# Cosmology in Terms of Wave Geometry (X). Observers on the Nebulae.

By

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## § 1. Introduction.

In previous papers[(1)] we have shown that the de-Sitter type of universe is the only one allowed as a model in the wave-goemetrical cosmology and in that space-time we have deduced Hubble's velocity-distance relation and the other physical properties of this model, starting from the line element

$$ds^2 = -\frac{dr^2}{1-k^2r^2} - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 + (1-k^2r^2)\,dt^2\,, \tag{1.1}$$

and the momentum-density vector

$$\left.\begin{aligned} u^r &= (-pe^{-kt}+qe^{kt})\,kr\sqrt{1-k^2r^2} \\ u^\theta &= u^\phi = 0 \\ u^t &= \frac{pe^{-kt}+qe^{kt}}{\sqrt{1-k^2r^2}}\,. \end{aligned}\right\} \tag{1.2}$$

In a subsequent paper,[(2)] furthermore, the physical properties for the time lapse of this model were discussed.

In this paper we shall show (i) that observers situated on nebulae are equivalent to one another in Milne's sense,[(3)] (ii) that the mathematical expression of light: $ds^2=0$, which was postulated in the derivation of Hubble's velocity-distance relation, is consistent with our theory of the universe previously established, and (iii) what transformation formulae exist between the coordinate systems of observers on the nebulae, provided the nebulae move in accordance with the vector (1.2).

## § 2. Condition for Equivalent Observers.

We define, after Milne,[(4)] that two observers $A$ and $B$ are equivalent

(1) Y. Mimura and T. Iwatsuki, this Journal **8** (1938), 193, (W. G. No. 28).
T. Sibata, this Journal **8** (1938), 199, (W. G. No. 29).
H. Takeno, this Journal **8** (1938), 223, (W. G. No. 30).
K. Itimaru, this Journal **8** (1938), 239, (W. G. No. 31).
(2) K. Itimaru, this Journal **10** (1940), 151, (W. G. No. 37).
(3) E. A. Milne: Relativity, Gravitation and World-Structure, (1935), 24.
(4) E. A. Milne: loc. cit.

when the totality of the observations which $A$ can make on $B$ can be described by $A$ in the same way as the totality of the observations which $B$ can make on $A$ can be described by $B$. Milne derived the condition of equivalence as follows:

Suppose that $A$ and $B$ are equivalent observers provided with identical clocks. Let $B$ send a light signal at epoch $t_1'$ by his clock, and let $A$ reflect it at epoch $t_A$ by $A$'s clock; call this event $E_A$. This reflected signal is reflected at $B$ again, at $t_B'$, by $B$'s clock; call this $E_B$. Lastly, let $A$ receive this signal at time $t_2$ by $A$'s clock (see Fig. 1). Thus observers $A$ and $B$ have two data, $(t_A, t_2)$ and $(t_1', t_B')$ respectively.

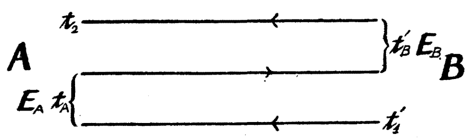


Fig. 1.

From these data, $A$ and $B$ now assign coordinates (epochs and distances) to events $E_B$ and $E_A$ as follows:

For Event $E_B$. $A$'s assignments:

$$T_B = \frac{1}{2}(t_2 + t_A), \qquad R_B = \frac{1}{2}c(t_2 - t_A). \tag{2.1}$$

For Event $E_A$. $B$'s assignments:

$$T_A' = \frac{1}{2}(t_B' + t_1'), \qquad R_A' = \frac{1}{2}c(t_B' - t_1'), \tag{2.2}$$

where $c$ is a positive number. (These assignments are the fundamental postulates in Milne's treatment.) From these equations, eliminating $t_2$ and $t_1'$, we have

$$T_B - \frac{R_B}{c} = t_A, \tag{2.3}$$

$$T_A' - \frac{R_A'}{c} = t_B'. \tag{2.4}$$

From repeated observations of event $E_B$ at $B$, $A$ can determine $R_B$ as a function of $T_B$, and also $t_B'$ as a function of $T_B$. Let these functions be

$$R_B = c\phi(T_B), \qquad t_B' = f(T_B). \tag{2.5}$$

Similarly $B$ can determine $R_A'$ and $t_A$ as functions of $T_A'$. Since $A$ and $B$ are equivalent, these functions must be expressed as

$$R_A' = c\phi(T_A'), \qquad t_A = f(T_A'). \tag{2.6}$$

By employing (2.5) and (2.6), from (2.3) and (2.4) it follows that

$$T_B - \phi(T_B') = f(T_A'), \tag{2.7}$$

$$T_A' + \phi(T_A') = f(T_B). \tag{2.8}$$

These equations are the condition of equivalence. In other words, if $A$ and $B$ are equivalent, by the substitution of $T_B$ obtained from (2.7) as a function of $T'_A$, (2.8) must be an identity in $T'_A$.

