+a_8\mu^2}{a_{11}+a_{12}\mu+a_{13}\mu^2}=K_2\,,$$

$$a_1+a_2\mu=0\,,\qquad a_9+a_{10}\mu=0\,,$$

K_1, and K_2, being constants. But $\mu\equiv\tanh\eta$ is a function of ρ and independent of the integration constants of ψ $(l, m, p, q\,;\ l', m', p', q')$, accordingly independent of a_i's. Therefore, we have

$$\mu = \text{constant}$$

or
$$\frac{a_3}{a_{11}} = \frac{a_4}{a_{12}} = \frac{a_5}{a_{13}} = K_1\,, \qquad \frac{a_6}{a_{11}} = \frac{a_7}{a_{12}} = \frac{a_8}{a_{13}} = K_2\,,$$

and u^i becomes as follows:

$$\begin{cases} u^\rho = K_1 \cos\varphi + K_2 \sin\varphi\,, \\ u^z = 0 \\ u^\varphi = \dfrac{1}{\rho}(K_2 \cos\varphi - K_1 \sin\varphi) \\ u^t = 1\,. \end{cases}$$

Case (B). $\quad a_3 + a_4\mu + a_5\mu^2 = 0\,, \qquad a_6 + a_7\mu + a_8\mu^2 = 0\,,$

$$\frac{(a_9 + a_{10}\mu)\mu k\rho}{a_{11} + a_{12}\mu + a_{13}\mu^2} = \frac{K_1}{\rho} \qquad (K_1 = \text{constant})$$

From the last equation above, we have the following two cases:

$$(\mathrm{B}_1) \quad \begin{cases} \mu = \dfrac{n_1}{\rho^2} & (n_1 = \text{constant}) \\ a_{13} = 0\,, & \dfrac{a_9}{a_{11}} = \dfrac{a_{10}}{a_{12}} = \dfrac{K_3}{n_1 k} \end{cases}$$

$$\text{or} \quad (\mathrm{B}_2) \quad \begin{cases} \mu = \dfrac{n_2}{\rho} & (n_2 = \text{constant}) \\ a_9 = a_{12} = a_{13} = 0\,, & \dfrac{a_{10}}{a_{11}} = \dfrac{K_3}{n_2 k}\,. \end{cases}$$

Corresponding to the two cases (B_1) and (B_2), we have

$$\begin{cases} u^\rho = \dfrac{(a_1 + a_2 n_1/\rho^2)k\rho}{a_{11} + a_{12} n_1/\rho^2}\,, \\ u^z = 0\,, \\ u^\varphi = \dfrac{K_3}{\rho^2}\,, \\ u^t = 1\,, \end{cases}$$

and

$$\begin{cases} u^\rho = \dfrac{1}{a_{11}}\left(a_1 + a_2 \dfrac{n_2}{\rho}\right)k\rho \\ u^z = 0 \\ u^\varphi = \dfrac{K_3}{p^2} \\ u^t = 1\,. \end{cases}$$

Case (C).

$$A_1=A_2=A_3=0\,,$$

$$\begin{cases} u^\rho=0 \\ u^z=0 \\ u^\varphi=\dfrac{(a_9+a_{10}\mu)\mu k\rho}{a_{11}+a_{12}\mu+a_{13}\mu^2} \\ u^t=1\,. \end{cases}$$

Case (D).

$$a_1+a_2\mu=0\,,$$

$$\text{(N. 13)}\quad \begin{cases} \dfrac{a_3+a_4\mu+a_5\mu^2}{a_{11}+a_{12}\mu+a_{13}\mu^2}=K_1 \\ \dfrac{a_6+a_7\mu+a_8\mu^2}{a_{11}+a_{12}\mu+u_{13}\mu^2}=K_2 \\ \dfrac{(a_9+a_{10}\mu)\mu k\rho}{a_{11}+a_{12}\mu+a_{13}\mu^2}=K_3\,. \end{cases}$$

From the last equation of (N. 13), we have the following three cases:

$$\text{(i)}\quad \begin{cases} \mu=\dfrac{m_1}{\rho} \qquad (m_1=\text{constant}) \\ a_{13}=0 \qquad K_3=\dfrac{a_{10}}{a_{12}}km_1=\dfrac{a_9}{a_{11}}km_1\,, \end{cases}$$

$$\text{(ii)}\quad \begin{cases} \mu=m_2\rho \qquad (m_2=\text{constant}) \\ a_{10}=a_{11}=a_{12}=0\,, \qquad K_3=\dfrac{a_9}{a_{13}}km_2\,, \end{cases}$$

$$\text{(iii)}\quad \begin{cases} \mu^2=\dfrac{m_3}{\rho} \qquad (m_3=\text{constant}) \\ a_9=a_{12}=a_{13}=0\,, \qquad K_3=\dfrac{a_{10}}{a_{11}}km_3\,. \end{cases}$$

Corresponding to these three cases, from the first and second equations of (N. 13) we have

$$\text{(i)}\quad a_5=a_8=0\,, \qquad K_1=\frac{a_3}{a_{11}}=\frac{a_4}{a_{12}}\,,$$

$$K_2=\frac{a_6}{a_{11}}=\frac{a_7}{a_{12}}\,,$$

$$\text{(ii)}\quad a_3=a_4=a_6=a_7=0\,, \quad K_1=\frac{a_5}{a_{13}}\,, \quad K_2=\frac{a_8}{a_{13}}\,,$$

$$\text{(iii)}\quad a_4=a_5=a_7=a_8=0\,, \quad K_1=\frac{a_3}{a_{11}}\,, \quad K_2=\frac{a_6}{a_{11}}\,.$$