At the beginning of the derivation of (2.7) and (2.8), as the assignments of $A$ and $B$ to each other, Milne postulated (2.1) and (2.2) respectively. This postulate implies the constancy of the velocity of the signal. Accordingly the constancy of the velocity of light follows immediately, although Milne deduced the constancy of the velocity of light as the consequence of the equivalence of the observers. Milne adopted Minkowski's space as the background, in which the velocity of light is necessarily constant, therefore postulates (2.1) and (2.2) may be allowed in Milne's case.

But since in our theory of the universe space is not Minkowskian but of the de-Sitter type, the postulate of constancy of the velocity of light must be abandoned. Accordingly, in order to derive the condition of equivalence of observers, Milne's method must be modified. Hence in our case, as the velocity of light we take provisionally the velocity deduced from $ds^2=0$ instead of Milne's postulates (2.1) and (2.2), which are equivalent to the postulate of constancy of the velocity of light as above mentioned. In our theory of the universe space is of de-Sitter type with line element

$$ds^2=-\frac{dr^2}{1-k^2r^2}-r^2d\theta^2-r^2\sin^2\theta d\phi^2+(1+k^2r^2)dt^2\,,$$

so that the radial velocity of light is

$$\frac{dr}{dt}=(1-k^2r^2)\,. \tag{2.9}$$

Integrating (2.9), we have

$$\int_0^t dt=\int_0^r\frac{dr}{1-k^2r^2}\,,$$

that is,

$$t=\frac{1}{2k}\log\frac{1+kr}{1-kr}\,; \tag{2.10}$$

and solving (2.10) with respect to $r$, we have

$$r=\frac{1}{k}\frac{e^{2kt}-1}{e^{2kt}+1}\,. \tag{2.11}$$

Thus $A$'s assignment of coordinates to the event $E_B$ becomes

$$T_B=\frac{1}{2}(t_2+t_A)\,,\qquad R_B=\frac{1}{k}\frac{e^{k(t_2-t_A)}-1}{e^{k(t_2-t_A)}+1}\,, \tag{2.12}$$

and similarly $B$'s assignment to the event $E_A$ becomes

$$T'_A=\frac{1}{2}(t'_B+t'_1)\,,\qquad R'_A=\frac{1}{k}\frac{e^{k(t'_B-t'_1)}-1}{e^{k(t'_B-t'_1)}+1}\,. \tag{2.13}$$

From (2.12) and (2.13), eliminating $t_2$ and $t'_1$ respectively, we have

$$R_B=\frac{1}{k}\frac{e^{2k(T_B-t_A)}-1}{e^{2k(T_B-t_A)}+1}\,, \tag{2.14}$$

and

$$R'_A=\frac{1}{k}\frac{e^{2k(t'_B-T'_A)}-1}{e^{2k(t'_B-T'_A)}+1}\,. \tag{2.15}$$

In like manner as in (2.5) and (2.6), we put

$$R_B=\Phi(T_B)\,,\qquad t'_B=F(T_B)\,, \tag{2.16}$$

and

$$R'_A=\Phi(T'_A)\,,\qquad t_A=F(T'_A)\,. \tag{2.17}$$

Substituting these relations into (2.14) and (2.15), we obtain

$$T_B-\frac{1}{2k}\log\frac{1+k\Phi(T_B)}{1-k\Phi(T_B)}=F(T'_A)\,; \tag{2.18}$$

and

$$T'_A+\frac{1}{2k}\log\frac{1+k\Phi(T'_A)}{1-k\Phi(T'_A)}=F(T_B)\,. \tag{2.19}$$

These two equations together are the required condition of equivalence of the observers in our case, corresponding to (2.7) and (2.8) in Milne's case.

Thus we see that, by adopting $\dfrac{dr}{dt}=1-k^2r^2$ as the radial velocity of light, the function $\Phi$, which is the representation of the motion of one observer $A$ (or $B$) by the coordinate system of the other observer $B$ (or $A$), and the function $F$, which represents the relation between the time by the clock of one observer $A$ (or $B$) for an event happening in his own place, and the time by the clock of the other observer $B$ (or $A$) for the same event, satisfy (2.18) and (2.19) simultaneously.

If we take formally $\phi=\dfrac{1}{2k}\log\dfrac{1+k\Phi}{1-k\Phi}$, (2.8) and (2.9) become (2.18) and (2.19) respectively, so that, at a glance, our condition of equivalence seems to be included in Milne's condition; but it is not so; Milne's condition is a limiting case of ours, because when $k$ tends to zero, the de-Sitter line element becomes Minkowskian, and (2.18) and (2.19) become (2.8) and (2.9).

## § 3. Derivation of the Functions $\Phi$ and $F$.

In this section we shall investigate the explicit form of the functions $\Phi$ and $F$ in our theory of the universe.

In the first place, we start from (1.2) in order to determine the from of $\Phi$. The velocity of a nebula $A$ (or $B$) referred to the coordinate system on the other nebula $B$ (or $A$) is, neglecting $p$ compared with $q$ (the case of red shift[1]), as follows:

$$\frac{dr}{dt}=\frac{u^r}{u^t}=kr(1-k^2r^2) \qquad \text{(from (1.2))}.$$

So that, by integration, we have

$$t+c=\frac{1}{k}\log\frac{kr}{\sqrt{1-k^2r^2}},$$

or

$$k\tau e^{kt}=\frac{kr}{\sqrt{1-k^2r^2}} \qquad \text{where } k\tau=e^{kc}.$$

Solving this with respect to $r$, we have

$$r=\frac{\tau e^{kt}}{\sqrt{1+k^2\tau^2e^{2kt}}}. \tag{3.1}$$

Since this equation represents the motion of an observer $A$ (or $B$) in the coordinate system of the other observer $B$ (or $A$), the right-hand side of (3.1) corresponds to the function $\Phi$ metioned in § 2. So we can put

$$r=\Phi(t)=\frac{\tau e^{kt}}{\sqrt{1+k^2\tau^2e^{2kt}}}. \tag{3.1a}$$

In the next place, to determine the form of $F$ we begin with (1.1). When the observers $A$ and $B$ are equivalent, it is necessary that the line element at $B$ expressed in the coordinate system of $A$ must have the same form as that of the line element at $A$ expressed in the coordinate system of $B$. Now, consider $ds^2$ at $B$; then, since $B$ is moving with the velocity

$$\frac{dr'}{dt'}=kr'(1-k^2r'^2),$$

in the coordinate system of $A$, $ds^2$ becomes, because of (1.1),

$$ds^2=\left[-\frac{\left(\frac{dr'}{dt'}\right)^2}{1-k^2r'^2}+(1-k^2r'^2)\right]dt'^2=(1-k^2r'^2)^2dt'^2,$$

in the coordinate system of $A$. On the other hand, in the coordinate system of $B$, the same $ds^2$ must be

$$ds^2=dt^2.$$

And since the two values of the $ds^2$ above must be the same, we have

$$dt^2=(1-k^2r'^2)dt'^2. \tag{3.2}$$

(1) K. Itimaru, this Journal **8** (1938), 239, (W. G. No. 31).

Substituting the relation (3.1) expressed in the coordinate system of $A$ $\left(\text{i. e., } r'=\frac{\tau e^{kt'}}{\sqrt{1+k^2\tau^2 e^{2kt'}}}\right)$ into (3.2), we have

$$dt=(1-k^2r'^2)dt'=\frac{dt'}{1+k^2\tau^2e^{2kt'}}\ .$$

Integrating this, and setting the zero points of $t$ and $t'$ to coincide, we have

$$t=t'-\frac{1}{2k}\log(1+k^2\tau^2e^{2kt'})\ . \qquad (3.3)$$

This is the relation between the time for event happening at $B$ expressed in $B$'s coordinate system and the time for the same event expressed in $A$'s coordinate system; hence the right-hand side of (3.3) must be the function $F$ mention in § 2. So we can put

$$t=F(t')=t'-\frac{1}{2k}\log(1+k^2\tau^2e^{2kt'})\ . \qquad (3.4)$$

Thus we have obtained the functins (3.1 a) and (3.4) as the explicit forms of $\Phi$ and $F$ in our theory of the universe

Substituting these explicit forms (3.1 a) and (3.4) into (2.18) and (2.19), we see that (2.18) and (2.19) become the same equation with regard to $T_B$ and $T'_A$, i. e.,

$$kT'_A+\log(\sqrt{1+k^2\tau^2e^{2kT'_A}}+k\tau e^{kT'_A})=kT_B-\frac{1}{2}\log(1+k^2\tau^2e^{2kT_B})\ .$$