But in all cases (i), (ii), and (iii), u^i is expressed as

$$\begin{cases} u^\rho = K_1 \cos\varphi + K_2 \sin\varphi \\ u^z = 0\,, \\ u^\varphi = \dfrac{1}{\rho}(K_2 \cos\varphi - K_1 \sin\varphi + K_3) \\ u^t = 1\,. \end{cases}$$

Putting together the results obtained above, the forms of u^i are classified as follows: (in all cases $u^z=0$)

Case (A): Disregarding the common factor, we have

$$\begin{cases} u^\rho = K_1 \cos\varphi + K_2 \sin\varphi \\ u^\varphi = \dfrac{1}{\rho}(K_2 \cos\varphi - K_1 \sin\varphi + K_3) \\ u^t = 0\,. \end{cases}$$

K_1, K_2, and K_3 being constants.

Case (B_1) $\mu = \dfrac{n_1}{\rho_2}$ (n_1=constant)

$$\begin{cases} u^\rho = (a_1 + a_2\mu)k\rho \\ u^\varphi = (a_9 + a_{10}\mu)k\mu \\ u^t = (a_{11} + a_{12}\mu) \end{cases} \quad \text{or} \quad \begin{cases} u^\rho = \dfrac{a_1 + a_2\mu}{a_{11} + a_{12}\mu}k\rho \\ u^\varphi = \dfrac{K_1}{\rho^2}\,, \\ u^t = 1\,. \end{cases}$$

$$\frac{a_9}{a_{11}} = \frac{a_{10}}{a_{12}} = \frac{K_3}{n_1 k} \qquad (K_3=\text{constant}).$$

Case (B_2) $\mu = \dfrac{n_2}{\rho}$ (n_2=constant)

$$\begin{cases} u^\rho = (a_1 + a_2\mu)k\rho \\ u^\varphi = a_{10}k\mu^2 \\ u^t = a_{11}\,. \end{cases}$$

Case (C) μ=arbitrary function of ρ.

$$\begin{cases} u^\rho = 0\,, \\ u^\varphi = (a_9 + a_{10}\mu k\,, \\ u^t = a_{11} + a_{12}\mu + a_{13}\mu^2\,. \end{cases}$$

Case (D)

$$\begin{cases} u^\rho = K_1 \cos\varphi + K_2 \sin\varphi \\ u^\varphi = (K_2 \cos\varphi - K_1 \sin\varphi + K_3) \\ u^t = 1\,, \qquad (K_1, K_2, K_3 = \text{constants}) \end{cases}$$

where K_1, K_2, K_3 are given by

$$K_1=\frac{a_3+a_4\mu+a_5\mu^2}{a_{11}+a_{12}\mu+a_{13}\mu^2}$$

$$K_2=\frac{a_6+a_7\mu+a_8\mu^2}{a_{11}+a_{12}\mu=a_{13}\mu^2}$$

$$K_3=\frac{(a_9+a_{10}\mu)\mu k\rho}{a_{11}+a_{12}\mu+a_{13}\mu^2}$$

and μ and a_i's are one of the following three forms:

(i) $\mu=\frac{m_1}{\rho}$ (m_1=constant)

$$\text{(N.14a)}\begin{cases} a_1=a_2=a_5=a_8=a_{13}=0 \\ \frac{a_{10}}{a_{12}}km_1=\frac{a_9}{a_{11}}km_1(=K_3) \\ \frac{a_3}{a_{11}}=\frac{a_4}{a_{12}}(=K_1), \qquad \frac{a_6}{a_{11}}=\frac{a_7}{a_{12}}(=K_2) \end{cases}$$

(ii) $\mu=m_2\rho$ (m_2=constant)

$$\text{(N.14b)}\begin{cases} a_1=a_2=a_3=a_4=a_6=a_7=a_{10}=a_{11}=a_{12}=0 \\ \frac{a_5}{a_{13}}=K_1, \quad \frac{a_8}{a_{13}}=K_2, \quad \frac{a_9}{a_{13}}km_2=K_3 \end{cases}$$

(iii) $\mu^2=\frac{m_3}{\rho}$ (m_3=constant)

$$\text{(N.14c)}\begin{cases} a_1=a_2=a_4=a_5=a_7=a_8=a_9=a_{12}=a_{13}=0 \\ \frac{a_3}{a_{11}}=K_1, \quad \frac{a_6}{a_{11}}=K_2, \quad \frac{a_{10}}{a_{11}}km_3=K_3. \end{cases}$$

For respective cases the trajectories of motion of a particle defined by $\frac{d\rho}{u^\rho}=\frac{d\varphi}{u^\varphi}=\frac{dt}{u^t}$ are given as follows:

Case (A): straight lines.
Case (B): some system of curves which are symmetric with respect to z-axis.
Case (C): circle (the circular velocity depends only on ρ).
Case (D): conical curves.

Cases (A) and (C) are included as special cases in case (D) as far as their trajectories are concerned. Therefore if we neglect the case (B), which represents axial-symmetric curve-systems, from the condition that u^i satisfies assumption ($\mathrm{I_s}$), ($\mathrm{II_s}$a), and ($\mathrm{II_s}$b), we have (4.1).

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