This shows that the two equations (2.18) and (2.19) are satisfied simultaneously by the expressions (3.1 a) and (3.4). Consequently, the functions $\Phi$ and $F$ in our theory satisfy the conditions of equivalence (2.18) and (2.19); that is to say, *all the observers situated on nebulae in our model of the universe are equivalent to one another.*

Remark: When we deduced the physical properties in the wave-geometrical cosmology,[1] especially Hubble's velocity-distance relation, we adopted $ds^2=0$ provisionally as the mathematical expression of light. But, as seen in § 2, in order that the two observers $A$ and $B$ may be equivalent by means of the light-signal characterized by $ds^2=0$, the functions $\Phi$ and $F$ must satisfy (2.18) and (2.19) simultaneously. But on the other hand, the forms of $\Phi$ and $F$ obtained from (1.1) and (1.2) are (3.1 a) and (3.4) respectively. From this we know, when we regard any two observers situated on different nebulae as equivalent, that the adoption of the equation $ds^2=0$ as the expression for the light-signal is not contradictory to the result that we have used (1.2) as the momentum-density vector for the

(1) K. Itimaru, this Journal **8** (1938), 239, (W. G. No. 31).
K. Itimaru, this Journal **10** (1940), 151, (W. G. No. 37).

motion of nebulae. So that we can say that *to regard $ds^2=0$ as the expression for light is consistent with our cosmology,* in which almost all properties have been derived by means of the line element (1.1) and the momentum-density vector (1.2).

## § 4. Transformation formulae.

Since we have appreciated that the observers in our model of the universe are equivalent, we proceed to consider the problem: What are the transformation formulae existing between the coordinate systems accompanied by observers? To answer this question we shall follow the method by which Milne obtained his transformation formulae.

For simplicity, we restrict ourselves to the one-dimensional case. Let $A$ and $B$ be equivalent observers, and $P$ a particle on the line joining $A$ and $B$ (Fig. 2). Let $A$ send, at time $t_1$ by his clock, a light signal which passes over $B$ as time $t_2'$ by $B$'s clock, is reflected at a particle $P$, and passes over $B$ at time $t_3'$ by $B$'s clock, returning to $A$ at time $t_4$ by his clock. So the data $t_1$, $t_4$ are directly obtained by $A$, the data $t_2'$, $t_3'$ by $B$. $A$ assigns to the event of reflection at $P$ the coordinate $(T, X)$ from his data $(t_1, t_4)$ in the same way as (2.12) as follows:

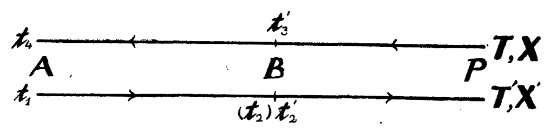


Fig. 2.

$$T=\frac{1}{2}(t_4+t_1)\,, \qquad X=\frac{1}{k}\,\frac{e^{k(t_4-t_1)}-1}{e^{k(t_4-t_1)}+1}\,. \tag{4.1}$$

Similarly, $B$ assigns to the same event the coordinate $(T'\ X')$ from his data $(t_2', t_3')$ as follows:

$$T'=\frac{1}{2}(t_3'+t_2')\,, \qquad X'=\frac{1}{k}\,\frac{e^{k(t_3'-t_2')}-1}{e^{k(t_3'-t_2')}+1}\,. \tag{4.2}$$

Hence we have, from (4.1) and (4.2),

$$T+\frac{1}{2k}\log\frac{1+kX}{1-kX}=t_4\,, \quad (4.3) \qquad T=\frac{1}{2k}\log\frac{1+kX}{1-kX}=t_1\,, \quad (4.4)$$

$$T'+\frac{1}{2k}\log\frac{1+kX'}{1-kX'}=t_3'\,, \quad (4.5) \qquad T'=\frac{1}{2k}\log\frac{1+kX'}{1-kX'}=t_2'\,. \quad (4.6)$$

If $t_2$ is the time in the ccordinate system of $A$ for $t_2'$ at $B$, then, from (2.16), between $t_2'$ and $t_2$ there holds good the relation:

$$t_2'=F(t_2)\,.$$

On the other hand, since $(t_2-t_1)$ is the lapse of time during the passage of light from $A$ to $B$ measured in the coordinate system of $A$, from (2.10) and (2.16) it must follow that

$$t_2-t_1=\frac{1}{2k}\log\frac{1+k\Phi(t_2)}{1-k\Phi(t_2)},$$

or

$$t_1=t_2=\frac{1}{2k}\log\frac{1+k\Phi(t_2)}{1-k\Phi(t_2)}. \tag{4.7}$$

Substituting the explicit forms (3.1a) and (3.4) of the functions $\Phi$ and $F$ into (4.7), we have the expression of $t_1$ as the function of $t_2'$; i. e.,

$$t_1=t_2'-\log(1+k\tau e^{kt_2'})\equiv p(t_2'). \tag{4.8}$$

Similarly, since $t_3'$ must be expressed as the same function of $t_4$ as (4.8), we have

$$t_3'=p(t_4). \tag{4.9}$$

Accordingly, if we substitute (4.4) for $t_1$ and (4.6) for $t_2$ in (4.8), we have

$$T-\frac{1}{2k}\log\frac{1+kX}{1-kX}=p\Big(T'-\frac{1}{2k}\log\frac{1+kX'}{1-kX'}\Big). \tag{4.10}$$

In the same way, from (4.3), (4.5), and (4.9), we have

$$T'+\frac{1}{2k}\log\frac{1+kX'}{1-kX'}=p\Big(T+\frac{1}{2k}\log\frac{1+kX}{1-kX}\Big). \tag{4.11}$$

But since we know that the explicit form of the function $p$ is (4.8), we can solve $T$ and $X$ as functions of $T'$ and $X'$;

$$e^{kT}=\frac{e^{kT'}}{\sqrt{1-k^2\tau^2e^{2kT'}-\dfrac{2k^2X'\tau e^{kT'}}{\sqrt{1-k^2X'^2}}}}, \tag{4.12}$$

and

$$X=X'+\tau e^{kT'}\sqrt{1-k^2X'^2}, \tag{4.13}$$

which are the required transformation formulae between the coordinate systems of observers $A$ and $B$.

Dr. T. Sibata,[(1)] on a basis of the homogeneity of space, deduced that de-Sitter space is the only model having the homogeneous property and fitted to explain the actual universe; and the transformation formulae between the coordinate systems of observers were obtained by him in the form:

$$\left.\begin{aligned} &z=z'+e^{kt'}\sqrt{1-k^2r'^2}\,\tau, \qquad x=x', \qquad y=y', \\ &e^{kt}=e^{kt'}[1-k^2\tau^2e^{2kt'}-2k^2z'e^{kt}\tau/\sqrt{1-k^2r'^2}]^{\frac{1}{2}}, \end{aligned}\right\} \tag{4.14}$$

In his definition of homogeneity of the universe it is fundamental that $ds^2$ must have the same form for each observer situated on a nebula who assigns

(1) T. Sibata, this Journal **11** (1941), 21, (W. G. No. 43).

the coordinates by the same manner; therefore our definition for the equivalent observers accords exactly with his definition of homogeneity; hence it is concluded that *a universe constructed of equivalent observers has the homogeneous property in Sibata's sense.* In fact, in the one-dimensional case Sibata's result (4.14) exactly coincides with our (4.12) and (4.13).

## § 5. Conclusion.

Milne abandoned the general relativistic idea for the structure of the universe, and assumed that observers resting on the nebulae in the universe have an intuitive notion of the lapse of time, but not of spatial distance, and are equivalent to one another. Thus he set up a kinematical theory of the universe which has no connection with general Relativity, and interpreted Hubble's velocity-distance relation.

On the other hand, general Relativity has established the elegant theory of the solar system, and explained the cosmological and other physical phenomena to some extent, but not yet completely. Therefore, for the unification of physical theories, it is desirable to unify Milne's and relativistic theories.

In the wave-geometrical cosmology, starting from the fundamental consideration that each of the nebulae is a test-particle in detecting the construction of the universe, as well as being a constituent of the universe,(1) we have deduced that the space of the universe is of de-Sitter type, and that the motion of a nebula is represented by the momentum-density vector (1.2). And, provided that the light-signal satisfied $ds^2=0$, observers resting on the nebulae are equivalent to one another. In other words, in the wave-geometrical cosmology the space of the universe is of de-Sitter type from the relativistic point of view, and observers resting on the nebulae in the universe are equivalent to each other from the kinematical point of view. So that we are led to the conclusion that *in our theory the kinematical theory and the relativistic theory are complementally unified.*

This problem was discussed at a special Seminar of Geometry and Theoretical Physics in the Hirosima University, and research into it has been carried on under the Scientific-Research Expenditure of the Department of Education.

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(1) Y. Mimura and T. Iwatsuki, loc. cit.